DEMONSTRATIO MATHEMATICA
Vol. XXVI No 1 1993

Galina B. Belyavskaya

ON CONCEPT OF CENTRE IN QUASIGROUPS

0. The author of this article introduced the concept of h-centre Z_h, for an element $h \in Q$ in a quasigroup $Q(\cdot)$. In particular, in [3] and [4] is proved that Z_h is a normal subset of $Q(\cdot)$, i.e. Z_h is a class of some normal congruence on $Q(\cdot)$. Moreover, for any $g,h \in Q$ centres Z_g and Z_h define the same congruence. This normal congruence is denoted by $\theta_z(\cdot)$ or by θ_z. It was also proved that if a θ_z-class is a subquasigroup, then it is a T-quasigroup and every θ_z-class of an idempotent quasigroup is a medial distributive subquasigroup. In a distributive quasigroup $Q(\cdot)$ we get $Z_h = N_m(h)$, for any $h \in Q$, where $N_m(h)$ is the middle nucleus (see [2]).

A quasigroup $Q(\cdot)$ is said to be a T-quasigroup iff

$$xy = \alpha(x) + \beta(y) + c,$$

where $Q(+)$ is an abelian group, α, β are its automorphisms and $c \in Q$. T-quasigroups were introduced and studied by T. Kepka and P. Nemec (see [8], [9]) as a generalization of medial quasigroups. Two simple identities characterizing a primitive T-quasigroup one can find in [3] and [4]. Basic definitions and theorems about quasigroups can be found in [2].

On the other hand, the concept of the centre of a universal algebra A was given by Freese and McKenzie (see for instance [6] p. 83). It is the set of all pairs $(a,b) \in A \times A$ such that for any term function $p(x,y_1,y_2,\dots,y_n)$ on A and for any $c_1,c_2,\dots,c_n,d_1,d_2,\dots,d_n \in A$

$$p(a,c_1,\dots,c_n) = p(a,d_1,\dots,d_n)$$

(1) implies

This note is a part of the lecture presented at the International Conference on Universal Algebra, Quasigroups and Related Systems, Poland, Jadwisin, May 23-28, 1989.

$$p(b,c_1,\ldots,c_n) = p(b,d_1,\ldots,d_n).$$

Denote this binary relation by δ_z. It is a congruence on A coinciding with the centre from [7], [12], [13]. In these papers the centre coincides with the centre for algebras in Mal'cev varieties and for algebras in modular varieties. We denote this centre of a primitive quasigroup $Q(\cdot,\backslash,/)$ (of a quasigroup $Q(\cdot)$) by $\delta_z(\cdot,\backslash,/)$ (by $\delta_z(\cdot)$ respectively).

The purpose of this note is to prove that

$$\theta_z(\cdot) = \theta_z(\backslash) = \theta_z(/) = \delta_z(\cdot,\backslash,/) \le \delta_z(\cdot) .$$

This result implies that ζ-quasigroups $Q(\cdot,\backslash,/)$ defined by J.D.H. Smith in [14] (i.e. quasigroups with $\delta_z(\cdot,\backslash,/) = Q\times Q$) are (primitive) T-quasigroups, and only they are nilpotent quasigroups of class 1 in the variety of all primitive quasigroups. We consider also quasigroups with $\delta_z(\cdot,\backslash,/) = \delta_z(\cdot)$.

1. Some notions and results

At first we remind some necessary notions and results from [2], [7], [12] and make several remarks.

Let $Q(\cdot)$ be a quasigroup. We use the following notations: $x\cdot y = xy$, $(xy)z = xy\cdot z$, $x(yz) = x\cdot yz$, $R_a x = xa$, $L_a x = ax$. A congruence θ on a quasigroup $Q(\cdot)$ is called normal iff each of the relations $xa\ \theta\ ya$ and $ax\ \theta\ ay$ implies $x\ \theta\ y$ (in other words, $x\ \theta\ y$ implies $R_a^{-1}x\ \theta\ R_a^{-1}y$ and $L_a^{-1}x\ \theta\ L_a^{-1}y$) (see [2], p. 54).

A homomorphic image of a quasigroup is a groupoid with division (see [2]). The quotient groupoid $Q/_\theta$ is a quasigroup iff θ is a normal congruence. A normal congruence is defined by any its class.

If $Q(\cdot,\backslash,/)$ is the primitive quasigroup for a quasigroup $Q(\cdot)$, then

$$(x/y)y = x, \quad x(x\backslash y) = y, \quad xy/y = x, \quad x\backslash xy = y.$$

All primitive quasigroups form a Mal'cev variety (see [13]).

Every congruence on the quasigroup $Q(\cdot,\backslash,/)$ is a normal congruence on $Q(\cdot)$ and conversely. Moreover, each normal congruence is invariant with respect to the parastrophy (see [1]).

The group of permutations on Q, generated by R_a, L_a,

$a \in Q$, is called the multiplication group of the quasigroup $Q(\cdot)$. It is denoted by $G_{(\cdot)}$. For an element $h \in Q$, the stabilizer of h in $G_{(\cdot)}$ is denoted by I_h. By Theorem 4.4 from [2] the group I_h is generated by the permutations

$$R^h_{x,y} = R^{-1}_{x\cdot y}R_yR_x\ ,\quad L^h_{x,y} = L^{-1}_{x\circ y}L_xL_y\ ,\quad T^h_x = L^{-1}_{\sigma x}R_x\ ,$$

where $x,y \in Q$ and $x\cdot y = L_h^{-1}(hx\cdot y)$, $x\circ y = R_h^{-1}(x\cdot yh)$, $\sigma = R_h^{-1} L_h$, i.e. $h(x\cdot y) = (hx)y$, $(x\circ y)h = x(yh)$, $ha = \sigma(a)h$.

According to Theorem 1 from [4], a subset $H \subseteq Q$ containing the element $h \in Q$ is normal in a quasigroup $Q(\cdot)$ iff:

(i) $\alpha H = H$ for all $\alpha \in I_h$,

(ii) $R_h^{-1}H\cdot H \subseteq H$ and $h \in R_h^{-1}a\cdot H$ for all $a \in H$,

where $R_h^{-1}H\cdot H = \{R_h^{-1}a\cdot b : a,b \in H\}$.

Remark 1. Let $a,b,x,h \in Q$, then

$R^h_{a,b}x = (xa\cdot b)/(h\backslash(ha\cdot b)) = p_1(x,h,a,b)$,

$L^h_{a,b}x = ((a\cdot bh)/h)\backslash(a\cdot bx) = p_2(x,h,a,b)$,

$T^h_a x = (ha/h)\backslash xa = p_3(x,a,h)$.

Thus $R^h_{a,b}$, $L^h_{a,b}$, T^h_a correspond to the term functions $p_1(x,y_1,y_2,y_3)$, $p_2(x,y_1,y_2,y_3)$, $p_3(x,y_1,y_2)$ on $Q(\cdot,\backslash,/)$, respectively. Namely, these permutations are translations of $Q(\cdot,\backslash,/)$ in the sense of Mal'cev (see [10]). Analogously, such permutations in $Q(\backslash)$ and $Q(/)$ also correspond to some term functions of $Q(\cdot,\backslash,/)$.

Remark 2. It is a simple exercise to show that in the quasigroup $(Q\times Q)(\cdot)$ the permutations

$$\tilde{R}^{(h,h)}_{(x,y),(u,v)}\ ,\quad \tilde{L}^{(h,h)}_{(x,y)(u,v)}\ ,\quad \tilde{T}^{(h,h)}_{(x,y)}$$

are expressed by the corresponding permutations in $Q(\cdot)$ as follows:

$$\tilde{R}^{(h,h)}_{(x,y),(u,v)}(a,b) = (R^h_{x,u}\ a,\ R^h_{y,v}\ b)\ ,$$

$$\tilde{L}^{(h,h)}_{(x,y),(u,v)}(a,b) = (L^h_{x,u}\, a,\ L^h_{y,v}\, b)\ ,$$

$$\tilde{T}^{(h,h)}_{(x,y)}(a,b) = (T^h_x\, a,\ T^h_y\, b)\ ,$$

since

$\tilde{R}_{(x,y)}(a,b) = (R_x\, a,\ R_y\, b)$ and $\tilde{L}_{(x,y)}(a,b) = (L_x\, a,\ L_y\, b)$.

Remark 3. The condition (1) from the definition of the centre δ_z of the algebra A states precisely that for any arbitrary $x \in A$

$$K_x^{\delta_z} = \{(y,y)\ :\ y\ \delta_z x\}$$

is a class of a some congruence on A×A (see [4],[7],[10]). Hence the centre $\delta_z(\cdot,\backslash,/) = \delta_z$ is the greatest congruence on $Q(\cdot,\backslash,/)$ such that for any $x \in Q$, $K_z^{\delta_z}$ is a class of a some congruence on $(Q\times Q)(\cdot,\backslash,/)$.

Note that there is one more characterization of the centre $\delta_z(\cdot,\backslash,/)$ in [14] as the (unique) maximal subquasigroup of $(Q\times Q)(\cdot,\backslash,/)$ containing the diagonal $\hat{Q} = \{(a,a)\ :\ a \in Q\}$ as a normal subquasigroup.

2. The centres $\theta_z(\cdot)$, $\delta_z(\cdot,\backslash,/)$ and $\delta_z(\cdot)$

The left (right) identity of $h \in Q$ is denoted by f_h (e_h - respectively).

Definition (cf. [3], [4]). The h-centre of a quasigroup $Q(\cdot)$ is the greatest subset H of Q such that:

(2) (i) $R^h_{x,y}a = ae_h$, $L^h_{x,y}a = f_h a$, $T^h_x\, a = \delta_a^{-1}$

for all $a \in H$ and $xy \in Q$,

(ii) $He_h = f_h H = h$ and $h \in R_h^{-1}a\cdot H$ for all $a \in H$

where $He_h = \{ae_h\ :\ a \in H\}$, $f_h H = \{f_h a\ :\ a \in H\}$.

The h-centre is denoted by Z_h. It contains the element h and, in according to [3] and [4], it is a normal subset in $Q(\cdot)$. Moreover, Z_h and Z_g define the same normal congruence on $Q(\cdot)$. This congruence is denoted by $\theta_z(\cdot)$ or by θ_z and is called the centre of the quasigroup $Q(\cdot)$.

The h-centre of a quasigroup $Q(\cdot)$ has the following characterization:

Lemma. The h-centre Z_h of a quasigroup $Q(\cdot)$ is the greatest normal subset of Q such that

(3) $$R^h_{x,y}\, a = R^h_{u,v}\, a\ ,\quad L^h_{x,y}\, a = L^h_{u,v} a\ ,\quad T^h_x\, a = T^h_u\, a$$

for all $a \in Z_h$ and $x,y,u,v \in Q$.

Proof. If the subset H of Q satisfies the conditions (3) and $a \in H$, then for all $x,y \in Q$ we have

$$R^h_{x,y}\, a = R^h_{e_h,e_h} a = R_{e_h} a\ ,$$

$$L^h_{x,y}\, a = L^h_{f_h,f_h} a = L_{f_h} a\ ,$$

$$T^h_x\, a = T^h_h\, a = \sigma^{-1} a\ ,$$

i.e. the condition (i) from the definition of Z_h holds.

Since H is a normal subset of Q then by Theorem 1 from [4] the condition (ii) holds as well.

Conversely, (3) follows immediately from (2). Normality of Z_h is proved in [3] and [4].

Now, let $\theta_z(\cdot)$, $\theta_z(\backslash)$, $\theta_z(/)$ are the centres of the quasigroups $Q(\cdot)$, $Q(\backslash)$, $Q(/)$, respectively. Let $\delta_z(\cdot,\backslash,/)$ be the centre of the primitive quasigroup $Q(\cdot,\backslash,/)$ (defined like in [6]).

Theorem 1. $\delta_z(\cdot,\backslash,/) = \theta_z(\cdot) = \theta_z(\backslash) = \theta_z(/)$.

Proof. At first we shall prove that $\delta_z(\cdot,\backslash,/) \le \theta_z(\cdot)$. If a,b,c,d,h are arbitrary elements from Q, then

$$R^h_{a,b}\, h = R^h_{c,d}\, h = h\ ,$$

or according to Remark 1

$$p_1(h,h,a,b) = p_1(h,h,c,d)\ .$$

Thus, (1) for $\delta_z(\cdot,\backslash,/)$ implies

$$p_1(x_0,h,a,b) = p_1(x_0,h,c,d)$$

for all $(h,x_0) \in \delta_z(\cdot,\backslash,/)$, i.e. $R^h_{a,b}\, x_0 = R^h_{c,d}\, x_0$ for every

$x_0 \in K_h$, where K_h is δ_z-class containing h.

Analogously, using the term functions $p_2(x,y_1,y_2,y_3)$ and $p_3(x,y_1,y_2)$ from Remark 1, for $a,b,c,d \in Q$ and $x_0 \in K_h$ we obtain

$$L^h_{a,b}\, x_0 = L^h_{c,d}\, x_0 \quad \text{and} \quad T^h_a\, x_0 = T^h_b\, x_0 \ .$$

According to the Lemma, $K_h \subseteq Z_h$, i.e.

$$\delta_z(\cdot,\backslash,/) \le \theta_z(\cdot) \ , \tag{4}$$

since $\delta_z(\cdot,\backslash,/)$ and $\theta_z(\cdot)$ are normal congruences on $Q(\cdot)$. But $\delta_z(\cdot,\backslash,/)$ is a normal congruence on the quasigroup $Q(\backslash)$ and $Q(/)$, too. And some term functions on $Q(\cdot,\backslash,/)$ correspond to the permutations $\overset{/}{L}{}^h_{a,b}$, $\overset{/}{R}{}^h_{a,b}$, $\overset{/}{T}{}^h_a$ and $\overset{\backslash}{L}{}^h_{a,b}$, $\overset{\backslash}{R}{}^h_{a,b}$, $\overset{\backslash}{T}{}^h_a$ in quasigroups $Q(/)$ and $Q(\backslash)$.

Consequently,

$$\delta_z(\cdot,\backslash,/) \le \theta_z(/) \quad \text{and} \quad \delta_z(\cdot,\backslash,/) \le \theta_z(\backslash) \ . \tag{5}$$

Let $\theta_z = \theta_z(\cdot)$. Now we shall show that

$$H = K_h^{\theta_z} = \{(a,a) : h\theta_z\, a\} = \{(a,a) : a \in Z_h\}$$

is a class of some normal congruence on the quasigroup $(Q\times Q)(\cdot)$ and hence H is a class of some congruence on $(Q\times Q)(\cdot,\backslash,/)$.

According to Theorem 1 from [4] we have to prove that

(i) $\tilde{\alpha}H = H$ for $\tilde{\alpha} \in \tilde{I}_{\bar{h}}$,

(ii) $\tilde{R}^{-1}_{\bar{h}} H\cdot H \subseteq H$ and $\bar{h} \in \tilde{R}^{-1}_{\bar{h}}\, \bar{a}\cdot H$ for $\bar{a} \in H$,

where $\bar{h} = (h,h)$, $\tilde{R}_{\bar{h}}\bar{x} = \overline{xh}$, $\bar{x} \in Q\times Q$, $\tilde{I}_{\bar{h}}$ is the stabilizer of $\bar{h}$ in $\tilde{G}_{(\cdot)}$ and $\tilde{G}_{(\cdot)}$ is the multiplication group of $(Q\times Q)(\cdot)$.

Let $a \in Z_h$ then $(a,a) \in H$. Using Remark 2 and the definition of Z_h we obtain:

$$\tilde{R}^{(h,h)}_{(x,y),(u,v)}(a,a) = (R^h_{x,u}\, a,\, R^h_{y,v}\, a) = (ae_h, ae_h) \in H \ ,$$

$$\tilde{L}^{(h,h)}_{(x,y),(u,v)}(a,a) = (L^h_{x,u}\, a,\, L^h_{y,v}\, a) = (f_h a, f_h a) \in H \ ,$$

$$\tilde{T}^{(h,h)}_{(x,y)}(a,a) = (T^h_x\, a,\, T^h_y\, a) = (\sigma^{-1}a,\, \sigma^{-1}a) \in H \ ,$$

since ae_h, $f_h a$, $\sigma^{-1}a = L_{f_h}^{-1} R_{e_h} a \in Z_h$. The last equality follows from $T_x^h\, a = \sigma^{-1}a$ if $x = e_h$.

Therefore $\tilde{\alpha}H \subseteq H$ for any $\tilde{\alpha} \in I_h$ and so $\tilde{\alpha}H = H$, which proves (i). To prove (ii) observe that for all $a,b \in Z_h$

$$\tilde{R}^{-1}_{(h,h)}(a,a)\cdot(b,b) = (R_h^{-1}a\ ,\ R_h^{-1}a)(b,b) = (R_h^{-1}a\cdot b,\ R_h^{-1}a\cdot b) \in H$$

so far as $R_h^{-1}Z_h \cdot Z_h \subseteq Z_h$.

By the definition of Z_h for every $a \in Z_h$ and some $z_0 \in Z_h$ we have $h \in R_a^{-1}a\cdot z_0$. Hence

$$(h,h) = (R_h^{-1}a\cdot z_0,\ R_h^{-1}a\cdot z_0) =$$
$$\tilde{R}^{-1}_{(h,h)}(a,a)\cdot(z_0,z_0) \in \tilde{R}^{-1}_{(h,h)}(a,a)\cdot H.$$

Thus H is a class of some congruence on $(Q\times Q)(\cdot,\backslash,/)$. Using Remark 3 now we conclude that

$$\theta_z(\cdot) \le \delta_z(\cdot,\backslash,/)\ .$$

This, together with (4), implies

$$\theta_z(\cdot) = \delta_z(\cdot,\backslash,/)\ . \tag{6}$$

Let ${}^r(\backslash)$ (${}^l(/)$ respectively) be the right (left) inverse operation for $(\backslash)$ (for $(/)$, respectively). It is clear that ${}^r(\backslash) = {}^l(/) = (\cdot)$. From (5) and (6) we get

$$\theta_z(\cdot) \le \theta_z(\backslash) \cap \theta_z(/)\ .$$

But then

$$\theta_z(\backslash) \le \theta_z{}^r(\backslash) = \theta_z(\cdot) \quad\text{and}\quad \theta_z(/) \le \theta_z{}^l(/) = \theta_z(\cdot)\ ,$$

which completes our proof.

Corollary 1. The centre $\theta_z(\cdot)$ of a quasigroup $Q(\cdot)$ is invariant by parastrophy. The centres of the primitive quasigroups $Q(\cdot,\backslash,/)$, $Q(\backslash,\cdot,{}^l(\backslash))$ and $Q(/,{}^r(/),\cdot)$ coincide.

According to Theorem 6 from [4], the h-centre Z_h is a T-quasigroup if Z_h is a subquasigroup of $Q(\cdot)$ and conversely, if $Q(\cdot)$ is a T-quasigroup then $Z_h = Q$ for all $h \in Q$.

Corollary 2. A ζ-quasigroup $Q(\cdot,\backslash,/)$ is a (primitive) T-quasigroup and conversely, every primitive T-quasigroup is a ζ-quasigroup.

Thus, in the variety of all primitive quasigroups only T-quasigroups are nilpotent quasigroups of the class 1.

Now we consider the centre $\delta_z(\cdot)$ of a quasigroup $Q(\cdot)$ (in the sense of [6]). In this case all term functions contain only one operation $(\cdot)$ and so form a part of all term functions of the primitive quasigroup $Q(\cdot,\backslash,/)$. We received

Corollary 3. $\delta_z(\cdot,\backslash,/) = \theta_z(\cdot) \le \delta_z(\cdot)$.

Describe some quasigroups with $\theta_z(\cdot) = \delta_z(\cdot)$. At first we remind that a loop $Q(\cdot)$ is called normal if all congruences on $Q(\cdot)$ are normal (see [5]). Let us extend this concept on quasigroups. Examples of normal quasigroups are TS-quasigroups, IP-loops and all finite quasigroups. We say (see [2]), that a quasigroup $Q(\cdot)$ is a IP-quasigroup if there are permutations I_r and I_l on Q (they have the order two) such that $I_l x\cdot xy = y$ and $xy\cdot I_r y = x$. All IP-quasigroups are normal. Indeed, let θ be a congruence of a IP-quasigroup $Q(\cdot)$. Then for any $a \in Q$ the condition $x\,\theta\,y$ implies $x\cdot I_r a\,\theta\,y\cdot I_r a$ and $x/a\,\theta\,y/a$. This gives $R_a^{-1}x\,\theta\,R_a^{-1}y$, because $x/y = x\cdot I_r y$. Analogously, $x\,\theta\,y$ implies $I_l a\cdot x\,\theta\,I_l a\cdot y$ and $a\backslash x\,\theta\,a\backslash y$. Thus $L_a^{-1}x\,\theta\,L_a^{-1}y$ because $x\backslash y = I_l x\cdot y$ and $a\backslash x = L_a^{-1}x$.

Theorem 2. If $Q(\cdot)$ and $(Q\times Q)(\cdot)$ are normal quasigroups then

$$\theta_z(\cdot) = \delta_z(\cdot,\backslash,/) = \delta_z(\cdot) \ .$$

Proof. Quasigroups $Q(\cdot)$ and $(Q\times Q)(\cdot)$ are normal, so lattices of all congruences of these quasigroups coincide with lattices of all congruences of the quasigroups $Q(\cdot,\backslash,/)$ and $(Q\times Q)(\cdot,\backslash,/)$, respectively. By Remark 3 $\delta_z(\cdot,\backslash,/) = \theta_z(\cdot) = \theta_z$ is the greatest congruence on $Q(\cdot,\backslash,/)$, such that for all $x \in Q$

$$K_x^{\theta_z} = \{(y,y) : x\,\theta_z\,y\}$$

is precisely a class of some congruence on $(Q\times Q)(\cdot,\backslash,/)$. Moreover, $\delta_z(\cdot) = \delta_z$ is the greatest congruence on $Q(\cdot)$ such that

$$K_x^{\delta_z} = \{(y,y) : x\,\delta_z\,y\}$$

is a class of some congruence on $(Q\times Q)(\cdot)$. The statement of the theorem follows from coincidence of the corresponding congruence lattices.

Corollary 4. If $Q(\cdot)$ is a IP-quasigroup, or a TS-quasigroup or a finite quasigroup then $\theta_z(\cdot) = \delta_z(\cdot,\backslash,/) = \delta_z(\cdot)$.

In particular, groups, IP-loops, Steiner quasigroups and CH-quasigroups (see [11]), have this property.

REFERENCES

[1] **V.D. Belousov:** On the group associative with a quasigroup, Mat. Issledov. 4(1969) 21-39 (Russian).

[2] **V.D. Belousov:** Foundations of the theory of quasigroups. Moscow 1967.

[3] **G.B. Beylavskaya:** The left, right, middle nuclei and the centre of a quasigroup, Institute of Mathematics of Mold. SSR, Kishiniev, Preprint 1988 (Russian).

[4] **G.B. Beylavskaya:** T-quasigroups and the centre of a quasigroup, Mat. Issledov. 111(1989), 24-43 (Russian).

[5] **G. Birkhoff:** Lattice Theory. Rhode Island 1967.

[6] **S. Burris, H.P. Sankappanavar:** A course in universal algebra. Springer-Verlag, 1981.

[7] **H.P. Gumm:** An easy way to the commutator in modular varieties, Archiv. Math. 34(1980), 220-228.

[8] **T. Kepka, P. Nemec:** T-quasigroups, I, Acta Univ. Carolinae 12(1971), 39-49.

[9] **T. Kepka, P. Nemec:** T-quasigroups, II, Acta Univ. Carolinae 12(1971), 31-49.

[10] **A.I. Mal'cev:** On the general theory of algebraic systems, Math. Sb. 35 (77) (1954), 3-20 (Russian).

[11] **Ju.I. Manin:** Cubic Forms. North-Holland, 1974.

[12] **A.G. Pinus:** Congruence-modular varieties of algebras. Irkutsk, 1986.

[13] J.D.H. Smith: Mal'cev varieties, Lecture Notes Math. 554(1976).
[14] J.D.H. Smith: Combinatorial characters of quasigroups, Inst. of Math. and its Applications, Minnesota, Preprint 425 (1988).

INSTITUTE OF MATHEMETICS, MOLDAVIAN ACADEMY OF SCIENCES,
ul. Grosula 5, 277 028 KISHINIEV, MOLDOVA

Received July 18, 1990.