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SOME PROPERTIES OF HERMITE POLYNOMIALS OF A WIENER PROCESS

We investigate a stochastic process $h_k(W_t,t) = t^{k/2}H_k(W_t/\sqrt{t})$, where $\{W_t, t\geq 0\}$ is a Wiener process, H_k - Hermite polynomial of degree k. We give formulas for moments, conditional moments and limit distribution of h_k when $t \to \infty$.

1. Introduction and formulation of the results

Let $\{W_t, t\geq 0\}$ be a Wiener process, H_k - Hermite polynomial defined by formula

$$\frac{d^k}{dx^k}\, e^{-x^2/2} = (-1)^k\, e^{-x^2/2}\, H_k(x).$$

Let

$$h_k(W_t,t) = t^{k/2}\, H_k(W_t/\sqrt{t}).$$

It is evident that $h_1 = W_t$, $h_2 = W_t^2 - t$, $h_3 = W_t^3 - 3tW_t$ and so on. We shall use the typical denotations for σ-fields: $\mathfrak{F}_{\leq t} = \sigma(W_s, s \leq t)$, $\mathfrak{F}_{\geq t} = \sigma(W_s, s \geq t)$.

It is well known (see e.g. [3], [6]) that

$$(1) \qquad E\{h_k(W_t,t) = 0, \qquad E\{h_k(W_T-W_t), T-t)\} = 0,$$

$$(2) \qquad E\{h_k^2(W_t,t)\} = k!t^k,$$

$$(3) \qquad h_k(W_t,t) = I_k(1_{[0,t]}^{\otimes t}) = \int_{\{t_1<\ldots<t_k<t\}} dW(t_1)\ldots dW(t_k),$$

$\{h_k(W_t,t), \mathfrak{F}_{\leq t}\}$ is a martingale and a Markov process.

The aim of this paper is to give formulas for moments, conditional moments and limit distribution of the process

$h_k(W_t,t)$. More precisely, we are going to prove that for every natural k, every t<T the following properties hold

(4) $$h_k(W_T,T) = \sum_{r=0}^{k}\binom{k}{r}h_r(W_t,t)h_{k-r}(W_T-W_t,T-t),$$

(5) $$E\left[h_k(W_t,t) \mid \mathfrak{F}_{\ge T}\right] = \left(\frac{t}{T}\right)^k h_k(W_T,T)$$

(Linear regression),

(6) $$E\left[h_k^2(W_T,T) \mid \mathfrak{F}_{\le t}\right] = \sum_{r=0}^{k}\binom{k}{r}^2(k-r)!\,(T - t)^{k-r}h_r^2(W_t,t),$$

(7) $$E\left(h_k^4(W_t,t)\right) = (k!)^2 t^{2k}\sum_{r=0}^{k}\binom{k}{r}^2\binom{2r}{r},$$

(8) $$E\left(h_k(W_T,T) - h_k(W_t,t)\right)^4 = (T - t)^2\varphi_{2(k-1)}(t,T),$$

where $\varphi_{2(k-1)}$ is a polynomial of degree $2(k-1)$.

Next we shall prove the following limit property: for every $w \neq 0$

(9) $$\lim_{\substack{t\to\infty\\ t/T\to 1}} \mathfrak{L}\left(\frac{h_k(W_T,T) - E\{h_k(W_T,T) \mid \mathfrak{F}_{\le t}\}}{\sigma_k(t.T,w)} \,\middle|\, \frac{W_t}{\sqrt{t}} = w\right) = N(0,1),$$

where $\sigma_k^2(t,T,w)$ is the conditional variance given by formula

$$\sigma_k^2\,(t,T,w) = \mathrm{Var}\{h_k(W_T,T) \mid W_t = w\sqrt{t})\},\qquad t < T.$$

It follows from the martingale property and (9) that for $0<T - t\ll t$, for given $w = W_t/\sqrt{t}$, we can write

$$\mathfrak{L}\{h_k(W_T,T)\} \approx N\{h_k(w\sqrt{t},t),\sigma_k(t,T,w)\},$$

(10) $$h_k(W_T,T) \approx h_k(w\sqrt{t},t) + Y\sigma_k(t,T,w),$$

where Y has a normal distribution N(0,1).

Formula (10) can be interpreted as a forecasting at the moment T when the state of the process at the moment t is known. This is of course a linear forecasting. Formula (10) can be treated as a basis for linear forecasting. Of course a basis of another kind then in ARIMA models (see [1]). Unconditional limit distributions (similar to (10)) were considered inthe ealier papers of the authors [8], [9].

The addition formula (4) is the main tool in the proofs of (5)-(9). (Relations (1)-(3) also follow immediately from (4)). This formula looks like binomial formula. On the other hand (4) can be treated as a version of Cameron-Martin formula given in Kallianpur book [4].

Different connections and applications of Hermite polynomials in random theory were considered by Ito [3], Wiener [11], McKean [6], Dym and McKean [2]. Especially homogenous chaos is described by Kallianpur [4].

Hermite polynomials of a Wienar process are a nice example of different known class of stochastic processes (martingale, Markov property, the linearity of regression).

2. Auxiliary results

By the definition of Hermite polynomial and formulas given in [11] we get the following relations

$$H_k(x) = \sum_{r=0}^{[k/2]} \frac{(-1)^r k!}{r!(k-2r)!2^r} x^{k-2r}, \tag{11}$$

$$h_k(x,t) = \sum_{r=0}^{[k/2]} \frac{(-1)^r k!}{r!(k-2r)!2^r} t^r x^{k-2r}, \tag{11'}$$

$$\left(c^2+1\right)^{k/2} H_k\left\{\frac{cx+y}{\sqrt{c^2+1}}\right\} = \sum_{r=0}^{k} \binom{k}{r} c^r H_r(x) H_{k-r}(y), \tag{12}$$

$$\frac{1}{\sqrt{2\pi}} \int H_k(x) H_r(x)\, e^{-x^2/2}\, dx = \begin{cases} k! & \text{for } k=r, \\ 0 & \text{for } k\neq r, \end{cases} \tag{13}$$

$$\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} H_k^4(x) dx = (k!)^2 \sum_{r=0}^{k} \binom{k}{r}^2 \binom{2r}{r}, \tag{14}$$

$$\frac{1}{\sqrt{2\pi}} \int \exp\{-\frac{(x-y)^2}{2}\} H_k(ax) dx = (1-a^2)^{k/2} H_k\left(\frac{ay}{\sqrt{1-a^2}}\right), \tag{15}$$

(16) $$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x),$$

(16') $$h_{k+1}(x,t) = xh_k(x,t) - kth_{k-1}(x,t).$$

The proof of (9) will be based on a limit theorem for sum of dependent random variables.

Let $\{X_{nr}\}$ be a sequence of random variables, $Y_{nr} = g(X_{nr})$, $E(Y_{nr}) = 0$, where $g \in C^0$. We denote $\mathcal{E}_0(\cdot) = E\{(\cdot)|X_{n0} = x_0)\}$, $s_n^2 = \sum_{r=1}^{n} \mathcal{E}_0(Y_{nr}^2)$.

Lemma 1. (see [8]). If

(17) $$\mathcal{E}_0(Y_{nr}|X_{n0},\dots,X_{n,r-1}) = 0,$$

(18) $$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{r=1}^{n} \mathcal{E}_0|\mathcal{E}_0(Y_{nr}^2|X_{n0},\dots,X_{n,r-1}) - \mathcal{E}_0(Y_{nr}^2)| = 0,$$

there exists $\delta > 0$ such that

(19) $$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{r=1}^{n} \mathcal{E}_0(|Y_{nr}^{2+\delta}|) = 0$$

then for every x_0

(20) $$\lim_{n\to\infty} \mathcal{L}\left(\frac{1}{s_n} \sum_{r=1}^{n} Y_{nr}|X_{n0} = x_0\right) = N(0,1).$$

Lemma 1 is a modified version of comparison theorem (see [5]).

3. Proofs of formulas (4) - (9)

We can get formula (4) by elementary methods. Namely we substitute in (12):

$$x = W_t/\sqrt{t}, \quad y = \frac{W_T - W_t}{\sqrt{T-t}}, \quad c = \sqrt{t/(T-t)}.$$

Then after some easy transformations we get (4). Formula (4) can be also deduced by non elementary methods based on representation (3). In virtue of (15) we have

$$E\left\{h_k\left(W_t,t\right)\middle|\mathfrak{F}_{\geq T}\right\} = \sqrt{\frac{T}{2\pi t(T-t)}}\int \exp\left\{-\frac{\left(x - W_T\frac{t}{T}\right)^2}{2(T-t)t/T}\right\}h_k(x,t)\,dx =$$

$$= \left(\frac{t}{T}\right)^k h_k\left(W_T,t\right) .$$

Therefore (5) is proved.

Formula (6) follows from (2), (4) and (13). Formula (7) from (14).

It is evident that (4) can be written in the following form

$$(21) \qquad h_k\left(W_T,t\right) - h_k\left(W_t,t\right) = \sum_{r=0}^{k-1} \binom{k}{r} h_r\left(W_T,T\right) h_{k-r}\left(W_T-W_t,T-t\right).$$

In virtue of (21) and the formulas for moments of the Wiener process we obtain (8).

Now we are going to show (9) using Lemma 1.

We introduce the following partition of the interval (t,T): $t_r = t + \frac{r}{n}(T - t)$, $r=0,\dots,n$. We put $X_{nr} = W_{t_r}$,

$$Y_{knr} = h_k\left(W_{t_r},t_r\right) - h_k\left(W_{t_{r-1}},t_{r-1}\right),$$

$$\mathcal{E}_0(\cdot) = \mathcal{E}_w(\cdot) = E\left((\cdot)\,\middle|\, \frac{W_t}{\sqrt{t}} = w\right).$$

It is evident that

$$h_k(W_T,t)-h_k(W_t,t) = \sum_{r=1}^{n} \left[h_k\left(W_{t_r},t_r\right) - h_k\left(W_{t_{r-1}},t_{r-1}\right)\right] = \sum_{r=1}^{n} Y_{knr}.$$

We will prove that random variables X_{nr}, Y_{knr} satisfy the assumptions of Lemma 1.

Condition (17) follows from (1). In virtue of (2) and (6) we have

$$(22) \qquad \mathcal{E}_0\{|\mathcal{E}_0(Y^2_{knr}|X_{n0},\dots,X_{n,r-1}) - \mathcal{E}_0(Y^2_{knr}|\} =$$

$$= \sum_{s=0}^{k-1} \binom{k}{s}^2 (k-s)!\,(t_r - t_{r-1})^{k-s}\mathcal{E}_w|h^2_s\left(W_{t_{r-1}},t_{r-1}\right) -$$

$$- \sum_{l=1}^{s} \binom{s}{l}^2 (s-1)!\,(t_{r-1} - t)^{s-l} h_l(W_t, t) | \le$$

$$\le \sum_{s=1}^{k-1} \binom{k}{s}^2 (k-s)! \left(\frac{T-t}{n}\right) \sqrt{(t_{r-1} - t) p_{2s-1}(t, t_{r-1})}$$

where p_{2s-1} is a polynomial of degree 2s-1.

Taking into account (1) we get

$$(23) \qquad s_n^2 = \sum_{r=1}^{n} \mathcal{E}_w(Y_{knr}^2) = \sum_{r=1}^{n} \left[\mathcal{E}_w\left\{h_k^2\left(W_{t_r}, t_r\right)\right\} - \right.$$

$$\left. - \mathcal{E}_w\left\{h_k^2\left(W_{t_{r-1}}, t_{r-1}\right)\right\}\right] =$$

$$= \mathcal{E}_w\{h_k^2(W_T, T)\} - \mathcal{E}_w\{h_k^2(W_t, t) = \mathcal{E}_w\{h_k(W_T, T) - h_k(W_t, t)\}^2.$$

By (22) and (23) we have

$$\frac{1}{s_n^2} \sum_{r=1}^{n} \mathcal{E}_w | \mathcal{E}_w(Y_{nkr}^2 | X_{n0}, \ldots, X_{n,r-1}) - \mathcal{E}_w(Y_{nkr}^2) | \le$$

$$\le \frac{\sum_{r=1}^{n} \sum_{s=0}^{k-1} \binom{k}{s}^2 (k-s)! \left(\frac{T-t}{n}\right)^{k-s} \sqrt{(t_{r-1} - t) p_{2s-1}(t, t_{r-1})}}{\sum_{s=0}^{k-1} \binom{k}{s}^2 (k-s)!\,(T-t)^{k-s} t^s H_s(w)} \le$$

$$\le \sqrt{\frac{T-t}{t}} \left[\sum_{s=0}^{k-1} c_{ks} \left(\frac{T-t}{n}\right)^{k-s-1} T^{s-1/2} \right] t^{\frac{3}{2}-k} \longrightarrow 0$$

as $\lim_{t\to\infty} \frac{T-t}{t} = 0$, c_{ks} - depend only on their subscripts. Thus (18) holds.

Condition (19) for $\delta = 2$ follows from (8).

Therefore all the assumptions of Lemma 1 are satisfied and (20) is true. Thus by Lemma 1 formula (9) holds.

4. Remark

Properties similar to (1) - (10) are true for Hermite polyno-mials of optional gaussian Markov processes.

REFERENCES

[1] **G.E.P. Box, G.N. Jenkins:** Time Series Analysis, Forecasting and Control, (Holden-Day, San Francisco 1976).

[2] **H. Dym, H.P. McKean:** Gaussian Processes, Function theory and the inverse spectral problem. Academic Press, New York (1976).

[3] **K. Ito:** Multiple Wiener Integral, J. Math.Soc. Japan 3 (1951), 157-169.

[4] **G. Kallianpur:** Stochastic Filtering Theory (Springer, New York 1980).

[5] **M. Loewe:** Probability Theory II. Springer, New York, 1978, Section 31.2.

[6] **H.P. McKean:** Stochastic Integrals, Academic Press, New York (1969).

[7] **A. Plucińska:** Some limit properties for processes with the linear regression, Theory Probab. Appl. (1991), in print.

[8] **A. Plucińska, E. Pluciński:** Some limit problem for dependent random variables, to appear.

[9] **A. Plucińska, E. Pluciński:** A note on central limit theorem for processes with the linear regression, Proceedings of the the 1989 COSMEX Meeting, Stochastics Method in Experi-mental Sciences, World Scientific 1990,pp.362-372.

[10] **A.P. Prudnikov, J.A. Bryckov, O.I. Maricev:** Integrals and Series, special functions, Science, Moscov 1983, (in Russian).

[11] **N. Wiener:** Non-linear Problems in Random Theory, John Willey, New York 1958.

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