DEMONSTRATIO MATHEMATICA
Vol. XXVI No 1 1993

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ON THE ORDER OF AN ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES REPRESENTED BY MULTIPLE DIRICHLET SERIES

1. Introduction and notation

Throughout the paper we denote complex n-space by C^n and real n-space by R^n and the elements $(s_1,s_2,\dots,s_n)$, $(m_1,m_2,\dots,m_n)$ etc. of C^n by their corresponding unsuffixed symbols s, m etc. For $x,y \in C^n$ we define

$$x = (x_1,x_2,\dots,x_n),\quad y = (y_1,y_2,\dots,y_n),$$
$$xy = (x_1y_1,x_2y_2,\dots,x_ny_n),\quad \|x\| = x_1 + x_2 + \dots + x_n.$$

Consider the multiple Dirichlet series [1].

$$f(s_1,s_2,\dots,s_n) = \sum_{m_1,m_2,\dots,m_n=0}^{\infty} a_{m_1,m_2,\dots,m_n} \times$$
$$\times \exp\left(s_1\lambda_{1m_1} + s_2\lambda_{2m_2} + \dots + s_n\lambda_{nm_n}\right),\ \text{i.e.}$$

$$(1.1)\qquad f(s) = \sum_{m=1}^{\infty} a_m \exp \|s\lambda_{nm_n}\|,\ (s_j = \sigma_j + 1t_j,\ j = 1,2,\dots,n),$$

where $a_m \in C$, λ_{nm_n} denotes the real tuple $(\lambda_{1m_1},\lambda_{2m_2},\dots,\lambda_{nm_n})$; $0 < \lambda_{p1} < \lambda_{p2} < \dots < \lambda_{pk} \longrightarrow \infty$ as $k \longrightarrow \infty$, for $p = 1,2,\dots,n$. We know [2] that if a tuple $q > \bar{0} = (0,0,\dots,0)$ such that

$$(1.2)\qquad \lim_{\|m\|\to\infty} \frac{\sum_{k=1}^{n} \log m_k}{\|q\lambda_{nm_n}\|} = 0$$

then the domain of absolute convergence of (1.1) coincides with its domain of convergence. Let

$$(1.3) \qquad \lim_{\|m\|\to\infty} \frac{\log|a_m|}{\|\lambda_{nm_n}\|} = -\infty .$$

Hence ([1] p. 1222) series (1.1) satisfying (1.2) becomes an entire function.

Throughout the paper F stands for the family of all multiple Dirichlet series of the form (1.1) satisfying (1.2) and (1.3).

We know ([1] p. 1226) $f(s) \in F$ is of Gol'dberg order ρ iff

$$\rho = \rho(D) = \lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\log\left\{\frac{1}{|a_m|\Phi_D(m)}\right\}} ,$$

where

$$\Phi = \Phi_D(m) = \sup_{s\in D} |\exp \|s\lambda_{nm_n}\| | ,$$

or

$$(1.4) \qquad \rho = \lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\log\left\{\frac{1}{\Phi|a_m|}\right\}} .$$

In this paper we have obtained some relations between the orders of entire functions of several complex variables represented by multiple Dirichlet series by taking asymptotic behaviour in their coefficients.

2. **Theorem 1.** Let

$$f_r(s) = \sum_{m=1}^{\infty} (a_m)_r \exp \|s^{\lambda} r, nm_n\|, (r = 1,2,\dots,q)$$

be q entire functions of non-zero orders $\rho_1, \rho_2, \dots, \rho_q$ respectively. Then the function

$$f(s) = \sum_{m=1}^{\infty} a_m \exp \|s^{\lambda} nm_n\| ,$$

where

$$\sum_{r=1}^{q} \alpha_r \left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1} \sim \left[\log\left\{\frac{1}{\Phi|a_m|}\right\}\right]^{-1}$$

$(0 < \alpha_r < 1, \sum_{r=1}^{q} \alpha_r = 1)$ and $\lambda_{r,nm_n} \sim \lambda_{nm_n}$; is an entire function such that

$$\rho \le \sum_{r=1}^{q} \alpha_r \rho_r \quad ,$$

where ρ is the order of f(s).

Proof. Since $f_r(s)$ is an entire function and $\lambda_{r,nm_n} \sim \lambda_{nm_n}$, therefore on using (1.3), we have for $\varepsilon > 0$ and large ℓ

$$(\ell - \varepsilon)^{\|\lambda_{nm_n}\|} < \frac{1}{|(a_m)_r|} < (\ell + \varepsilon)^{\|\lambda_{nm_n}\|} \;, \quad \|m\| > N_r \;,$$

or

$$\alpha_r\left[\log\left\{\frac{(\ell+\varepsilon)}{\Phi}^{\|\lambda_{nm_n}\|}\right\}\right]^{-1} < \alpha_r\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1} <$$

$$< \alpha_r\left[\log\left\{\frac{(\ell-\varepsilon)}{\Phi}^{\|\lambda_{nm_n}\|}\right\}\right]^{-1}, \quad \|m\| > N_r.$$

Putting r = 1,2,...,q and adding q inequalities thus obtained, we get

$$\sum_{r=1}^{q}\alpha_r\left[\log\left\{\frac{(\ell+\varepsilon)}{\Phi}^{\|\lambda_{nm_n}\|}\right\}\right]^{-1} < \sum_{r=1}^{q}\alpha_r\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1} <$$

$$< \sum_{r=1}^{q}\alpha_r\left[\log\left\{\frac{(\ell-\varepsilon)}{\Phi}^{\|\lambda_{nm_n}\|}\right\}\right]^{-1}$$

for $\|m\| > N = \max(N_1, N_2, \ldots, N_q)$. Since,

$$\sum_{r=1}^{q}\alpha_r\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1} \sim \left[\log\left\{\frac{1}{\Phi|a_m|}\right\}\right]^{-1}$$

therefore,

$$\log\left\{\frac{(\ell-\varepsilon)}{\Phi}^{\|\lambda_{nm_n}\|}\right\} < \log\left\{\frac{1}{\Phi|a_m|}\right\} < \log\left\{\frac{(\ell+\varepsilon)}{\Phi}^{\|\lambda_{nm_n}\|}\right\}$$

or

$$(\ell-\varepsilon) < \left\{\frac{1}{|a_m|}\right\}^{1/\|\lambda_{nm_n}\|} < (\ell+\varepsilon)$$

or

$$\lim_{\|m\|\to\infty} \frac{\log|a_m|}{\|\lambda_{nm_n}\|} = -\infty.$$

Hence f(s) is an entire function.

Now using (1.4) for the function $f_r(s)$, we have

$$\lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}} = \rho_r.$$

Therefore, for $\varepsilon > 0$ and $\|m\| > N_r$ we get

$$\alpha_r\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1} \left[\|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|\right] < \alpha_r(\rho_r + \varepsilon).$$

Putting r = 1,2,...,q and adding the q inequalities thus obtained, we get

$$\sum_{r=1}^{q}\alpha_r\left[\log \left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1}\left[\|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|\right] < \sum_{r=1}^{q}\alpha_r(\rho_r+\varepsilon),$$

$\|m\| > N = \max(N_1,N_2,...,N_q)$. Since,

$$\sum_{r=1}^{q}\alpha_r\left[\log \left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{-1} \sim \left[\log \left\{\frac{1}{\Phi|a_m|}\right\}\right]^{-1}$$

and

$$\lambda_{r,nm_n} \sim \lambda_{nm_n}$$

we have

$$\lim_{\|m\|\to\infty} \sup \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\log\left\{\frac{1}{\Phi(a_m)}\right\}} \le \sum_{r=1}^{q} \alpha_r\rho_r .$$

Hence, we get

$$\rho \le \sum_{r=1}^{q} \alpha_r \rho_r$$

where ρ is the order of f(s).

3. **Theorem 2.** Let

$$f_r(s) = \sum_{m=1}^{\infty} (a_m)_r \exp \|s\lambda_{r,nm_n}\| , \quad (r = 1,2,\ldots,q)$$

be q entire function of finite non zero orders $\rho_1,\rho_2,\ldots,\rho_q$ respectively. Then the function

$$f(s) = \sum_{m=1}^{\infty} a_m \exp \|s\lambda_{nm_n}\| ,$$

where

$$\sum_{r=1}^{q} \alpha_r \log \left\{\frac{1}{\Phi|(a_m)_r|}\right\} \sim \log \left\{\frac{1}{\Phi|a_m|}\right\} ,$$

$\left(0 < \alpha_r < 1, \sum_{r=1}^{q} \alpha_r = 1\right)$ and $\lambda_{r,nm_n} \sim \lambda_{nm_n}$, is an entire function such that

$$\frac{1}{\rho} \ge \sum_{r=1}^{q} \frac{\alpha_r}{\rho_r}$$

where ρ is the order of f(s).

Proof. Since $f_r(s)$ is an entire function and $\lambda_{r,nm_n} \sim \lambda_{nm_n}$ therefore on using (1.3) we have for $\varepsilon > 0$ and large ℓ

$$(\ell-\varepsilon)^{\|\lambda_{nm_n}\|} < \frac{1}{|(a_m)_r|} < (\ell+\varepsilon)^{\|\lambda_{nm_n}\|} , \quad \|m\| > N_r$$

or

$$\alpha_r \log\left\{\frac{(\ell-\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\} < \alpha_r \log\left\{\frac{1}{\Phi|(a_m)_r|}\right\} < \alpha_r \log\left\{\frac{(\ell+\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\},$$

$$\|m\| > N_r .$$

Putting r = 1,2,...,q, and adding q inequalities thus obtained, we get

$$\sum_{r=1}^{q}\alpha_r \log\left\{\frac{(1-\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\} < \sum_{r=1}^{q}\alpha_r \log\left\{\frac{1}{\Phi|(a_m)_r|}\right\} <$$

$$< \sum_{r=1}^{q}\alpha_r \log\left\{\frac{(1+\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\} \text{ for } \|m\| > N = \max(N_1,N_2,\ldots,N_q).$$

Since,

$$\sum_{r=1}^{q}\alpha_r \log\left\{\frac{1}{\Phi|(a_m)_r|}\right\} \sim \log\left\{\frac{1}{\Phi|a_m|}\right\}$$

therefore

$$\log\left\{\frac{(\ell-\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\} < \log\left\{\frac{1}{\Phi|a_m|}\right\} < \log\left\{\frac{(\ell+\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\}$$

or

$$(\ell-\varepsilon) < \left\{\frac{1}{|a_m|}\right\}^{1/\|\lambda_{nm_n}\|} < (\ell+\varepsilon)$$

or

$$\lim_{\|m\|\to\infty} \frac{\log|a_m|}{\|\lambda_{nm_n}\|} = -\infty \ .$$

Hence f(s) is an entire function.

Now using (1.4) for the function $f_r(s)$, we have

$$\lim_{\|m\|\to\infty} \inf \frac{\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}}{\|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|} = \frac{1}{\rho_r} .$$

Therefore, for an arbitrary $\varepsilon > 0$ we get

$$\frac{\alpha_r \log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}}{\|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|} > \alpha_r\left(\frac{1}{\rho_r} - \varepsilon\right)$$

for $\|m\| > N_r$. Putting $r = 1,2,\ldots,q$ adding q-inequalities thus obtained we have

$$\sum_{r=1}^{q} \frac{\alpha_r \log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}}{\|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|} > \sum_{r=1}^{q} \alpha_r\left(\frac{1}{\rho_r} - \varepsilon\right)$$

for $\|m\| > N = \max(N_1,N_2,\ldots,N_q)$. Since,

$$\sum_{r=1}^{q} \alpha_r \log \left\{\frac{1}{\Phi|(a_m)_r|}\right\} \sim \log \left\{\frac{1}{\Phi|a_m|}\right\}$$

and

$$\lambda_{r,nm_n} \sim \lambda_{nm_n}$$

we have

$$\liminf_{\|m\|\to\infty} \frac{\log \left\{\frac{1}{\Phi|a_m|}\right\}}{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|} \geq \sum_{r=1}^{q} \frac{\alpha_r}{\rho_r} .$$

Hence

$$\frac{1}{\rho} \geq \sum_{r=1}^{q} \frac{\alpha_r}{\rho_r}$$

where ρ is the order of $f(s)$.

4. **Theorem 3.** Let

$$f_r(s) = \sum_{r=1}^{q} (a_m)_r \exp \|s\lambda_{r,nm_n}\| , \quad (r = 1,2,\ldots,q)$$

be q entire functions of finite non-zero orders $\rho_1,\rho_2,\ldots,\rho_q$ respectively. Then the function

$$f(s) = \sum_{m=1}^{\infty} a_m \exp \|s\lambda_{nm_n}\| ,$$

where

$$\prod_{r=1}^{q} \left[\log \left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r} \sim \log \left\{\frac{1}{\Phi|a_m|}\right\} ,$$

$\left(0 < \alpha_r < 1, \sum_{r=1}^{q} \alpha_r = 1\right)$ and

$$\lambda_{r,nm_n} \sim \lambda_{nm_n}$$

is an entire function such that

$$\rho \leq \prod_{r=1}^{q} \rho_r^{\alpha_r} ,$$

where ρ is the order of $f(s)$.

Proof. Since $f_r(s)$ is an entire function and $\lambda_{r,nm_n} \sim \lambda_{nm_n}$

therefore on using (1.3), we have for $\varepsilon > 0$ and large ℓ

$$(\ell-\varepsilon)^{\|\lambda_{nm_n}\|} < \frac{1}{|(a_m)_r|} < (\ell+\varepsilon)^{\|\lambda_{nm_n}\|}, \quad \|m\| > N_r$$

or

$$\left[\log\left\{\frac{(\ell-\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\}\right]^{\alpha_r} < \left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r} <$$

$$< \left[\log\left\{\frac{(\ell+\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\}\right]^{\alpha_r}, \quad \|m\| > N_r .$$

Putting $r = 1,2,..,q$ and multiplying q inequalities thus obtained we get

$$\prod_{r=1}^{q}\left[\log\left\{\frac{(\ell-\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\}\right]^{\alpha_r} < \prod_{r=1}^{q}\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r} <$$

$$< \prod_{r=1}^{q}\left[\log\left\{\frac{(\ell+\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\}\right]^{\alpha_r}$$

for $\|m\| > N = \max(N_1, N_2, ..., N_q)$. Since

$$\prod_{r=1}^{q}\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r} \sim \log\left\{\frac{1}{\Phi|a_m|}\right\}$$

therefore

$$\log\left\{\frac{(\ell-\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\} < \log\left\{\frac{1}{\Phi|a_m|}\right\} < \log\left\{\frac{(\ell+\varepsilon)^{\|\lambda_{nm_n}\|}}{\Phi}\right\}$$

or

$$(\ell-\varepsilon) < \left\{\frac{1}{|a_m|}\right\}^{1/\|\lambda_{nm_n}\|} < (\ell+\varepsilon)$$

or

$$\lim_{\|m\|\to\infty} \frac{\log|a_m|}{\|\lambda_{nm_n}\|} = -\infty .$$

Hence $f(s)$ is an entire function.

Now using (1.4) for the function $f_r(s)$, we have

$$\limsup_{\|m\|\to\infty} \frac{\|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|}{\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}} = \rho_r .$$

Therefore, for $\varepsilon > 0$ and $\|m\| > N$ we get

$$\frac{\left[\ \|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|\right]^{\alpha_r}}{\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r}} < (\rho_r+\varepsilon)^{\alpha_r}.$$

Putting $r = 1,2,\ldots,q$ and multiplying the q inequalities thus obtained, we get

$$\prod_{r=1}^{q} \frac{\left[\ \|\lambda_{r,nm_n}\| \log \|\lambda_{r,nm_n}\|\right]^{\alpha_r}}{\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r}} < \prod_{r=1}^{q} (\rho_r+\varepsilon)^{\alpha_r}$$

for $\|m\| > N = \max\,(N_1,N_2,\ldots,N_q)$. Since

$$\prod_{r=1}^{q}\left[\log\left\{\frac{1}{\Phi|(a_m)_r|}\right\}\right]^{\alpha_r} \sim \log\left\{\frac{1}{\Phi|a_m|}\right\}$$

and

$$\lambda_{r,nm_n} \sim \lambda_{nm_n}$$

therefore

$$\limsup_{\|m\|\to\infty} \frac{\|\lambda_{nm_n}\| \log \|\lambda_{nm_n}\|}{\log\left\{\frac{1}{\Phi|a_m|}\right\}} \le \prod_{r=1}^{q} \rho_r^{\alpha_r}.$$

Hence

$$\rho \le \prod_{r=1}^{q} \rho_r^{\alpha_r}$$

where ρ is the order of $f(s)$.

It is my privilege to thank Dr. S. N. Srivastava for his valuable suggestions and guidance in the preparation of this paper.

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Received April 1987; new version June 8, 1992.