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PREDICTION AND TESTING IN A GENERALISED LIFE TEST

A generalised life test model involving Laguerre Polynomials is considered. Using this model, a procedure to test a hypothesis concerning the equality of parameters in (p+1) Weibull populations is developed. Secondly, the prediction problem is considered. Using this above model, the sum total of the observations in a future sample from a Weibull population is predicted in terms of earlier samples and for this purpose, a predictive distribution is obtained.

1. Introduction

Life test models are often represented by general functions, in addition to being represented by simpler models such as exponential and gamma distributions. One situation is Zelen and Donnamiller [12] where they have introduced a generalised life test model involving Laguerre polynomials and they have obtained the distribution of the sum of n-independent observations from a population represented by this general model. They have also shown that the distribution of the sum of n-independent Weibull variables can be approximated by the above distribution. Basu and Lochner [1] have used this approximate distribution to develop a procedure to test the hypothesis of the equality of parameters in two independent Weibull populations. In this paper another test procedure is developed in which the distribution of the statistics turns out to be a linear combination of Dirichlet distributions. This statistic is used to test whether the parameters in (p+1) independent

Weibull populations are equal or not. In this connection, it may be noted that Glaser [2], [3] has developed a statistic in terms of the ratio of geometric mean to the arithmetic mean of gamma variables and the distribution of such a statistic in terms of Dirichlet distribution is used to test the equality of variances in k normal populations. This author has dealt with some properties and generalisations of Dirichlet distribution in [8], [9]. Next, the problem of prediction is taken up. Literature on this subject of prediction is quite large. For example, Lawless [5] predicts an order statistics in terms of the sum of order statistics. Lingappaiah [6], [7] predicts an order statistics in terms of another order statistics. Kaminsky [4] gives the extension of the results of Lingappaiah [6] and Lawless [5]. Lingappaiah [10] uses this Bayesian approach with reference to reliability. Here, p independent samples from a Weibull population are considered. First by using a suitable prior $g(\theta)$ and the data from sample 1, a predictive distribution is obtained at the second stage (sample 2), by taking the posterior at stage one (sample 1) as the prior for second stage. Similarly continuing on this way, the posterior at stage (p-1) is taken as the prior for stage p (sample p) and a predictive distribution is obtained at stage p, from which the sample total at stage p can be predicted by using similar totals at earlier stages (Samples).

2. The test procedure

Generalised life test model as given by Zelen and Donnamiller [12] in terms of Laguerre polynomials is

$$f(x) = \frac{e^{-x}x}{\Gamma(\alpha+1)} \sum_{s=0}^{k} a_s L_s^{\alpha}(x), \quad x > 0 \tag{1}$$

$\alpha > -1$, k a positive integer and $L_s^{\alpha}(x)$ is the Laguerre Polynomial

(2) $$L_s^{\alpha}(x) = \sum_{j=0}^{s} \binom{s+\alpha}{s-j} (-x)^j / j!$$

and a_s's in (1) are suitably chosen with $a_0 = 1$. Also from Basu and Lochner [1], we have the approximate distribution of the $y = c_0 \sum_{i=1}^{n} x_i$, [$c_0$ is as in (7a) below], where x has the Weibull model

(3) $$f(x) = \frac{\rho x^{\rho-1}}{\theta} \exp\left[-x^{\rho}/\theta\right] \quad x, \theta, \rho > 0.$$

as

(4) $$f(y) = \sum_{s=0}^{nk} \alpha_s \frac{s!}{\Gamma(N+s)} \bar{e}^{y} y^{N-1} L_s^{N-1}(y),$$

where $N = n(\alpha+1)$ and α_s is the coefficient of t^s in the expansion of

(5) $$\left[1 + \binom{\alpha+1}{1} a_1 t + \binom{\alpha+2}{2} a_2 t^2 + \ldots + \binom{\alpha+k}{k} a_k t^k\right]^n.$$

From (5) it follows, for computational purposes,

(6) $$\alpha_{s,n} = \sum_{j=0}^{s} \alpha_{s-j,n-1} \binom{\alpha+j}{j} a_j.$$

Basu and Lochner [1] use (4) to develop a procedure to test $\theta_1 = \theta_2$ where θ_1 and θ_2 are the parameters in two populations following (3). Suppose, we have (p+1) independent populations each represented by (3) with parameters $\theta_1, \theta_2, \ldots, \theta_{p+1}$ and common ρ, the sum y_i in the sample of size n_i from the i-th population has the pdf similar to (4) with k_i replaced for k in (4) and $N_i = n_i(\alpha_i+1)$, (may be common α). Then the joint distribution of these p+1 sample totals can be expressed as (taking N_i as integer, $i=1,2,\ldots,p+1$)

$$(7)\quad f(y_1,\dots,y_{p+1})=\prod_{i=1}^{p+1}\sum_{s_i=0}^{n_ik_i}\sum_{j_i=0}^{s_i}\alpha_{s_i,i}\binom{s_i}{j_i}\frac{(-1)^{j_i}e^{y_i}y_i^{N_i+j_i-1}}{\Gamma(N_i+j_i)}$$

where

$$(7a)\quad y_i=\left[\rho\sum_{j=1}^{n_i}x_{ij}\right]\Big/\theta_i^{1/\rho}\,\Gamma(1+1/\rho)=c_0\sum x_{ij},$$

and $\alpha_{s_i,i}$ is the coefficient of t^{s_i} in the expansion of

$$(8)\quad \left[1+\binom{\alpha_i+1}{1}a_{1(i)}t+\binom{\alpha_i+2}{2}a_{2(i)}t^2+\dots+\binom{\alpha_i+k_i}{k_i}a_{k(i)}t^{k_i}\right]^{n_i}$$

$a_{1(i)},\dots a_{k(i)}$ are similar to $a_1,\dots a_k$ in (5).

Let

$$(9)\quad z_i=y_i/y_{p+1},\quad i=1,2,\dots p,\; z_{p+1}=y_{p+1}.$$

The Jacobian of transformation is $|J|=z_{p+1}^p$ and using $|J|$ in (7) and integrating out z_{p+1}, we have the joint pdf of $z_1,z_2,\dots z_p$ as

$$(10)\quad f(z_1,\dots,z_p)=\prod_{i=1}^{p+1}\sum_{s_i}\sum_{j_i}\left[\alpha_{s_i,i}\binom{s_i}{j_i}(-1)^{j_i}\Big/\Gamma(N_i+j_i)\right]\times$$

$$\times\left[\frac{z_1^{N_1+j_1-1}\dots z_p^{N_p+j_p-1}\,\Gamma(N+j)}{(1+z_1+\dots+z_p)^{N+j}}\right],$$

where $N=N_1+\dots+N_{p+1}$, $j=j_1+\dots+j_{p+1}$.

From (7a), we have

$$(11) \qquad z_i = \left[n_i \hat{\theta}_i / \theta_i^{1/\rho}\right] \Big/ \left[n_{p+1} \hat{\theta}_{p+1} / \theta_{p+1}^{1/\rho}\right],$$

where $\hat{\theta}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and now we have to evaluate $c_1, \ldots c_p$ (say with $n_1 = \ldots = n_{p+1}$) as

$$(12) \qquad \int_0^{c_1} \ldots \int_0^{c_p} f(z_1, \ldots z_p;\ \theta_1 = \ldots = \theta_{p+1}) dz_p \ldots dz_1 = 1-\alpha$$

and

$$(13) \qquad \int_0^{c_1} \ldots \int_0^{c_\rho} f(z_1, \ldots z_p;\ \theta_1, \theta_2, \ldots, \theta_{p+1}) dz_p \ldots dz_1 = \beta .$$

Now using

$$(14) \qquad \frac{1}{B(a,b)} \int_0^y \frac{x^{a-1} dx}{(1+x)^{a+b}} = \left[1 - \sum_{m=0}^{a-1} \binom{b+m-1}{m} \frac{y^m}{(1+y)^{b+m}}\right]$$

we can write (12) as

$$(15) \qquad 1-\alpha = \prod_{i=1}^{p+1} \left[\sum_{s_i=0}^{n_i k_i} \sum_{j_i=0}^{s_i} \binom{s_i}{j_i} \alpha_{s_i,i} (-1)^{j_i} \Big/ \Gamma(d_i)\right] \left[\Gamma(D_\rho)\right] \times$$

$$\times \left[\sum_{l=0}^{p} \sum \sum_{j=1}^{l} (-1)^l \frac{\Phi(l) c_{i_j}^{m_{i_j}} \begin{pmatrix} D_{i_j} - 1 = \sum_{q=1}^{j} m_{i_q} - 1 \\ m_{i_j} \end{pmatrix}}{\left(1 + \sum_{j=1}^{l} c_{i_j}\right)^{D_0 + \sum_{j=1}^{l} m_{i_j}}}\right]$$

where $i_1, \dots i_p = 1,2,\dots p$ and $i_1 > i_2 > \dots > i_p$,

$$d_i = N_i + j_i,\ d_o = D_o = d_{p+1},\ D_i = \sum_{j=0}^{i} d_j$$

and $\Phi(1)$ is the product of complete beta functions of the type $B(a,b)$ and the summation $\sum$ is on the permutations of $i_1, \dots i_1$. For example if $p = 2$ (to test $\theta_1 = \theta_2 = \theta_3$), $n_i = 2$, $k_i = 1$, $\alpha_i = 1$ $(N_i = 4)$, $d_i = N_i + j_i = 4 + j_i$, $i=1,2,3$ we get the terms in the last braces in (15) as

$$(16) \qquad B(d_2,D_1)B(d_1,D_0)\left[1 - \sum_{m_1=0}^{d_1-1}\binom{D_0+m_1-1}{m_1}\frac{c_1^{m_1}}{(1+c_1)^{D_0+m_1}}\right] -$$

$$- \sum_{m_2=0}^{d_2-1}\binom{D_1+m_2-1}{m_2} c_2^{m_2}\, B(d_2,D_1)B(d_1,D_0+m_2)\times$$

$$\times\left[\frac{1}{(1+c_2)^{D_0+m_2}} - \sum_{m_1=0}^{d_1-1}\binom{D_0+m_1+m_2-1}{m_1}\frac{c_1^{m_1}}{(1+c_1+c_2)^{D_0+m_1+m_2}}\right].$$

Similarly β in (13) can be computed. In (16), c_1, c_2 such that (with $d_0 = D_0 = d_3$)

$$(16a) \qquad \int_0^{c_1}\int_0^{c_2} f(z_1,z_2,\theta_1=\theta_2)dz_2dz_1 = 1-\alpha.$$

C o m m e n t s : For simplicity, we can set all $n_i = n$ and $k_i = k$, $i = 1,2,\dots,p+1$, though in practice, it may not be feasible. If these are not equal, computation will be slightly complex. Also, if all the n_i's are equal, it may set a limit on the power of the test. We have also set $n_i(\alpha_i+1)=N_i$, an integer which may not be a severe ristriction. When α_i's are not integers, the problem may be the choice of $a_{i(j)}$'s, $j = 1,2,\dots k_i$, $i = 1,2,\dots,p+1$ in which one way may be the method of moments used by Basu and Lochner [1].

3. Predictive distribution

Now consider (4) where

$$y = \rho z/\theta^{1/\rho}\Gamma(1 + 1/\rho) \tag{17}$$

with $z = \sum_{i=1}^{n} x_i$. Set $c = 1/\rho$, $d = \Gamma(1+c)$. Draw a sample of size n from the Weibull population denoted by (3) and using a prior for θ as

$$g(\theta) = e^{1/\theta^c}\left(c/\theta^{c+1}\right), \quad \theta > 0. \tag{18}$$

Now we can write (4) using (17) as

$$f(z_1|\theta) = \sum_{s_1=0}^{n_1 k} \sum_{j_1=0}^{s_1} \left[\binom{s_1}{j_1} \alpha_{s_1,1} (-1)^{j_1}(1/cd\theta^c)^{d_1}\right] \times \tag{19}$$

$$\times \left[e^{-z_1/\theta^c(cd)} z_1^{d_1-1}/\Gamma(d_1)\right].$$

Now integrating out θ from $f(z|\theta)g(\theta)$, we get

$$f(z_1) = \sum_{s_1=0}^{n_1 k} \sum_{j_1=0}^{s_1} \frac{\alpha_{s_1,1}\binom{s_1}{j_1}(-1)^{j_1}(z_1/cd)^{d_1-1}\Gamma(d_1+1)}{[\Gamma(d_1)]\left[1 + \frac{z_1}{cd}\right]^{d_1+1}[cd]} \tag{20}$$

with $z_1 = \sum_{i=1}^{n_1} x_i$ from the sample 1 and $\alpha_{s_1,1}$ is similar to (8) with k and α instead k_1 and α_1. Next take a second sample of size n_2 and the corresponding pdf of z_2, that is, $f(z_2|\theta)$ which is similar to (19) and taking the posterior at stage 1 (sample 1), that is, $f(\theta|z_1)$ as the prior for the second stage and along with $f(z_2|\theta)$, we get the predictive distribution at stage 2 by integrating out θ from $f(z_2|\theta)f(\theta|z_1)$ as

$$(21) \qquad f(z_2|z_1) = \prod_{i=1}^{2} \sum_{s_i=0}^{n_i k} \sum_{j_i=0}^{s_i} \left[\alpha_{s_i,i} \binom{s_i}{j_i} (-1)^{j_i} \left(\frac{z_i}{cd}\right)^{d_i-1}\right] \times$$

$$\times \left[\frac{1}{\Gamma(d_1)\Gamma(d_2)}\right] \left[\Gamma(d_1+d_2+1)\right] \Big/ \left[1 + \frac{z_1+z_2}{cd}\right]^{d_1+d_2+1} \quad [cd]^2$$

$$\div \left[\text{Expression (20)}\right]$$

with $d_i = N_i + j_i$, $i = 1,2$.

Continuing on this line and taking the prediction at stage $(p-1)$ as the prior for stage p and using the pdf of z_p at stage p, that is $f(z_p|\theta)$ which is similar to (19) and integrating out θ in $f(z_p|\theta)f(\theta|z_1,\ldots,z_{p-1})$, we get the predictive distribution at stage p (sample p) as

$$(22) \qquad f(z_p|z_1,\ldots,z_{p-1}) = \psi(z_1,\ldots,z_p)/\psi(z_1,z_2,\ldots,z_{p-1}),$$

where

$$(23) \quad \psi(z_1,\ldots,z_m) = \prod_{i=1}^{m} \left[\sum_{s_i=0}^{n_i k} \sum_{j_i=0}^{s_i} \alpha_{s_i,i} \binom{s_i}{j_i} (-1)^{j_i} \cdot (z_i/cd)^{d_i-1}\right] \left[\phi(z_1,\ldots,z_m)\right]$$

with

(24) $$\phi(z_1,\ldots,z_m) = \frac{\left[\Gamma(d_1+\ldots+d_m+1)/\Gamma(d_1)\ldots\Gamma(d_m)\right]}{(cd)^m\left[1+(z_1+\ldots+z_m)/cd\right]^{d_1+\ldots+d_m+1}}\,.$$

In (23) $\alpha_{s_i,i}$'s are similar to (8) except k_i,α_i replaced by k and α. Now using (14), we can write the probability $P(z_p > a_p) = 1-\alpha$ as, that is

(25) $$\int_0^{a_p} f(z_p|z_1,\ldots,z_{p-1})dz_p = 1-\alpha$$

(26) $$(1-\alpha) = \left[\sum_{s_p=0}^{kn_p}\sum_{j_p=0}^{s_p}(-1)^{j_p}\binom{s_p}{j_p}\alpha_{s_p,p}\left\{\psi(z_1,\ldots,z_{p-1}) - \right.\right.$$

$$\left.\left. - \sum_{m_p=0}^{d_p-1}\binom{D'_{p-1}+m_p-1}{m_p}\psi_1(z_1,\ldots,z_{p-1})\left(\frac{a_p}{cd}\right)^{m_p}\right\}\right]\Big/\left[\psi(z_1,\ldots,z_{p-1})\right],$$

where $\psi_1(z_1,\ldots,z_{p-1})$ is $\psi(z_1,\ldots,z_{p-1})$ with $\phi(z_1,\ldots,z_{p-1})$ replaced by $\phi_1(z_1,\ldots,z_{p-1})$ with

(27) $$\phi_1(z_1,\ldots,z_{p-1}) =$$

$$= \Gamma(D'_{p-1})\Big/\left[1+\frac{a_p}{cd}+\bar{S}_{p-1}\right]^{D'_{p-1}+m_p}\left[\prod_{j=1}^{p-1}\Gamma(d_j)\right](cd)^{p-1},$$

where $\bar{S}_{p-1} = (z_1+\ldots+z_{p-1})/cd$, $D'_{p-1} = d_1+\ldots+d_{p-1}$.

The procedure to operate (22) and (26) is as follows:

a. Draw a series of (p-1) independent samples from the Weibull population denoted by (3) of sizes $n_1, n_2, \ldots, n_{p-1}$.
b. Compute $z_1, \ldots, z_{p-1}$ from these (p-1) samples.
c. Set them all in (22) which turns out to be a function of z_p alone or put them directly into (26).
d. Now obtain a prediction interval for z_p in the future sample of size n_p using (26) or (22) of size $(1-\alpha)$, for chosen α.

C o m m e n t s : Now, in this section, we have only one k instead of (p+1), k_i's in the previous section. Similarly α for α_i's. We have now $n_1, \ldots, n_p$ similar to previous section. But now p itself is a variable which shows how many samples we are going to consider for the prediction in the future sample. The smaller the p (that is, less number of samples to consider) less will be the computational work load, since now, we will be dealing with less number of summations in our computation. However, the larger the p, prediction result will be better, since the prediction depends now on larger number of observations. In the same context, one may consider whether it is better to take small n_i's and large p or the other way, that is, small p and large n_i's. This may very well depend on the nature of the experiment and the available economic resources for the experiment and also on the amount of accuracy desired on the predicted value. Also, the choice of the prior may be of interest. We have chosen the simplest form of the prior to keep the algebra and the computation simple, though the central features of the procedure remains unaltered with more complicated priors. Finally, our procedure of taking priors at a certain stage as the posterior of the earlier stage is quite logical in the sense, we carry along all the available information up to the point of prediction. Assumption that p is known may not be too restrictive. Otherwise, the analysis and computation becomes complex.

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