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ANALYSIS OF BIFURCATION OF PERIODIC SOLUTIONS BY POINCARÉ'S METHOD

1. Introduction

This paper is concerned with the study of bifurcations of periodic solutions of nonlinear nonautonomous differential equations depending on a small parameter. We perform this study using Poincaré's method [1]; that means that we compute formal Taylor expansions of the solution near a periodic solution of the reduced system. This reduced system, the system obtained for a zero value of the parameter, is not asked to be linear.

We study equations of the form

$$\dot{x} = X(x,t) + \varepsilon f(x,t,\varepsilon) \tag{1.1}$$

where

$$x : \mathbb{R} \longrightarrow \mathbb{R}^n,$$

$$X : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n,$$

$$f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^n.$$

The maps X and f are assumed to be analytic in x, eventually in a definition domain smaller than the whole space, and periodic in t of period T. The map f has to be analytic in ε near $\varepsilon = 0$. If we assume that the system is nonautonomous, and thus the period fixed, the maps X and f can not be both independent of t.

The system (1.1) with $\varepsilon = 0$ is called the *reduced system*:

$$\dot{x} = X(x,t). \tag{1.2}$$

We assume that this reduced system has a k-dimensional family of T-periodic solutions ($1 \le k \le n$)

$$x^{(0)} = \Phi(t,h), \quad h \in G \subset \mathbb{R}^k \tag{1.3}$$

which depends differentiably on the parameter h. This parameter corresponds to the initial conditions. We assume that

$$\mathrm{rk}\ \frac{\partial\Phi}{\partial h} = k$$

so that the set

$$\Phi = \{x \in \mathbb{R}^n \mid \exists t \in \mathbb{R},\ \ \exists h \in G,\ \ x = \Phi(t,h)\}$$

is a k-dimensional immersed submanifold of $\mathbb{R}^n$. We are looking for T-periodic solutions of (1.1) (depending on ε) such that they are equal to (1.3) for $\varepsilon = 0$.

Let h be fixed. The variation equation corresponding to this solution is

$$\dot{y} = \frac{\partial X}{\partial x}\bigg|_0 y \tag{1.4}$$

where $|_0$ denotes that the values are taken at $x^{(0)}$. This is a linear n-dimensional equation with periodic coefficients. We assume that we know its principal fundamental matrix Y(t) so that the solutions are

$$y(t) = Y(t)\cdot a$$

where $y(0) = a$.

As there is no general method to find the solutions of such an equation, the requirements of knowing the matrix Y(t) reduces seriously the interest of our method. Nevertheless, it is sometimes possible, as we shall see below, to obtain interesting results without knowing Y(t).

We can already notice that, if $\Phi(t,h)$ is a solution of (1.2), we obtain solutions of (1.4) by taking the derivatives

$$\frac{\partial\Phi}{\partial h_i}(t,h)$$

as X depends on h_i only via Φ. We then have k periodic solutions of the variation equation (1.4). This implies that $\mathrm{rk}(Y(T) - E) \le n - k$. This means that $Y(T) - E$ has at least k eigenvalues which are equal to zero. In order to avoid bifurcations *outside* the family $\Phi(t,h)$, we assume that

$$\mathrm{rk}(Y(T) - E) = n - k. \tag{1.5}$$

It follows that there exists a $(n - k)\cdot(n - k)$-submatrix of $Y(T) - E$ of maximal rank. We can assume that this submatrix is in the right-hand down corner (eventually after a linear transformation of the coordinates). This means that the subspace $\left\{0_{\mathbb{R}^k}\right\} \times \mathbb{R}^{n-k}$ is transverse to the tangent space of the manifold Φ.

2. Calculation of the power expansions

As we know that $\left\{0_{\mathbb{R}^k}\right\} \times \mathbb{R}^{n-k}$ is the transverse to $T_{\Phi(0,h)}\Phi$, we can look for initial conditions of the perturbed equation (1.1) under the form

(2.1) $$x(0) = \Phi(0,h + \beta) + \Psi(\varepsilon,h + \beta)$$

where $\beta \in \mathbb{R}^k$ and $\Psi = (0,\ldots,0,\Psi_{k+1},\ldots,\Psi_n)$ is an unknown map. We expand Ψ in powers of ε to obtain

(2.2) $$\Psi = B^{(1)}\varepsilon + B^{(2)}\varepsilon^2 + \ldots$$

where $B^{(1)} = {}^t(0,\ldots,0,B^{(i)}_{k+1},\ldots,B^{(i)}_n)$ depends uniquely on $h + \beta$.

Finally, we expand x in powers of ε and β to obtain

$$x(t,h + \beta,\varepsilon) = \sum_{s=0}^{\infty} c^{(s)}(t,h + \beta)\varepsilon^s =$$

$$= \sum_{s=0}^{\infty}\left(c^{(s)} + \frac{\partial c^{(s)}}{\partial\beta}\bigg|_{\beta=0}\beta + \frac{1}{2}\,\frac{\partial^2 c^{(s)}}{\partial\beta^2}\bigg|_{\beta=0}\beta^2 + \ldots\right)\varepsilon^s$$

where we write $c^{(s)}$ for $c^{(s)}(t,h)$. As $c^{(s)}(t,h + \beta)$ depends only on $h + \beta$ and not on h and β separately, we can replace

$$\frac{\partial^k c^{(s)}}{\partial\beta^k}\bigg|_{\beta=0} \quad \text{by} \quad \frac{\partial^k c^{(s)}}{\partial h^k}\bigg|_{h=h_0}$$

where h_0 is fixed value of h. Then,

(2.3) $$x(t,h+\beta,\varepsilon)=\sum_{s=0}^{\infty}\left(c^{(s)}+ \frac{\partial c^{(s)}}{\partial h}\,\beta+\frac{1}{2}\,\frac{\partial^2 c^{(s)}}{\partial h^2}\,\beta^2+\ldots\right)\varepsilon^s$$

so that we need uniquely to obtain equations for the $c^{(s)}(t,h)$, but not for all the derivatives.

We put (2.3) into the equation (1.1) and identify the terms of equal order in ε. The zeroth order relation gives us

$$C^{(0)} = \Phi(t,h)$$

and the higher order relation gives us linear nonhomogeneous equations which are to be verified by $C^{(s)}$

(2.4) $$\frac{dC^{(s)}}{dt} = \left(\frac{\partial X}{\partial x}\right)_0 C^{(s)} + H^{(s)}.$$

The initial conditions are

$$C_i^{(s)}(0) = 0_{(i=1,\dots,k)},\ C_{k+j}^{(s)}(0) = B_{k+j}^{(s)} \quad (j=1,\dots,n-k).$$

The nonhomogeneous term $H^{(s)}$ contains uniquely the $C^{(s)}$ of order < s so that we can solve these equations iteratively. The general expression for $H^{(s)}$ is

(2.5) $$H^{(s)} = \frac{1}{(s-1)!}\left.\frac{d^{s-1}f}{d\varepsilon^{s-1}}\right|_{\varepsilon=0,\beta=0} + \frac{1}{s!}\left(\frac{d^s X}{d\varepsilon^s} - \left(\frac{\partial X}{\partial x}\right)_0 \frac{d^s x}{d\varepsilon^s}\right)_{\varepsilon=0,\beta=0}$$

and the four first terms are

(2.6) $$H^{(1)} = f(t,\Phi,0),$$

(2.7) $$H^{(2)} = \frac{1}{2}\left.\frac{\partial^2 X}{\partial x^2}\right|_0 \left(C^{(1)}\right)^2 + \left.\frac{\partial f}{\partial x}\right|_0 C^{(1)} + \left.\frac{\partial f}{\partial \varepsilon}\right|_0,$$

(2.8) $$H^{(3)} = \frac{1}{6}\left.\frac{\partial^3 X}{\partial x^3}\right|_0 \left(C^{(1)}\right)^3 + \left.\frac{\partial^2 X}{\partial x^2}\right|_0 C^{(1)}C^{(2)} + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 \left(C^{(1)}\right)^2 +$$
$$+ \left.\frac{\partial f}{\partial x}\right|_0 C^{(2)} + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x\,\partial\varepsilon}\right|_0 C^{(1)} + \frac{1}{2}\left.\frac{\partial^2 f}{\partial \varepsilon^2}\right|_0,$$

(2.9) $$H^{(4)} = \frac{1}{24}\left.\frac{\partial^4 X}{\partial x^4}\right|_0 \left(C^{(1)}\right)^4 + \frac{1}{2}\left.\frac{\partial^3 X}{\partial x^3}\right|_0 \left(C^{(1)}\right)^2 C^{(2)} +$$
$$+ \left.\frac{\partial^2 X}{\partial x^2}\right|_0 \left[\frac{1}{2}\left(C^{(2)}\right)^2 + C^{(1)}C^{(3)}\right] + \frac{1}{6}\left.\frac{\partial^3 X}{\partial x^3}\right|_0 \left(C^{(1)}\right)^3 + \left.\frac{\partial^2 f}{\partial x^2}\right|_0 C^{(1)}C^{(2)} +$$
$$+ \left.\frac{\partial f}{\partial x}\right|_0 C^{(3)} + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x\,\partial\varepsilon}\right|_0 C^{(2)} + \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^2\,\partial\varepsilon}\right|_0 \left(C^{(1)}\right)^2 + \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^2\,\partial\varepsilon^2}\right|_0 C^{(1)} +$$
$$+ \frac{1}{6}\left.\frac{\partial^3 f}{\partial\varepsilon^3}\right|_0.$$

The homogeneous part of (2.4) is nothing else than the variation equation (1.4). The solution of (2.4) is

(2.10) $$C^{(s)} = Y(t)B^{(s)} + \Gamma^{(s)}$$

where $B^{(s)}$ is given in (2.2) and $\Gamma^{(s)}$ is a particular solution of the non-homogeneous equation (2.4) corresponding to zero initial conditions

$$\Gamma^{(s)} = Y(t) \int_0^t Y^{-1}(r) H^{(s)}(r) dr. \tag{2.11}$$

3. Conditions of periodicity

We use the periodicity condition

$$x(T, h + \beta, \varepsilon) = \Phi(0, h + \beta) + \Psi(\varepsilon, h + \beta) \tag{3.1}$$

which is written using the expansions

$$\Phi(T,h+\beta)+\sum_{s=1}^{\infty}\left\{C^{(s)}(T) + \frac{\partial C^{(s)}}{\partial h}(T)\beta + \frac{1}{2}\frac{\partial^2 C^{(s)}}{\partial h^2}(T)\beta^2+\dots\right\}\varepsilon^s = \tag{3.2}$$

$$= \Phi(0,h+\beta) + \sum_{s=1}^{\infty}\left\{B^{(s)} + \frac{\partial B^{(s)}}{\partial h}\beta + \frac{1}{2}\frac{\partial^2 B^{(s)}}{\partial h^2}\beta^2 + \dots\right\}\varepsilon^s.$$

We shall consider separately the k first components and the n - k last ones to take advantage of the special form of $B^{(s)}$. Let

$$C^{(s)} = \left(C_1^{(s)}, C_2^{(s)}\right)^t$$

where $C_1^{(s)} \in \mathbb{R}^k$, $C_2^{(s)} \in \mathbb{R}^{n-k}$, and similar notations for the other variables. The relation (2.10) at $\tau = T$ becomes, after replacing $C_i^{(s)}(T,h)$ by $B_i^{(s)}(h)$,

$$\begin{pmatrix} 0 \\ B_2^{(s)}(T,h) \end{pmatrix} = \begin{pmatrix} Y_{11}(T,h) & Y_{12}(T,h) \\ Y_{21}(T,h) & Y_{22}(T,h) \end{pmatrix}\begin{pmatrix} 0 \\ B_2^{(s)}(T,h) \end{pmatrix} + \begin{pmatrix} \Gamma_1^{(s)}(T,h) \\ \Gamma_2^{(s)}(T,h) \end{pmatrix}. \tag{3.3}$$

As $Y_{22}(T) - E$ is invertible, we can solve this equation to $B_2^{(s)}$ and obtain

$$B_2^{(s)}(h) = \left(E - Y_{22}(T,h)\right)^{-1}\Gamma_2^{(s)}(T,h). \tag{3.4}$$

So we can compute $B_2^{(s)}$ and afterwards $C_2^{(s)}(t,h)$ and $x_2^{(s)}(t,h)$, and it appears that the last n - k components - the *non-resonant* ones - are periodic independly of β, even if we need to know β to actually compute $x_2^{(s)}$.

The periodic condition for the first k components is written

$$(3.5)\qquad \sum_{s=1}^{\infty}\left\{c_1^{(s)}(T,h) + \frac{\partial c_1^{(s)}}{\partial h}(T,h)\beta + \frac{1}{2}\,\frac{\partial\, c_1^{(s)}}{\partial h^2}(T,h)\beta^2 + \dots\right\}\varepsilon^s = 0.$$

We know that, for $\varepsilon = 0$, $\beta = 0$ so that the power expansion of β begins with a term of first order at least. Then, assuming $\varepsilon \neq 0$, the zeroth order relation gives

$$(3.6)\qquad c_1^{(s)}(T,h) = 0 .$$

This condition gives us the acceptable values of h so that we can determine $h = h_0$ and $\Phi(t,h_0)$. It is called the *main amplitude equation* or the *bifurcation equation* [2].

Assuming $\varepsilon \neq 0$, we can divide (3.5) by ε and obtain, after reordering,

$$(3.7)\qquad \sum_{s=2}^{\infty} c_1^{(s)}(T,h_0)\varepsilon^{s-1} + \left(\sum_{s=1}^{\infty}\frac{\partial c_1^{(s)}}{\partial h}(T,h_0)\varepsilon^{s-1}\right)\beta + $$

$$+ \frac{1}{2}\left(\sum_{s=1}^{\infty}\frac{\partial^2 c_1^{(s)}}{\partial h^2}(T,h_0)\varepsilon^{s-1}\right)\beta^2 + \dots = 0.$$

This equation gives us $\beta = \beta(\varepsilon)$ as implicite function and the final (formal) solution of (1.1).

a) First Case.

$$\operatorname{dtm}\frac{\partial c_1^{(1)}}{\partial h}(T,h_0) \neq 0.$$

In this case, the implicit function theorem says us that there is one and only one solution of (3.7) and that this solution is analytic in a neighborhood of $\beta(\varepsilon)$. If

$$(3.8)\qquad \beta(\varepsilon) = b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + \dots ,$$

the first four coefficients will be

$$(3.9)\qquad b_1 = -\left(\frac{\partial c_1^{(1)}}{\partial h}\right)^{-1} c_1^{(2)},$$

$$(3.10)\qquad b_2 = -\left(\frac{\partial c_1^{(1)}}{\partial h}\right)^{-1}\left\{\frac{1}{2}\frac{\partial^2 c_1^{(1)}}{\partial h^2}b_1^2 + \frac{\partial c_1^{(2)}}{\partial h}b_1 + c_1^{(3)}\right\},$$

$$(3.11)\qquad b_3 = -\left(\frac{\partial c_1^{(1)}}{\partial h}\right)^{-1}\left\{\frac{\partial^2 c_1^{(1)}}{\partial h^2}b_1 b_2 + \frac{1}{6}\frac{\partial^3 c_1^{(1)}}{\partial h^3}b_1^3 + \frac{\partial c_1^{(2)}}{\partial h}b_2 + \right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 c_1^{(2)}}{\partial h^2}b_1^2 + \frac{\partial c_1^{(3)}}{\partial h}b_1 + c_1^{(4)}\right\},$$

$$(3.12)\qquad b_4 = -\left(\frac{\partial c_1^{(1)}}{\partial h}\right)^{-1}\left\{\frac{1}{2}\frac{\partial^2 c_1^{(1)}}{\partial h^2}\left(b_2^2 + 2b_1b_2\right) + \frac{1}{2}\frac{\partial^3 c_1^{(1)}}{\partial h^3}b_1^2 b_2 + \right.$$
$$+ \frac{1}{24}\frac{\partial^4 c_1^{(1)}}{\partial h^4}b_1^4 + \frac{\partial c_1^{(2)}}{\partial h}b_3 + \frac{\partial^2 c_1^{(2)}}{\partial h^2}b_1b_2 + \frac{1}{6}\frac{\partial^3 c_1^{(2)}}{\partial h^3}b_1^3 +$$
$$\left. + \frac{\partial c_1^{(3)}}{\partial h}b_2 + \frac{1}{2}\frac{\partial^2 c_1^{(3)}}{\partial h^2}b_1^2 + \frac{\partial c_1^{(4)}}{\partial h}b_1 + c_1^{(5)}\right\}.$$

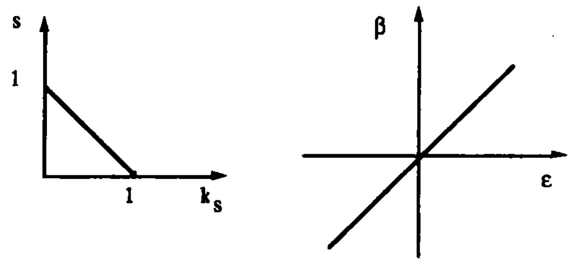
b) Second Case.

$$\mathrm{rk}\frac{\partial c_1^{(1)}}{\partial h}(T,h_0) = \ell < k.$$

The implicit function theorem does not allow us any more to obtain a unique solution of (3.6) and (3.7). Nevertheless, assuming $\mathrm{rk}\left(\partial c_1^{(s)}/\partial(h_1,\ldots,h_\ell)\right) = \ell$, we can use the theorem to eliminate the first ℓ variables in (3.6) to obtain a completely degenerate equation in the variables $(h_{\ell+1},\ldots,h_k)$. It is then possible to use Newton's diagram method to obtain the solutions of the bifurcations equation as fractional power series. As the computations become quite tedious for higher order, we shall limit ourselves to consider the 1-dimensional case $\ell = k - 1$, where the reduced bifurcation equation is a one-variable equation.

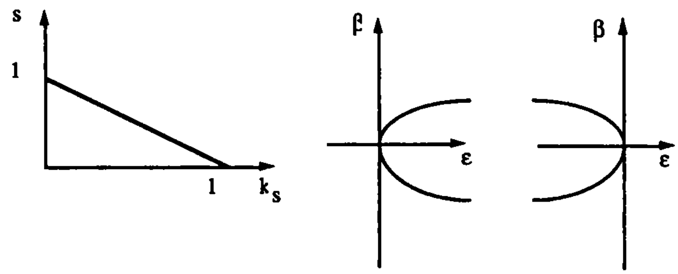
We shall now consider a few examples of Newton's diagrams.

a) $B_{01}\beta + B_{10}\varepsilon = 0.$



This is the regular case.

b) $B_{10}\varepsilon + B_{02}\beta^2 = 0.$



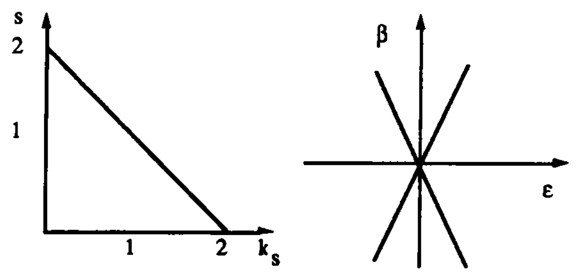
$\beta = \sqrt{-(B_{10}/B_{02})}\varepsilon.$ $B_{10}/B_{02} < 0.$ $B_{10}/B_{02} > 0.$

This is a fold-type bifurcation (using Thom's denomina tions) [4].

c) $B_{20}\varepsilon^2/B_{02}\beta^2 = 0.$

When $B_{20}/B_{02} > 0$, we obtain no solution.

When $B_{20}/B_{02} < 0$, we obtain two straight lines.

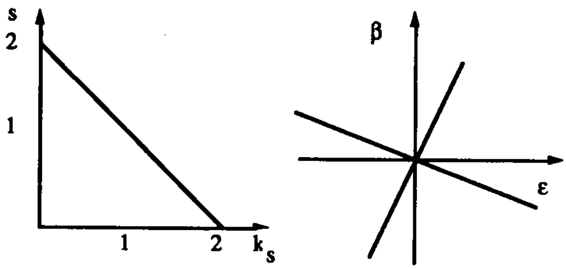


$$\beta = \pm \sqrt{(B_{20}/B_{02})}\varepsilon$$

d) $B_{20}\varepsilon^2 + B_{11}\varepsilon\beta + B_{02}\beta^2 = 0.$

When $B_{11}^2 - 4B_{20}B_{02} < 0$, we obtain no solution.

When $B_{11}^2 - 4B_{20}B_{02} > 0$, we obtain two straight lines.



$$\beta = \varepsilon(-B_{11} \mp \sqrt{(B_{11}^2 - 4B_{20}B_{02})}/2B_{02}.$$

When $B_{11}^2 - 4B_{20}B_{02} = 0$, we obtain one straight line, but we have to compute the second order to describe completely the bifurcation.

Let us denote the cases c) and d) are not structurally stable, while a) and b) are.

4. Stability of the solutions

The stability of a solution $x(t,h + \beta,\varepsilon)$ is studied considering the related variation equation

$$\text{(4.1)} \qquad \dot{y} = \left(\frac{\partial X}{\partial x}\bigg|_{x(t,h+\beta,\varepsilon)} + \frac{\partial f}{\partial x}\bigg|_{x(t,h+\beta,\varepsilon)} \right) y.$$

As usual, we look for a solution $y = y(t,h + \beta,\varepsilon)$ as a power expansion

$$\text{(4.2)} \qquad y = \sum_{i=0}^{\infty} Y_i(t,h + \beta)\varepsilon^i.$$

We can use the same technique as before to avoid to evaluate the coefficients of the Taylor expansion of $Y_i(t,h)$; we then obtain

$$\text{(4.3)} \qquad \dot{Y}_0(t) = \frac{\partial X}{\partial x}\bigg|_0 Y_0(t),$$

(4.4) $$\dot{Y}_1(t) = \frac{\partial X}{\partial x}\Big|_0 Y_1(t) + \frac{\partial^2 X}{\partial x^2}\Big|_0 C_1^{(1)} Y_0(t),$$

(4.5) $$\dot{Y}_2(t) = \frac{\partial X}{\partial x}\Big|_0 Y_2(t) + \frac{\partial^2 X}{\partial x^2}\Big|_0 C_1^{(1)} Y_1(t) + \left[\frac{\partial^2 X}{\partial x^2}\Big|_0 C_1^{(2)} + \right.$$

$$\left. + \frac{1}{2}\frac{\partial^3 X}{\partial x^3}\Big|_0 \left(C_1^{(1)}\right)^2 + \frac{\partial f}{\partial x}\Big|_0 C_1^{(1)}\right] Y_0(t).$$

Generally,

$$\dot{Y}_s(t) = \frac{\partial X}{\partial x}\Big|_0 Y_s(t) + G_s$$

where G_s depends uniquely on Y_k, $k < s$ and C_j, $j \le s$.

Taking the initial conditions

$$Y_0(0,h+\beta) = E; \quad Y_i(0,h+\beta) = 0, \quad i \ge 1,$$

we see that

(4.6) $$Y_0(0,h+\beta) = Y(t),$$

(4.7) $$Y_1(0,h+\beta) = Y(t) \int_0^t Y^{-1}(\tau) \frac{\partial^2 X}{\partial x^2}\Big|_0 C_1^{(1)} Y(\tau)d\tau,$$

and in general

(4.8) $$Y_s(0,h+\beta) = Y(t) \int_0^t Y^{-1}(\tau) G_s(\tau)d\tau.$$

So the computation of the characteristic factors - and thus of the stability of the solution - can be performed by solving the characteristic equation

$$\text{dtm}\left|Y(t) + Y_1\varepsilon + Y_2\varepsilon^2 + \ldots - \lambda E\right| = 0$$

to obtain λ as a power expansion of ε. Nevertheless, the amount of computation required tends to become exceedingly large, although we do not need to introduce the actual value of β as a series of power of ε before the end of the computation of λ.

5. Partial solutions of the variation equation

As we have seen before, the variation equation of the reduced system is usually hard to solve, and all our study is based upon it. Nevertheless, even if we cannot compute effectively the fundamental matrix $Y(t)$, it is sometimes possible to obtain the bifurcation equation.

The bifurcation equation (3.6) comes from the periodicity conditions on the first order term, which is a solution of the equation

(5.1) $$\dot{c}_1^{(1)} = \frac{\partial X}{\partial x}\bigg|_0 c_1^{(1)} + f\bigg|_0 .$$

However, it can be easily shown [5] that the periodicity condition for such an equation is equivalent to the orthogonality of the nonhomogeneous term and the periodic solution of the adjoint equation, that is

(5.2) $$\int_0^T (z(\tau)|f\big|_0(\tau)) = 0$$

where $z(\tau)$ is any periodic solution of

(5.3) $$\dot{z} = -\frac{\partial X}{\partial x}\bigg|_0' z .$$

The resolution of (5.3) is also very difficult in general, but we already know the periodic solutions

$$\partial\Phi(t,h)/\partial h_i, \quad i = 1,\dots,k$$

of the variation equation. If we assume that the T-periodic solutions of (5.3) are given by a linear combination with constant coefficients of the T-periodic solutions of the variation equation, that is

$$Z = (\partial\Phi/\partial h)A$$

where $Z = (z_j^{(i)})\in\mathbb{R}^{n\times k}$, $\partial\Phi/\partial h = (\partial\Phi/\partial h_i)_j\in\mathbb{R}^{n\times k}$, $A\in\mathbb{R}^{k\times k}$. We obtain as a condition on $\frac{\partial X}{\partial x}\big|_0$ that the diagram

$$\begin{array}{ccc}
\mathbb{R}^{n\times k} & \xrightarrow{\ A^*\ } & \mathbb{R}^{n\times k} \\
\Big\downarrow \frac{\partial X}{\partial x}\Big|_0 & & \Big\downarrow \frac{\partial X}{\partial x}\Big|_0' \\
\mathbb{R}^{n\times k} & \xrightarrow{\ A^*\ } & \mathbb{R}^{n\times k}
\end{array}$$

commutes for every $t \in [0,T]$, where $A^*(\partial\Phi/\partial h) = (\partial\Phi/\partial h)A$. This will happen trivially if the variation equation is autoadjoint, with $A = E$, but also when the reduced system is Hamiltonian.

If the reduced system is written, with $J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$,

$$J\dot{x} = \frac{\partial H}{\partial x}',$$

the variation equation is

$$J\dot{y} = \left.\frac{\partial^2 H}{\partial x^2}\right|_0 y$$

and the adjoint equation is

$$\dot{z} = -\left.\frac{\partial^2 H}{\partial x^2}\right|_0 Jz.$$

Finally, we find that $z = Jy$.

In a coordinate frame where $y_i(0) = e_i$, $y_{n+i}(0) = e_i$ for $i = 1,\dots,k$, the matrix A will be $\begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix}$, that is

$$\begin{pmatrix} z_1 \\ \vdots \\ z_k \\ z_{n+1} \\ \vdots \\ z_{n+k} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ y_{n+1} \\ \vdots \\ y_{n+k} \end{pmatrix} \begin{pmatrix} 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{pmatrix}.$$

It is also possible to find some solutions of the adjoint equation if the reduced system has one or more integrals: if

$$F(t,x) = \text{const}$$

is an integral of the reduced system, then

$$H = \left.\frac{\partial F}{\partial x}\right|_0 y$$

is an integral of the variation equation.

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