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ON SOLUTIONS OF A FUNCTIONAL EQUATION IN A SPECIAL CLASS OF FUNCTIONS

Many papers have been devoted to the problem of the existence and uniqueness of solution of the iterative functional equation

$$\varphi(x) = h(x,\varphi[f(x)]), \tag{1}$$

where φ is an unknown real-valued function of a real variable (cf. [4],[5] and the references quoted therein). The problem has been considered by many authors in several class of functions for instance: continuous, differentiable, integrable, analytic, absolutely continuous, of bounded variation, lipschitzian. The present paper is devoted to this problem in a special function class $W^r_\gamma\langle a,b\rangle$ which is defined below. Our results are related to ones obtained by B.Choczewski [2], J. Matkowski [6] and [7] (cf.also [4],Ch.4,[5],Ch.5). The methods used in order to prove the existence and uniqueness of the solution are classical. However, there are some new problems connected with the examined class of functions.

Let $\langle a,b\rangle$ be a closed interval, $a<b$, $a,b\in R$, $d:=b-a$. Assume that

$$(\Gamma)\quad \begin{cases} \gamma:\langle 0,d\rangle \longrightarrow \langle 0,\infty) \text{ is increasing, concave, } \gamma(0)=0, \\ \gamma(t)\longrightarrow 0, \text{ as } t\longrightarrow 0^+ \text{ and } \gamma(t)\longrightarrow\gamma(d), \text{ as } t\longrightarrow d^-. \end{cases}$$

Let $W^r_\gamma\langle a,b\rangle$ denote the class of r times differentiable functions defined as follows: a function $\varphi:\langle a,b\rangle\longrightarrow R$ belongs to $W^r_\gamma\langle a,b\rangle$ if and only if its r-th derivative satisfies the following condition: there exists constant $M \ge 0$ such that

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \le M\,\gamma(|x-\bar{x}|),\quad x,\bar{x}\in\langle a,b\rangle, \tag{2}$$

where γ fulfils the condition (Γ).

Remark 1. Denote by $\gamma'_+(0)$ the right derivative of γ at the point t=0. If it is finite, then $W^r_\gamma\langle a,b\rangle$ = Lip $C^r\langle a,b\rangle$, where Lip $C^r\langle a,b\rangle$ denotes the set of all r times differentiable functions $\varphi:\langle a,b\rangle\longrightarrow R$ whose r-th derivative fulfils a Lipschitz condition in $\langle a,b\rangle$ (cf.[3]). Therefore, we proceed in the sequel as $\gamma'_+(0)=+\infty$. Clearly, the class $W^r_\gamma\langle a,b\rangle$ is wider than that of functions with lipschitzian r-th derivative.

Remark 2. The functions of the form $\gamma(t)=t^\alpha$, where $0<\alpha<1$, fulfil the assumption (Γ) and moreover $\gamma'_+(0)=+\infty$. Therefore, we call condition (2) the generalized Hölder condition or the γ-Hölder condition.

Remark 3. By the condition $\gamma'_+(0)=+\infty$, we have $\gamma(t)\geq t$ in a right neighbourhood of zero. If moreover $\gamma(d)\geq d$, then, by (Γ), we obtain $\gamma(t)\geq t$ for $t\in\langle 0,d\rangle$. If $\gamma(d)<d$, then there exists a point $c\in(0,d)$ such that $\gamma(c)=c$ and $\gamma(t)\geq t$ for $t\in\langle 0,c\rangle$, $\gamma(t)<t$ for $t\in(c,d\rangle$. In the latter case we define the function $\gamma_1:\langle 0,d\rangle\longrightarrow\langle 0,\infty)$ by the formula

$$\gamma_1(t):=\frac{d}{\gamma(d)}\,\gamma(t),\quad t\in\langle a,b\rangle. \tag{3}$$

The function γ_1 fulfils the condition (Γ), $\gamma'_1(0)=+\infty$ and $\gamma(t)\leq\gamma_1(t)$ for $t\in\langle 0,d\rangle$. Moreover, $\gamma_1(t)\geq t$ for $t\in\langle 0,d\rangle$. Clearly, if a function φ fulfils the condition (2) with the function γ such that $\gamma(d)<d$ and with constant M, then φ fulfils this condition with the function (3) and the same constant M. Therefore, we my assume in the sequel that $\gamma(t)>t$ for $t\in\langle 0,d\rangle$.

Remark 4. Observe that, if $\varphi\in W^r_\gamma\langle a,b\rangle$, then the functions $\varphi^{(k)}$, k=0,...,r-1, fulfil in $\langle a,b\rangle$ ordinary Lipschitz conditions with some constants L_k. From Remark 3 they fulfil the following condition

$$|\varphi^{(k)}(x)-\varphi^{(k)}(\bar{x})|\leq L_k\,\gamma\,(|x-\bar{x}|),\quad x,\bar{x}\in\langle a,b\rangle, \tag{4}$$

for k=0,...,r-1.

Remark 5. The class $W^r_\gamma\langle a,b\rangle$ with the norm

$$(5)\quad \|\varphi\| := \sum_{i=0}^{r} |\varphi^{(i)}(\xi)| + \sup\left\{\frac{|\varphi^{(r)}(x)-\varphi^{(r)}(\bar{x})|}{\gamma(|x-\bar{x}|)};\ x,\bar{x}\in\langle a,b\rangle,\ x\neq\bar{x}\right\}$$

is a Banach space (cf. [8],Ch.7 and 8). In this formula $\xi\in\langle a,b\rangle$ is arbitrarily fixed.

Now, consider the equation (1) in the space $W^r_\gamma\langle a,b\rangle$ and assume that the given functions h and f fulfil the following conditions:

(i) $h:\langle a,b\rangle\times R\longrightarrow R$ is of the class $C^r(\langle a,b\rangle\times R)$ and each of its r-th partial derivatives fulfil γ-Hölder condition with respect to the first variable and an ordinary Lipschitz condition with respect to the second one;

(ii) $f:\langle a,b\rangle\longrightarrow\langle a,b\rangle, \sup\{|f'(x)|, x\in\langle a,b\rangle\}\le 1,\ f\in W^r_\gamma\langle a,b\rangle$.

Let us define the functions $h_k:\langle a,b\rangle\times R^{k+1}\longrightarrow R$, $k=0,\dots,r$, by the recurrent relations (cf.[2] or [4] p.84 and [5] p.209)

$$(6)\quad \begin{cases} h_0(x,y_0) = h(x,y_0), \\ h_{k+1}(x,y_0,\dots,y_{k+1}) = \dfrac{\partial h_k}{\partial x} + f'(x)\left(\dfrac{\partial h_k}{\partial y_0}y_1+\dots+\dfrac{\partial h_k}{\partial y_k}y_{k+1}\right), \end{cases}$$
$$k=0,\dots,r-1.$$

We have the following lemma.

Lemma 1. (cf.[2] and [4] p.85) Suppose that hypotheses (i),(ii) are fulfiled. Then the functions h_k defined by (6) are of the class $C^{r-k}(\langle a,b\rangle\times R^{k+1})$. Moreover

1) if r=1 then

$$(7)\qquad h_1(x,y_0,y_1) = \frac{\partial h}{\partial x}(x,y_0) + \frac{\partial h}{\partial y}(x,y_0)\ f'(x)\ y_1$$

2) if $r\ge 2$ then

$$(8)\quad h_k(x,y_0,\dots,y_k) = p_k(x,y_0,y_1) + q_k(x,y_0,\dots,y_k) + $$
$$+\frac{\partial h}{\partial y}(x,y_0)\ f^{(k)}(x)\ y_1 + \frac{\partial h}{\partial y}(x,y_0)\ (f'(x))^k\ y_k,$$ [1)]

where

1) Decomposition (8) is a little more detailed than that given in [2]. Namely we additionally distinguish here the component $\frac{\partial h}{\partial y}(x,y_0)\ f^{(k)}(x)y_1$. Therefore, we omit a simple inductive proof of (8).

$$p_k(x,y_0,y_1) = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k h}{\partial x^{k-i}\partial y^i}(x,y_0)\ (f'(x))^i\ y_1^i\ ,$$

$q_2 \equiv 0$ and $q_k(x,y_0,\dots,y_{k-1})$, $k=3,\dots,r$, are polynomials (of the variables $y_1,\dots,y_{k-1}$) whose coefficients depending on the variables $(x,y_0)\in I\times R$ are of the class $C^{r-k+1}(I\times R)$.

Lemma 2. Let hypotheses (i),(ii) be fulfiled and let $\bar{a},\bar{b}\in\langle a,b\rangle$, $\bar{a}<\bar{b}$, $a_k< b_k$, for $k=0,\dots,r$, be arbitrary constants and $Z := \langle\bar{a},\bar{b}\rangle\times\langle a_0,b_0\rangle\times\dots\times\langle a_r,b_r\rangle$. Then there exist constants $m,l_0,\dots,l_{r-1}$ and

$$l_r = \sup\{|\ \frac{\partial h}{\partial y}(x,y)(f'(x))^r|,\ x\in\langle\bar{a},\bar{b}\rangle, y\in\langle a_0,b_0\rangle\}$$

such that for $(x,y_0,\dots,y_r),(\bar{x},\bar{y}_0,\dots,\bar{y}_r)\in Z$ we have

(9) $$|h_r(x,y_0,\dots,y_r) - h_r(\bar{x},\bar{y}_0,\dots,\bar{y}_r)| \le$$
$$\le m\ \gamma\ (|x-\bar{x}|) + l_0|y_0-\bar{y}_0| + \dots + l_r|y_r-\bar{y}_r|\ ,$$

where h_r is defined by (6).

Proof. Let us fix arbitrary numbers $\bar{a}<\bar{b}$, $a_k<b_k$, $k=0,\dots,r$, and an $x\in\langle\bar{a},\bar{b}\rangle$. It follows from [2] (cf.also [4] p. 94) that the function h_r fulfils in the set Z a Lipschitz condition with respect to the variables $y_0,\dots,y_r$. Let us consider the function h_r in the set Z with y_k fixed, $k=0,\dots,r$. We shall show that h_r fulfils the γ-Hölder condition with respect to x. By Lemma 1, the function h_r can be expressed in the form (8). Thus, by (i),(ii), the function p_r fulfils the γ-Hölder condition with respect to x . The function $\frac{\partial h}{\partial y}$ satisfies the γ-Hölder condition in $\langle\bar{a},\bar{b}\rangle$ with respect to the x (cf. Remarks 3 and 4) and, in view of (ii), the function $f^{(r)}$fulfils the γ-Hölder condition in $\langle\bar{a},\bar{b}\rangle$. Since the function q_r is a polynomial of variables $y_1,\dots,y_r$ with coefficients being of the class C^1 with respect to x, so, by Remark 4, it fulfils the γ-Hölder condition in $\langle\bar{a},\bar{b}\rangle$. Therefore condition (9) is fulfiled in the set Z, and the lemma follows.

Lemma 3. Let hypotheses (i),(ii) be fulfiled. If $\varphi\in W_\gamma^r\langle a,b\rangle$ and ψ: $\langle a,b\rangle\longrightarrow R$ is defined by the formula

$$\psi(x) := h(x,\varphi[f(x)]), \quad x \in \langle a,b\rangle,$$

then the derivatives of ψ satisfy following equations

(10) $$\psi^{(k)}(x) = h_k(x,\varphi[f(x)],\dots,\varphi^{(k)}[f(x)]), \ k=1,\dots,r,$$

for $x\in\langle a,b\rangle$. Moreover $\psi\in W^r_\gamma\langle a,b\rangle$.

Proof. The first part of this lemma is due to B. Choczewski [2] (cf. [4],p.85). On account of (9), hypotheses (i), (ii) and Lemma 2, we obtain $\psi\in W^r_\gamma\langle a,b\rangle$.

We now assume that the function f fulfils the following condition:

(iii) there exists a $\xi \in\langle a,b\rangle$ such that $\lim_{n\to\infty} f^n(x) = \xi$ for $x\in\langle a,b\rangle$, where f^n is the n-th iterate of f.

Remark 6. Suppose that the assumptions (i)-(iii) are fulfiled. If $\varphi \in W^r_\gamma\langle a,b\rangle$ is a solution of the equation (1) in $\langle a,b\rangle$ and $\eta_k := \varphi^{(k)}(\xi)$, $k=0,\dots,r$, then, by Lemma 3, the numbers η_k, $k=0,\dots,r$, fulfil the equations

(11) $$\eta_k = h_k(\xi,\eta_0,\dots,\eta_k), \ k = 0,\dots,r,$$

where h_k are defined by (6).

In the next lemma we give conditions under which every local W^r_γ-solution of (1), defined in a neighbourhood of a fixed point ξ of f, can be extended to a global W^r_γ-solution.

Lemma 4. Let hypotheses (i)-(iii) be fulfiled and let f be monotonic in $\langle a,b\rangle$. If U is a neighbourhood of ξ such that $f(U)\subset U$ and $\varphi_0\in W^r_\gamma(\bar{U})$ is a solution of (1) in U, then there exists exactly one function $\varphi\in W^r_\gamma\langle a,b\rangle$ satisfying the equation (1) in $\langle a,b\rangle$ and fulfiling the condition

(12) $$\varphi(x) = \varphi_0(x) \text{ for } x \in U.$$

Proof. Let d_0:=diam U and $N_0=N\cup\{0\}$. Obviously, by the Remark 3, we have $\gamma(d_0)\ge d_0$. Put $U_0 := U$, $U_{n+1} := f^{-1}(U_n)$, $n\in N_0$. It is easily seen that

(13) $$\langle a,b\rangle = \bigcup_{n=0}^{\infty} U_n,$$

(14) $$f(U_{n+1}) \subset U_n \subset U_{n+1}, \quad n \in N_0.$$

On account of Baron's extension theorem (cf.[1] and also [5], Section 7.1), there exists exactly one continuous extension φ of φ_0 fulfiling the equation (1) in <a,b> and defined by the formula $\varphi(x) = \varphi_k(x)$, $x \in U_k$, $k \in N_0$, where

$$\varphi_k\colon U_k \longrightarrow R,\ k \in N_0, \quad \varphi_{k+1}(x) = h(x, \varphi_k[f(x)]).$$

We shall prove that $\varphi^{(r)}$ satisfies the γ-Hölder condition in every interval U_n, $n \in N_0$. The function $\varphi_0^{(r)}$ fulfils the generalized Hölder condition in $U_0 = U$, by assumption. Assume that the function $\varphi^{(r)}$ fulfils this condition in U_n with the constant M_n, $n \geq 0$. Functions $\varphi^{(i)}$, $i=0,\dots,r-1$, fulfil in U_n the γ-Hölder condition, in virtue of Remark 4. Let us put

$$a_i := \sup\{|\varphi^{(i)}(x)|,\ x \in \langle a,b\rangle\},\ i=0,\dots,r,$$

$$Z_r := \langle a,b\rangle \times \langle -a_0, a_0\rangle \times \dots \times \langle -a_r, a_r\rangle.$$

Let $L_{n,i}$ denote the Lipschitz constant in U_n, for each function $\varphi^{(i)}$, $i=0,\dots,r-1$. It follows from (14) that $f(x), f(\bar{x}) \in U_n$ for $x, \bar{x} \in U_{n+1}$. Lemmas 2 and 3 imply that there exist constants $m, l_0, \dots, l_r$ such that in the set Z_r the following inequality holds:

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| = |h_r(x, \varphi[f(x)], \dots, \varphi^{(r)}[f(x)]) -$$
$$- h_r(\bar{x}, \varphi[f(\bar{x})], \dots, \varphi^{(r)}[f(\bar{x})])| \leq m\,\gamma(|x-\bar{x}|) +$$
$$+ l_0|\varphi[f(x)] - \varphi[f(\bar{x})]| + \dots + l_r|\varphi^{(r)}[f(x)] - \varphi^{(r)}[f(\bar{x})]|.$$

Now, by the induction hypothesis and by (ii), we have

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \leq (m + l_0 L_{n,0} + \dots + l_{r-1} L_{n,r-1} + l_r M_n)\ \gamma(|x-\bar{x}|)$$

which means that $\varphi^{(r)}$ fulfil the γ-Hölder condition in U_{n+1}.

Thus, the function $\varphi^{(r)}$ fulfils the γ-Hölder condition in every interval U_n, $n \in N$. In view of (13) and (14), there is an index $k_n \in N$ such that $\langle a,b\rangle \subset U_{k_n}$. Therefore $\varphi \in W_\gamma^r\langle a,b\rangle$. This completes the proof.

Remark 7. Suppose that hypotheses (i)-(iii) are fulfiled and the numbers η_k, $k=0,\dots,r$, fulfil the system of equations (11). By a suitable change of functions one may reduce the problem to the case, where $\xi=0$ and $\eta_k=0$, $k=0,\dots,r$. The proof of this fact is analogous to that given in [6] (cf.also [5], Lemma 5.6.3).

Theorem 1. Suppose that hypotheses (i),(ii) are fulfiled and the function f, monotonic in $\langle a,b\rangle$, fulfils hypothesis (iii) for $\xi=0$. If

$$h_k(0,\dots,0) = 0,\ k=0,\dots,r, \tag{15}$$

and

$$\left|(f'(0))^r \frac{\partial h}{\partial y}(0,0)\right| < 1, \tag{16}$$

then equation (1) has the unique solution $\varphi \in W^r_\gamma\langle a,b\rangle$ fulfiling conditions

$$\varphi^{(k)}(0) = 0,\ k=0,\dots,r. \tag{17}$$

Proof. By inequality (16), from the continuity of the functions f' and $\frac{\partial h}{\partial y}$ and in view of (ii) and Remark 3, there exist a neighbourhood U_0 of $\xi=0$ and numbers d_1,θ : $d_1>0$, $0<\theta<1$, such that the following conditions hold

$$f(U_0) \subset U_0,\ c_0 := \operatorname{diam} U_0,\ c_0 \le \gamma(c_0), \tag{18}$$

$$\sup\left\{\left|(f'(x))^r \frac{\partial h}{\partial y}(x,y)\right|,\ x\in\bar U_0,\ y\in\langle -d_1,d_1\rangle\right\} \le \theta < 1. \tag{19}$$

In virtue of Lemma 2 and (19), there exist constants $m, l_0,\dots, l_{r-1}$, $l_r=\theta$ such that the following inequality

$$\begin{aligned}|h_r(x,y_0,\dots,y_r) - h_r(\bar x,\bar y_0,\dots,\bar y_r)| &\le \\ \le m\,\gamma(|x-\bar x|) + l_0|y_0-\bar y_0| + \dots + \theta\,|y_r-\bar y_r|\end{aligned} \tag{20}$$

holds in the set $\bar U_0\times\langle -d_1,d_1\rangle^{r+1}$. Choose a number c_1, $0<c_1\le d$, in such a manner that

$$c_1 \le \gamma(c_1) \le 1 \tag{21}$$

and

$$\gamma(c_1)\sum_{i=0}^{r-1} l_i c_1^{r-i-1} < 1-\theta. \tag{22}$$

Put

(23) $$M := m\,[1-\theta-\gamma(c_1)\sum_{i=0}^{r-1} c_1^{r-i-1}\, l_i]^{-1}$$

and fix a number c, $0 < c \le c_1$, such that

(24) $$\gamma(c) \le \min\,(\,\gamma(c_1),\ \frac{d_1}{M}).$$

Next, take a neighbourhood $U \subset U_0$ of the point $\xi=0$ such that $f(U) \subset U$ and diam $U \le c$ (it is possible, by hypothesis (ii)).

Let $C^r(\bar{U})$ ($\bar{U}$=clU) be a space of all functions $\varphi:\bar{U} \longrightarrow R$, r times differentiable in $\bar{U}$ with the continuous r-th derivative. Define the norm in $C^r(\bar{U})$ by the formula

(25) $$\|\varphi\| := \sum_{i=0}^{r} |\varphi^{(i)}(0)| + \sup_{\bar{U}} |\varphi^{(r)}|.$$

Of course $(C^r(\bar{U}),\|\ \|)$ is a Banach space.

Let A be the subset of the space $C^r(\bar{U})$ consisting of those functions $\varphi\in C^r(\bar{U})$ which fulfil the following conditions

(26) $$\varphi(0) = 0,\ \ldots\ ,\ \varphi^{(r)}(0) = 0,$$

(27) $$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \le M\,\gamma(|x-\bar{x}|),\ x,\bar{x} \in \bar{U}.$$

By (25) and (26), we get

(28) $$\|\varphi\| = \sup\,\{|\varphi^{(r)}(x)|,\ x\in\bar{U}\ \}.$$

Functions of the family $A_r := \{\varphi^{(r)};\ \varphi \in A\}$ are equibounded in $\bar{U}$. Indeed, setting in (27) $\bar{x}=0$, from (26) and (24), by hypothesis (Γ), we get

$$|\varphi^{(r)}(x)| \le M\,\gamma(|x|) \le M\,\gamma(c) \le M\,\frac{d_1}{M} = d_1,\ x\in\bar{U}.$$

Moreover, A_r is equicontinuous. Indeed, fix $\varepsilon > 0$ and put $\varepsilon_0 = \frac{\varepsilon}{M}$. The function γ, by (Γ), is right continuous at the point $t=0$. Thus, for ε_0 there exists $\delta > 0$ such that for $0 \le t < \delta$ we have $\gamma(t) \le \varepsilon$. Therefore, for $\varphi^{(r)} \in A_r$ and $x,\bar{x}\in\bar{U}$ such that $|x-\bar{x}|<\delta$ we get

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \le M\,\gamma(|x-\bar{x}|) \le M\,\frac{\varepsilon}{M} = \varepsilon,$$

where δ depends only on the choice of ε. Since the limit of a convergent sequence of functions fulfiling the γ-Hölder condition with a constant M fulfils this condition with the same constant, we get that A is compact. The convexity of A is obvious.

Let us note that, if $\varphi \in A$, then, by (27), we have

$$\sup \{ |\varphi^{(r)}(x)|,\ x \in \bar{U} \} \le M\ \gamma(c). \tag{29}$$

According to the mean-value theorem, we obtain for $x, \bar{x} \in \bar{U}, k=0,\dots,r-1$:

$$|\varphi^{(k)}(x) - \varphi^{(k)}(\bar{x})| \le \sup_{\bar{U}} |\varphi^{(k+1)}(x)|\ |x-\bar{x}|, \quad x,\bar{x} \in \bar{U}. \tag{30}$$

The relation (30) and (29),(24),(26) yields

$$\sup_{\bar{U}} |\varphi^{(k)}(x)| \le M\ c^{r-k}\gamma(c) \le M\ \frac{d_1}{M} = d_1,\ k=0,\dots,r. \tag{31}$$

Thus, if $\varphi \in A$, then $\varphi^{(k)} \in \langle -d_1, d_1 \rangle$, $k=0,\dots,r$, $x \in \bar{U}$.

Define the transformation $T: A \longrightarrow W^r_\gamma(\bar{U})$ by the formula

$$(T\varphi)(x) := h(x, \varphi[f(x)]), \quad x \in \bar{U}, \quad \varphi \in A. \tag{32}$$

We shall prove that (32) maps A into itself. It follows from Lemma 3 that $\psi := T\varphi$ is of the class C^r and its derivatives fulfil the system of equations (10). Putting x=0 in (10), we get

$$\psi^{(k)}(0) = h_k(0, \varphi[f(0)], \dots, \varphi^{(k)}[f(0)]\ ), \quad k=0,\dots,r.$$

By hypothesis (iii), we have f(0)=0 and from (15),(26) we obtain that $\psi^{(k)}(0)=0$, $k=0,\dots,r$. Thus, the function ψ fulfils condition (26). Let L_k be the Lipschitz constant in $\bar{U}$ of the function $\varphi^{(k)}$, $k=0,\dots,r-1$. In view of (30) and (31), we get

$$L_k \le M\ c^{r-k-1}\gamma(c), \quad k=0,\dots,r-1.$$

Thus, according to (20), by Remark 4, hypothesis (ii) and from (21)-(23),(31), we get

$$|\psi^{(r)}(x) - \psi^{(r)}(\bar{x})| = |h_r(x, \varphi[f(x)], \dots, \varphi^{(r)}[f(x)]) -$$

$$- h_r(\bar{x}, \varphi[f(\bar{x})], \dots, \varphi^{(r)}[f(\bar{x})])| \le m\ \gamma(|x-\bar{x}|) +$$

$$+ l_0|\varphi[f(x)] - \varphi[f(\bar{x})]| + \dots + l_{r-1}|\varphi^{(r-1)}[f(x)] - \varphi^{(r-1)}[f(\bar{x})]| +$$

$$+ \theta|\varphi^{(r)}[f(x)] - \varphi^{(r)}[f(\bar{x})]| \le m\ \gamma(|x-\bar{x}|) + l_0 L_0|f(x) - f(\bar{x})| +$$

$$+\dots+ l_{r-1}L_{r-1}|f(x)-f(\bar{x})| + \theta\, M\, \gamma(|f(x)-f(\bar{x})|) \le$$

$$\le (m + M(\gamma(c_1)\sum_{i=0}^{r-1} l_i c_1^{r-i-1} + \theta))\, \gamma(|x-\bar{x}|) = M\, \gamma(|x-\bar{x}|),$$

where $\varphi\in A$, $x,\bar{x}\in\bar{U}$. Therefore, (27) is fulfiled, i.e. $T(A)\subset A$. Transformation T is continuous in A. The proof of this fact is analogous to that given in [2] (cf.[4],p.94), then the convergence of the sequence φ_n to the function φ in the sense of the norm (25) means the uniform convergence of the sequence $\varphi_n^{(k)}$ to the function $\varphi^{(k)}$ in $\bar{U}$ for $k=0,\dots,r$.Thus, we have proved that the continuous transformation T maps the compact and convex subset A of the Banach space $C^r(\bar{U})$ into itself. On account of Schauder's theorem ,there exists in A a fixed point of transformation (32), i.e. there exists a solution φ_0 of the equation (1) fulfiling conditions (26) and (27).

We shall prove that this solution is unique (cf.[6]). Let $\varphi_1,\varphi_2 \in W_\gamma^r(\bar{U})$ be some solutions of the equation (1), fulfilling condition (26). By Taylor's formula,there exist functions $\psi_i:\bar{U} \longrightarrow R$, $i=1,2$, such that

$$\varphi_i(x) = x^r\, \psi_i(x) \quad \text{and} \quad \lim_{x\to 0} \psi_i(x) = 0, \quad i=1,2.$$

Moreover, ψ_i fulfil γ-Hölder condition in $\bar{U}$ with constants M_i, $i=1,2$. Let us note that ψ_i, $i=1,2$, fulfil in $\bar{U}-\{0\}$ the following equation

$$\psi(x) = H(x,\psi[f(x)]),$$

where $H(x,w) = h(x,[f(x)]^r w)\ (x^r)^{-1}$. Similarily as in [6], we prove that there exists the neighbourhood $\bar{W}\subset U$, where the following inequality holds

$$|\psi_1(x) - \psi_2(x)| \le |\psi_1[f^n(x)] - \psi_2[f^n(x)]|.$$

Since ψ_i,$i=1,2$, are continuous in $\bar{W}$ and $\lim_{n\to\infty} f^n(x)=0$, we obtain $\psi_1(x)=\psi_2(x)$ for $x\in \bar{W}-\{0\}$, thus $\varphi_1\equiv\varphi_2$ in $\bar{W}$. Denote by φ_0 this unique solution in $\bar{W}$. By Lemma 4, there exists the unique extension φ of φ_0 to the whole interval <a,b> such that $\varphi(x)=\varphi_0(x)$ for $x\in\bar{W}$, $\varphi\in W_\gamma^r$<a,b> and φ satisfies the equation (1) in <a,b>. This completes the proof.

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