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Efstathios Vassiliou

ON A CLASS OF PRINCIPAL BUNDLES OVER SYMPLECTIC BASES ON EUCLIDEAN SPACES

Introduction

In this paper we are concerned with a certain class of principal fibre bundles over symplectic manifolds, namely bundles equipped with a connection whose curvature form is, roughly speaking, the lift of a symplectic form.

This situation appears e.g. in the geometric (pre)quantization where the structural group is assumed to be abelian. However, the commutativity of the group is a simplifying assumption, which conceals a great deal of the geometric aspect underlying the amphidromy between the global and local structure of such bundles.

The purpose of the present note is precisely to exhibit the previous geometric aspect and to characterize equivalent bundles of the above said type in terms of appropriate local functions and forms. This is stated in the Theorem of Section 2.

We should like to add that, beside the non-commutativity of the structure group, our proof does not rely on particular open coverings (such as 1-simple) of the base space, a fact which allows to set the whole approach within the framework of infinite-dimensional manifolds and bundles.

1. (G,λ)-bundles and local systems

Throughout this paper we are working in the category of smooth Banach manifolds and bundles. The main notations and terminology are those of [1] and [4]. Particular cases of our

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approach, explicitly mentioned in the sequel, are related with [7] and [8].

Let (B,θ) be a *symplectic manifold* in the sense of [2], G a (not necessarily abelian) *Lie group* with corresponding *Lie algebra* g and let $\lambda:\mathbb{R}\longrightarrow g$ be an *injective linear map.* For a manifold M and any linear space $\mathbb{F}$, we denote by $\Lambda^k(M,\mathbb{F})$ the space of $\mathbb{F}$-valued differential k-forms on M.

Definition 1. A (G,λ)-*bundle* over (B,θ) is a principal fiber bundle (P,G,B,π) equipped with a connection (form) $\omega\in\Lambda^1(P,g)$ such that

$$(*) \qquad \Omega = \lambda(\pi^*\theta),$$

where Ω is the *curvature form* of ω.

In $(*)$ λ denotes also (for simplicity) the mapping $\Lambda^2(P,\mathbb{R})\longrightarrow\Lambda^2(P,g)$ induced by the linear injection λ. A bundle as in Definition 1 is denoted by $Q=(P,G,B,\pi;\omega,\lambda,\theta)$.

An example of this bundles provide the quantizing over (B,θ), where G is assumed to be abelian. Note that in this case, $(*)$ reduces to $d\omega=\lambda(\pi^*\theta)$ (cf. also [7] and [8] for further details).

Let now $U=\{U_i,\ i\in I\}$ be an open cover of B and let $\{(U_i,\Phi_i),\ i\in I\}$ be a trivializing cover of P, where $\Phi_i:\pi^{-1}(U_i)\longrightarrow U_i\times G$. The natural sections $s_i:U_i\longrightarrow P:x\longmapsto s_i(x):=$ $:=\Phi_i^{-1}(x,e)$ of P determine ω completely by means of the corresponding *local connection forms* $\omega_i:=s_i^*\omega$ (cf. e.g. [4], [5]). Setting also $\Omega_i:=s_i^*\Omega$, we obtain the following consequence of Condition $(*)$.

Lemma 1. With the previous notations, equality

$$\Omega=\pi^*\Omega_i$$

holds on $\pi^{-1}(U_i)$, for every $i\in I$.

Proof. First we prove the equality for $p\in\pi^{-1}(U_i)$ with $pr_2(\Phi(p))=e$. If $\pi(p)=x$, for any $u\in T_pP$, we set

$$T_p\Phi_i(u) = (u_x,u_e) = (u_x,0) + (0,u_e),$$

where $u_x=T_p\pi(u)\in T_xB$ and $u_e=pr_2(T_p\Phi_i(u))\in T_eG$. Since $(\Phi_i^{-1})^*\omega$ is a connection on $U_i\times G$ and each $(0,U_e)\in 0\times T_eG$ is a vertical vector, we have that, for each $u,v\in T_pP$,

$$\Omega_p(u,v) = ((\Phi_i^{-1})^*\Omega)_{(x,e)}[(u_x,u_e),(v_x,v_e)] =$$
$$= ((\Phi_i^{-1})^*\Omega)_{(x,e)}[(u_x,0),(v_x,0)] = \Omega_p[T_xs_i(u_x),T_xs_i(v_x)] =$$
$$= (s_i^*\Omega)_x[T_p\pi(u),T_p\pi(v)] = (\pi^*\Omega_i)_p(u,v).$$

For an arbitrary $q\in\pi^{-1}(U_i)$, we may set $q=p.g$ with $g\in G$ and p as before. Also, for any $u_q\in T_qP$, there exists $u_p\in T_pP$ with $u_q=T_pR_g(u_p)$, where R_g denotes the right translation (by g) of G. Hence, the R_g-invariance of Ω (an immediate consequence of (*)) and the first case imply that

$$\Omega_q(u_q,v_q) = \Omega_p(u_p,v_p) = \Omega_{i,\pi(p)}[T_p\pi(u_p),T_p\pi(v_p)] =$$
$$\Omega_{i,\pi(p)}[(T_p\pi\circ T_qR_g-1)(u_q),\ (T_p\pi\circ T_qR_g-1)(v_q)\} = (\pi^*\Omega_i)_q(u_q,v_q),$$

for every $u_q,v_q\in T_qP$, thus completing the proof. □

Corollary 1. ([7], Proposition 4). If G is abelian, then $d\omega=\pi^*d\omega_i$ holds on each $\pi^{-1}(U_i)$.

Fix again an open cover $U=\{U_i,\ i\in I\}$ of (B,θ).

Definition 2. *A local* (G,λ,U,θ)*-system* is a collection $S=\{g_{ij},\omega_i;\ i,j\in I\}$, where $g_{ij}:U_i\cap U_j\longrightarrow G$ is a cocycle over U and $\omega_i\in\Lambda^1(U_i,g)$ satisfy the following conditions:

(1.1) $$\omega_j = Ad(g_{ij}^{-1})\omega_i + g_{ij}^{-1}.dg_{ij},$$

(1.2) $$d\omega_i + 1/2.[\omega_i,\omega_j] = \lambda\theta,$$

on $U_i\cap U_j$ and U_i respectively.

Note that now λ is the mapping $\Lambda^2(B,\mathbb{R})\longrightarrow\Lambda^2(B,g)$ induced by the original linear injection λ.

In the terminology of (abelian) quantization a system S as above is called a system of (G,λ,U,θ)-quantizing functions.

The following result connects the bundles and systems of the previous definitions. Namely we have

Lemma 2. Any (G,λ)-bundle $Q=(P,G,B,\pi;\omega,\lambda,\theta)$ determines a local (G,λ,U,θ)-system over an arbitrary trivializing open cover U of B and vice versa.

Proof. Let Q be a (G,λ)-bundle. If $\{g_{ij};\ i,j\in I\}$ is the cocycle determined by the local sections s_i $(i\in I)$ over any U and ω_i are the local connection forms of ω (with respect to the same sections), then $S=\{g_{ij},\omega_i;\ i,j\in I\}$ is a local

$(G,\lambda,\mathcal{U},\theta)$-system. Indeed, Lemma 1 and (*) imply that

$$\Omega_i = s_i^*(\lambda\pi^*\theta) = \lambda(\pi\circ s_i)^*\theta = \lambda\theta.$$

Thus, Cartan's structural equation yields (1.2). On the other hand, (1.1) is the compatibility condition of the local connection forms on overlappings.

Conversely, by standard arguments (see e.g. [1], [4]) such a system S determines a pair (P,ω), where P is the principal bundle corresponding to the cocycle $\{g_{ij}\}$ and ω the connection on P having local connection forms the given ω_i's. Condition (*) is now a consequence of Lemma 1 and equality (1.2). □

Definition 3. Two bundles $Q=(P,G,B,\pi;\omega,\lambda,\theta)$ and $Q'=(P',G',B',\pi';\omega',\lambda',\theta')$ are said to be (f,φ,h)-*equivalent* if (f,φ,h) is a principal bundle isomorphism of (P,G,B,π) onto (P',G',B',π') such that h is a symplectomorphism of (B,θ) onto (B',θ') and the following conditions hold:

(1.3) $$\lambda' = \varphi_*\circ\lambda,$$

(1.4) $$f^*\omega' = \varphi_*\omega.$$

Note that φ_* is the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$-valued forms associated to the Lie group isomorphism $\varphi:G\longrightarrow G'$. Condition (1.3) refers to both interpretations of λ and λ' as mappings of appropriate forms used in Definitions 1 and 2.

On the other hand, let $S=\{g_{ij},\omega_i;\ i,j\in I\}$ and $S'=\{g'_{\alpha\beta},\omega'_\alpha;\ \alpha,\beta\in J\}$ be two local $(G,\lambda,\mathcal{U},\theta)$ and $(G',\lambda',\mathcal{U}',\theta')$-systems respectively, where $\mathcal{U}=\{(U_i,\Phi_i),\ i\in I\}$ and $\mathcal{U}'=\{(V_\alpha,\psi_\alpha),\ \alpha\in J\}$. Denoting, for simplicity, by Ad the adjoint representation of G and G′, we give the following.

Definition 4. Two local systems S and S', as above, are said to be (φ,h)-*equivalent* if there exists a Lie group isomorphism $\varphi:G\longrightarrow G'$, a symplectomorphism $h:(B,\theta)\longrightarrow(B',\theta')$ and smooth morphisms $h_{\alpha i}:U_i\cap h^{-1}(V_\alpha)\longrightarrow G'$ $(\alpha\in J,\ i\in I)$ such that the following conditions hold:

(1.5) $$h_{\alpha j} = h_{\alpha i}\cdot(\varphi\circ g_{ij}) \quad \text{on} \quad U_i\cap U_j\cap h^{-1}(V_\alpha),$$

(1.5′) $$h_{\alpha i} = (g'_{\alpha\beta}\circ h)\cdot h_{\beta i} \quad \text{on} \quad U_i\cap h^{-1}(V_\alpha\cap V_\beta),$$

(1.6) $$\varphi_*\cdot\omega_i = \mathrm{Ad}(h_{\alpha i}^{-1})(h^*\omega'_\alpha)+h_{\alpha i}^{-1}\cdot dh_{\alpha i} \quad \text{on} \quad \pi^{-1}(U_i\cap h^{-1}(V_\alpha)),$$

(1.7) $$\lambda' = \varphi_*\circ\lambda.$$

Note that in (1.6) $h_{\alpha i}^{-1}.dh_{\alpha i}$ denotes the left total differential of $h_{\alpha i}$.

The next result is crucial for the proof of the main Theorem and generalizes the case of ([1], n° 6.4.4).

Lemma 3. Let (P,G,B,π) and (P',G',B',π') be two (f,φ,h)--isomorphic bundles. If $C=\{g_{ij};\ i,j\in I\}$ and $C'=\{g'_{\alpha\beta};\ \alpha,\beta\in J\}$ are the cocycles of P and P′ over the covers $U=\{U_i,\ i\in I\}$ and $U'=\{V_\alpha,\ \alpha\in J\}$ respectively, then there exist smooth morphisms $h_{\alpha i}:U_i\cap h^{-1}(V_\alpha)\longrightarrow G'$ (whenever U_i and $h^{-1}(V_\alpha)$ meet) such that equalities (1.5) and (1.5′) hold along together with

(1.8) $$f\circ s_i = (s_\alpha\circ h).h_{\alpha i} \quad \text{on } U_i\cap h^{-1}(V_\alpha).$$

Conversely, if two cocycles C and C', as above, satisfy equalities (1.5) and (1.5′) for a Lie group isomorphism $\varphi:G\longrightarrow G'$ and a diffeomorphism $h:B\longrightarrow B'$, then there exists an isomorphism (f,φ,h) between the corresponding principal bundles such that (1.8) is satisfied.

Proof. The first part is an obvious consequence of a principal bundle (iso)morphism together with the usual combinations between local sections and transition functions.

Conversely, we define the mappings $f_{\alpha i}:\pi^{-1}\big(U_i\cap h^{-1}(V_\alpha)\big)\longrightarrow\pi'^{-1}\big(h(U_i)\cap V_\alpha\big)$ given by

$$f_{\alpha i}(p) = s'_\alpha(h(x)).h_{\alpha i}(x).\varphi(\underline{k}(p,s_i x)))^{-1},$$

where $x=\pi(p)$ and $\underline{k}:P\times_B P\longrightarrow G$ is the smooth morphism defined by $q=p.\underline{k}(p,q)$. We routinely check that, gluing together the $f_{\alpha i}$'s, we obtain a diffeomorphism f such that (f,φ,h) satisfies the stated properties. □

Remark. Assume that G is abelian. Then (1.2) takes the form $d\omega_i=\lambda\theta$, as in [7].

Moreover, if U and U' are 1-connected, then (1.1) reduces to $\omega_j=\omega_i+d\omega_{ij}$, where $\omega_{ij}:U_i\cap U_j\longrightarrow g$ is a smooth morphism such that $g_{ij}=\exp\circ\omega_{ij}$. Similarly, (1.6) reduces to $\varphi_*\omega_i=h^*\omega_\alpha+df_{\alpha i}$ with $f_{\alpha i}:U_i\cap h^{-1}(V_\alpha)\longrightarrow G'$ determined by $h_{\alpha i}=\exp'\circ f_{\alpha i}$. Hence, if $G=G'$ (abelian), $(B,\theta)=(B',\theta')$ and $\varphi=id_G$ $h=id_B$, then we obtain the formulas of ([7], Definition 2 and [8], §2).

2. The main result

We shall show that the classification of (equivalent) (G,λ)-bundles can be described by means of local systems. Our proof is based on an approach elaborating [6].

Theorem. Let $Q=(P,G,B,\pi;\omega,\lambda,\theta)$ and $Q'=(P',G',B',\pi';\omega',\lambda',\theta')$ be two bundles with corresponding (by Lemma 2) local systems $S=\{g_{ij},\omega_i;\ i,j\in I\}$ and $S'=\{g'_{\alpha\beta},\omega_\alpha;\ \alpha,\beta\in J\}$ over the covers U and U' of (B,θ) and (B',θ') respectively. Then Q and Q' are (f,φ,h)-equivalent if and only if S and S' are (φ,h)-equivalent.

Proof. Let (f,φ,h) be an isomorphism as in Definition 3. For the first part of the theorem, in virtue of Lemmas 2 and 3, it suffices to prove (1.6). To this end we denote by δ the (right) action of G' on P', by R_a the right translation of G', for any $a\in G'$, and by e the unit of both G and G'. Therefore, differentiation of (1.8) yields

$$T_x(f\circ s_i) = T_{s'(h(x))}R_{h(x)}\circ T_x(s'_\alpha\circ h) + \tag{2.1}$$
$$+ T_e[\delta(s'_\alpha(h(x)).h_{\alpha i}(x),.)\circ(h^{-1}_{\alpha i}.dh_{\alpha i})$$

where α and i are omitted whenever appear as subscripts of a subscript (for example, the subscript h(x) of R is in fact $h_{\alpha i}(x)$ etc.). Applying now (1.8) and (2.1) in (1.4), for every $x\in U_i\cap h^{-1}(V_\alpha)$ and $u\in T_xB$, we have that

$$(\varphi_*\omega_i)(u) = [(f\circ s_i)^*\omega'](u) =$$
$$= \omega'((T_{s'(h(x))}R_{h(x)}\circ T_x(s'_\alpha\circ h))(u)) + \omega'((T_e[\delta(f(s_i(x)),.)]\circ$$
$$\circ(h^{-1}_i.dh_{\alpha i}))(u)) =$$
$$=[R^*_{h(x)}\omega'](T_x(s'_\alpha\circ h)(u)) + \omega'([(h^{-1}_{\alpha i}.dh_{\alpha i})_x(u)]^*(f(s_i(x))) =$$
$$= Ad(h_{\alpha i}(x)^{-1})((h^*\omega'_\alpha)_x(u)) + (h^{-1}_{\alpha i}\cdot dh_{\alpha i})_x(u),$$

which achieves the first part of the proof. Recall that $[(h^{-1}_{\alpha i}\cdot dh_{\alpha i})_x(u)]^*$ is the fundamental vector field corresponding to the vector of g figuring between the brackets.

Conversely, assume that S and S' are (φ,h)-equivalent. Then, by Lemma 3, the corresponding principal bundles are (f,φ,h)-equivalent. Hence, it remains to show (1.4). This will be based on the construction of a connection (form) via the associate local connection forms (cf. e.g [5]): for every

$p\in\pi^{-1}(U_i)$, we have that

(2.2) $$\omega_p = Ad(g_i(p)^{-1})\circ(\pi^*\omega_i)_p + (g_i^{-1}.dg_i)_p ,$$

where $g_i:\pi^{-1}(U_i)\longrightarrow G$ are the smooth morphisms determined by

(2.3) $$p = s_i(\pi(p)).g_i(p);\ p\in\pi^{-1}(U_i).$$

The analogous expression for ω' is given by

(2.4) $$\omega'_{p'} = Ad(g'_\alpha(p')^{-1})\circ(\pi'^*\omega'_\alpha)_{p'} + (g'^{-1}_\alpha.dg'_\alpha)_{p'},$$

with $g'_\alpha:\pi'^{-1}(U_\alpha)\longrightarrow G'$ satisfying

(2.5) $$p' = s'_\alpha(\pi'(p')).g'_\alpha(p');\ p'=\pi'^{-1}(V_\alpha).$$

Thus, on $\pi^{-1}(U_i\cap h^{-1}(V_\alpha))$, setting $p'=f(p)$ in (2.5) and applying (2.3) along with Lemma 3, we obtain

(2.6) $$g'_\alpha\circ f = (h_{\alpha i}\circ\pi).(\varphi\circ g_i).$$

Taking the left total differential of (2.6), for any $p\in\pi^{-1}(U_i\cap h^{-1}(V_\alpha))$, we have that

(2.7) $$(g'^{-1}.dg')_{f(p)}\circ T_pf =$$
$$= Ad(\varphi(g_i(p))^{-1})\circ((h_{\alpha i}\circ\pi)^{-1}.d(h_{\alpha i}\circ\pi))_p + ((\varphi\circ g_i)^{-1}.d(\varphi\circ g_i))_p.$$

Hence, for every p as before, conditions (1.6) and (2.4) imply that

$$(f^*\omega')_p = Ad(g'_\alpha(f(p))^{-1})\circ(\pi'^*\omega_\alpha)_{f(p)}\circ T_pf + (g'^{-1}_\alpha.dg'_\alpha)_{f(p)}\circ T_pf=$$
$$= Ad(g'_\alpha(f(p))^{-1})\circ(h^*\omega_\alpha)_{\pi(p)}\circ T_p\pi + (g'^{-1}_\alpha.dg'_\alpha)_{f(p)}\circ T_pf =$$
$$= Ad(g'_\alpha(f(p))^{-1}\circ Ad(h_{\alpha i}(\pi(p)))\circ(\varphi_*\omega_i)_{\pi(p)}\circ T_p\pi -$$
$$- Ad(g'_\alpha(f(p))^{-1})\circ(dh_{\alpha i}.h_{\alpha i}^{-1})_{\pi(p)}\circ T_p\pi + (g'^{-1}_\alpha.dg'_\alpha)_{f(p)}\circ T_pf =$$
$$= a+b+c,$$

where a, b, c denote respectively the terms of the right hand side sum. Using (2.6) we check that

$$a = \varphi_*\circ Ad(g_i(p)^{-1})\circ(\pi^*\omega_i)_p,$$
$$b = Ad(\varphi(g_i(p))^{-1})\circ[(h_{\alpha i}\circ\pi)^{-1}.d(h_{\alpha i}\circ\pi)]_p.$$

Hence, in virtue of (2.7) and (2.2), we conclude that

$$(f^*\omega')_p = \varphi_*\circ Ad(g_i(p)^{-1})\circ(\pi^*\omega_i)_p + \varphi_*\circ(g_i^{-1}.dg_i)_p = \varphi_*\circ\omega_p$$

and the proof is complete. □

Concluding remarks

1) If $(B,\theta)=(B',\theta')$, $G=G'$ and $\varphi=id_G$, $h=id_B$, $\lambda=\lambda'$, under the additional assumption that G is an abelian group, the main Theorem reduces to the second part of ([7] Theorem 2).

Obviously, the previous assumptions simplify considerably the given proof.

2) Under the same circumstances as before, using standard techniques (cf. e.g. [3]), we can easily prove that the set of equivalence classes of (G,λ)-bundles over the same symplectic base (B,θ) is in a bijective correspondence with the 1-cohomology set derived from (equivalent) $(G,\lambda,\mathcal{U},\theta)$-systems.

3) In the finite-dimensional case one could choose as $\mathcal{U}$ a 1-connected covering of B (compare with the analogue of ([7], Theorem 2). However, since in infinite-dimensional manifolds the existence of special (good) coverings is not always assured, the described approach may be particularly useful.

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UNIVERSITY OF ATHENS, INSTITUTE OF MATHEMATICS,
PANEPISTIMIOPOLIS, ATHENS 157 84, GREECE

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