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ON UNIQUE COMMON FIXED POINT FOR COMPATIBLE MAPPINGS OF TYPE (A)

Let S and T be two self mappings of a metric space (X,d). Sessa [6] defines S and T to be weakly commuting if $d(STx,TSx) \leq d(Tx,Sx)$ for all x in X. Jungck [1] defines S and T to be compatible if $\lim_{n\to\infty} d(STx_n,TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implication is reversible (Ex. 1 [7] and Ex. 2.2 [1]).

Recently, Jungck, Murthy and Cho [2] defines S and T to be compatible of type (A) if $\lim_{n\to\infty} d(TSx_n,S^2x_n) = 0$ and $\lim_{n\to\infty} d(STx_n,T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in X$. Clearly, weakly commuting mappings are compatible of type (A). By (Ex. 2.2 [2]) follows that implications is not reversible. By (Ex. 2.1 and Ex. 2.2 [2]) follows that the notions of compatible mappings and compatible mappings of type (A) are independent.

LEMMA 1 [2]. *Let $S,T:(X,d)\to(X,d)$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible.*

LEMMA 2 [1]. *Let S and T be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Then $\lim_{n\to\infty} TSx_n = St$ if S is continuous.*

LEMMA 3 [2]. *Let $S,T:(X,d)\to(X,d)$ be mappings. If S and T are compatible of type (A) and $S(t)=T(t)$ for some $t\in X$, then $ST(t) = TT(t) = SS(t) = TS(t)$.*

Let R_+ be the set of all non-negative reals numbers and $f: R_+^3 \to R_+$ be a real function.

DEFINITION 1 [5]. We say that $f : R_+^3 \to R_+$ satisfies property (h) if there exists $h \geq 1$ such that for every $u, v \in R_+$ with $u \geq f(v,u,v)$ or $u \geq f(v,v,u)$, we have $u \geq h.v$.

DEFINITION 2 [3]. We say that $f : R_+^3 \to R_+$ satisfies property (u) if $f(u,0,0) > u$, $\forall\, u > 0$.

The following theorem is proved in [5].

THEOREM 1. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions:*

1° *A and B are surjective,*

2° *One of A, B, S, T is continuous,*

3° *A and S as well as B and T are compatible of type (A),*

4° *The inequality*

$$d(Ax,By) \geq f(d(Sx,Ty)), d(Ax,Sx), d(By,Ty)) \tag{1}$$

holds for all x, y in X where f satisfied the property (h) with $h \geq 1$. If the property (u) holds and f is continuous then A, B, S and T have a unique common fixed point.

The purpose of this paper is to extend Theorem 1.

Let $\mathcal{H}$ the set of real continuous functions $g(x_1, \ldots, x_5) : R_+^5 \to R_+$ satisfying the following conditions:

H_1: g is decreasing in variables x_4 and x_5,

H_2: there exists $h > 1$ such that for every $u, v \geq 0$ with

$$H_a : u \geq g(v,v,u,0,u+v) \quad \text{or} \quad H_b : u \geq g(v,u,v,u+v,0),$$

we have $u \geq h \cdot v$.

DEFINITION 3. We say that $g : R_+^5 \to R_+$ satisfies property (U) if $g(u,0,0,u,u) > u$, $\forall\, u > 0$.

Ex. $g(x_1, \ldots, x_5) = \left[ax_1^2 + \dfrac{bx_2^2 + cx_3^2}{x_4x_5 + 1}\right]^{1/2}$, where $a > 0$; $b, c \geq 0$ and $a + b + c > 1$.

H_1. Obvious.

H_2. If $u \geq g(v,v,u,0,u+v)$ then $u^2 \geq av^2 + bv^2 + cu^2$ and $u \geq \left(\frac{a+b}{1-c}\right)^{1/2} \cdot v = h_1 \cdot v$, where $h_1 > 1$.

If $u \geq g(v,u,v,u+v,0)$ then $u^2 \geq av^2 + bu^2 + cv^2$ and $u \geq \left(\frac{a+c}{1-b}\right)^{1/2} \cdot v = h_2 \cdot v$, where $h_2 > 1$. Thus H_2 holds for $h = \min\{h_1, h_2\}$.

THEOREM 2. *Let (X,d) be a metric space and A, B, S, T four mappings of X satisfying the inequality*

$$d(Ax,By) \geq g(d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Ax,Ty), d(By,Sx)) \tag{2}$$

for all x, y *in* X, *where* g *satisfies property* (U). *Then* A, B, S, T *have at most one common fixed point.*

P r o o f. Suppose that A, B, S, T have two common fixed points z and z', with $z \neq z'$. Then

$$\begin{aligned} d(z, z') &= d(Az, Bz') \\ &\geq g(d(Sz, Tz'), d(Az, Sz), d(Bz', Tz'), d(Az, Tz'), d(Bz', Sz)) \\ &= g(d(z, z'), 0, 0, d(z, z'), d(z', z)) > d(z, z'), \end{aligned}$$

a contradiction.

THEOREM 3. *Let* A, B, S *and* T *be mappings from a complete metric space* (X, d) *into itself satisfying the condition* 1°, 2° *and* 3° *of Theorem* 1. *If the inequality* (2) *holds for all* x, y *in* X, *where* $g \in \mathcal{H}$ *satisfies property* (U), *then* A, B, S *and* T *have a unique common fixed point.*

P r o o f. Let $x_0 \in X$ be arbitrary. By (1°) we can choose a point x_1 in X such that $Ax_1 = Tx_0 = y_0$ and for this point x_1 there exists a point x_2 in X such that $Bx_2 = Sx_1 = y_1$. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$(3) \qquad Ax_{2n+1} = Tx_n = y_{2n} \quad \text{and} \quad Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}.$$

By (2) and (3) we have

$$\begin{aligned} d(y_0, y_1) &= \\ &= d(Ax_1, Bx_2) \\ &\geq g(d(Sx_1, Tx_2), d(Sx_1, Tx_1), d(Tx_2, Bx_2), d(Ax_1, Tx_2), d(Bx_2, Sx_1)) \\ &= g(d(y_1, y_2), d(y_0, y_1), d(y_2, y_1), d(y_0, y_2), 0) \\ &\geq g(d(y_1, y_2), d(y_1, y_0), d(y_2, y_1), d(y_0, y_1) + d(y_1, y_2), 0). \end{aligned}$$

By H_b we have

$$d(y_0, y_1) \geq h \cdot d(y_2, y_1).$$

Thus

$$d(y_1, y_2) \leq \frac{1}{h} \cdot d(y_0, y_1).$$

Similarly, by (2), (3), (H_a) and (H_b) we have

$$d(y_n, y_{n+1}) \leq \left(\frac{1}{h}\right)^n \cdot d(y_0, y_1).$$

Then by a routine calculation one can show that $\{y_n\}$ is a Cauchy sequence and since X is complete, there is a $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Consequently, the subsequences $\{Ax_{2n+1}\}$, $\{Bx_{2n}\}$, $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ converge to z.

Now, suppose that A is continuous. Since A and S are compatible of type (A) and A is continuous, then, by Lemma 1., A and S are compatible. Lemma 2 implies that $A^2x_{2n+1} \to Az$ and $SAx_{2n+1} \to Az$ as $n \to \infty$.

By (2) we have

$$d(A^2_{2n+1}, Bx_{2n}) \geq g(d(SAx_{2n+1}, Tx_{2n}), d(SAx_{2n+1}, A^2x_{2n+1}), \\ d(Tx_{2n}, Bx_{2n}), d(A^2x_{2n+1}, Tx_{2n}), d(SAx_{2n+1}, Bx_{2n})).$$

Letting $n \to +\infty$ we have, by continuity of g, that

$$d(Az, z) \geq g(d(Az, z), 0, 0, Az, z), d(Az, z)).$$

By the property (U), it follows that $d(Az, Az) > d(Az, z)$ if $Az \neq z$. Thus $Az = z$. By (2) we have

$$d(Az, Bx_{2n}) \\ \geq g(d(Sz, Tx_{2n}), d(Sz, Az), d(Tx_{2n}, Bx_{2n}), d(Az, Tx_{2n}), d(Bx_{2n}, Sz)).$$

Letting $n \to +\infty$ we have, by continuity of g, that

$$0 = d(Az, z) \geq g(d(Sz, z), d(Sz, z), 0, 0, d(Sz, z)).$$

By (H_a) we have $0 \geq h \cdot d(Sz, z)$ which implies $z = Sz$. Let $u = By$ for some $u \in X$. Then by (2) we have

$$d(A^2x_{2n+1}, Bu) \geq g(d(SAx_{2n+1}, Tu), d(SAx_{2n+1}, A^2x_{2n+1}), \\ d(Tu, Bu), d(A^2x_{2n+1}, Tu), d(Ax_{2n+1}, Bu)).$$

Letting $n \to +\infty$ we have, by continuity of g, that

$$0 = d(Az, Bu) \geq (g(d(Az, Tu), 0, d(Tu, Bu), d(Az, Tu), d(Az, Bu)) \\ = g(d(z, Tu), 0, d(z, Tu), d(z, Tu), 0).$$

By (H_b) we have $0 \geq h \cdot d(z, Tu)$, which implies that $z = Tu$. Since B and T are compatible of type (A) and $Bu = Tu = z$, then by Lemma 3., $Bz = BTu = TBu = Tz$. Moreover, by (2), we have

$$d(Ax_{2n+1}, Bz) \geq g(d(Sx_{2n+1}, Tz), d(Sx_{2n+1}, Ax_{2n+1}), d(Tz, Bz), \\ d(Ax_{2n+1}, Tz), d(Bz, Sx_{2n+1})).$$

Letting $n \to +\infty$ we have, by continuity of g, that

$$d(z, Tz) \geq g(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)).$$

From the property (U), it follows that $d(z, Tz) > d(z, Tz)$ if $z \neq Tz$. Thus $z = Tz$. Therefore z is a common fixed point of A, B, S and T. Similarly, we can complete the proof in the case of continuity of B.

Next, suppose that S is continuous. Since A and S are compatible of type A and S is continuous, then, by Lemma 1, A and S are compatible.

Lemma 2 implies $S^2x_{2n+1} \to Sz$ and $ASx_{2n+1} \to Sz$ as $n \to \infty$. By (2), we have

$$d(ASx_{2n+1}, Bx_{2n} \geq g(d(S^2x_{2n+1}, Tx_{2n}), d(S^2x_{2n+1}, ASx_{2n+1}), \\ d(Tx_{2n}, Bx_{2n}), d(ASx_{2n+1}, Tx_{2n}), d(S^2x_{2n+1}, Bx_{2n})).$$

Letting $n \to +\infty$, we have, by continuity of g, that

$$d(Sz, z) \geq g(d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)).$$

By the property (U), we have $d(Sz, z) > d(Sz, z)$ if $z \neq Sz$. Thus $z = Sz$. Let $z = Av$ and $z = Bw$ for some v and w in X, respectively. Then, by (2), we have

$$d(ASx_{2n+1}, Bw) \geq g(d(S^2x_{2n+1}, Tw), d(S^2x_{2n+1}, ASx_{2n+1}), \\ d(Tw, Bw), d(ASx_{2n+1}, Tw), d(Bw, S^2x_{2n+1})).$$

Letting $n \to +\infty$, we have, by continuity of g, that

$$0 = d(Sz, z) \geq g(d(Sz, Tw), 0, d(Bw, Tw), d(Sz, Tw), d(Bw, Sz)) \\ = g(d(z, Tw), 0, d(z, Tw), d(z, Tw), 0).$$

By (H_b) we have $0 \geq h \cdot d(z, Tw)$, which implies that $z = Tw$. Since B and T are compatible of type (A) and $Bw = Tw = z$, by Lemma 3., we see that $Bz = BTw = TBw = Tz$.

Moreover, by (2), we have

$$d(Ax_{2n+1}, Bz) \geq g(d(Sx_{2n+1}, Tz), d(Sx_{2n+1}, Ax_{2n+1}), \\ d(Bz, Tz), d(Ax_{2n+1}, Tz), d(Bz, Sx_{2n+1})).$$

Letting $n \to +\infty$ we have, by continuity of g, that

$$d(z, Tz) = d(z, Bz) \geq g(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)).$$

By the property (U), it follows that $d(z, Tz) > d(z, Tz)$ if $z \neq Tz$. Thus $z = Tz$. Further, we have, by (2), that

$$d(Av, Bz) \geq g(d(Sv, Tz), d(Av, Sv), d(Tz, Bz), d(Av, Tz), d(Bz, Sv)).$$

and

$$0 = d(z, z) \geq g(d(Sv, z), d(Sv, z), 0, 0, d(Sv, z)).$$

By (H_a) we have $0 \geq h \cdot d(Sv, z)$ and thus $Sv = z$. Since A and S are compatible of type (A) and $Av = Sv = z$, then, by Lemma 3., $Az = ASv = SAv = Sz$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can complete the proof in the case of continuity of T.

From Theorem 2 follows that z is a unique common fixed point of A, B, S and T.

THEOREM 4. *Let S, T and $\{f_i\}_{i \in N}$ be mappings from a complete metric space (X, d) into itself satisfying the following conditions*:

1° $\{f_i\}_{i\in N}$ *are surjective;*

2° S *or* T *or every* $\{f_i\}_{i\in I}$ *is continuous;*

3° S *and* $\{f_i\}_{i\in N}$ *are compatible of type* (A), T *and* $\{f_i\}_{i\in N}$ *are compatible of type* (A),

4° *the inequality*

$$d(f_i x, f_{i+1} y) \geq g(d(Sx,Ty), d(f_i x, Sx), d(f_{i+1}y, Ty), d(f_i x, Ty), d(f_{i+1}y, Sx))$$

holds for all x *and* y *in* X, $\forall\ i \in N$, *where* $g \in \mathcal{H}$ *and satisfies the property* (U). *Then* $\{f_i\}_{i\in N}$, S *and* T *have a unique common fixed point.*

Proof. It is similar to the proof of ([4], Theorem 4.).

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