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FIELD OF COMPLEX LINEAR FRAMES ON REAL SPACE-TIME MANIFOLD AS DYNAMICAL VARIABLE FOR GENERALLY-COVARIANT MODELS

Abstract. A model of the self-interacting field of complex linear frames E on an n -dimensional real manifold is investigated. The model is generally-covariant and $GL(n, \mathbb{C})$ -invariant. If $n = 4$, the components of E can be interpreted as dynamical variables for the gravitational field. A Lagrangian of E is constructed, the Euler-Lagrange equations are derived and a wide class of their solutions is found. The solutions are built of left invariant vector fields on a real semisimple Lie group "deformed" by a complex factor of a natural exponential structure.

1. Introduction

In [9], [10] and [11], the author presented an alternative model of the gravitational field in which degrees of freedom are described by components of a field of real linear frames e on a "space-time" manifold M of dimension n (a tetrad field if $n = 4$). The group $GL(n, \mathbb{R})$ can act on e . In contrast to the Einstein theory (formulated in terms of the tetrad) or the general metric-teleparallel theories of gravitation ([4], [5], [6], [7]), which are invariant under the action of pseudo-Euclidean subgroups $SO(k, n - k)$ ($SO(1, 3)$ in the physical case), the model proposed by Ślawianowski is $GL(n, \mathbb{R})$ -invariant. A price one has to pay for this extension of the symmetry group is a very strong nonlinearity of the model. More precisely, the Lagrangian density of e has the form

$$(1.1) \quad L = \sqrt{|\det[t_{\mu\nu}]|},$$

where $t_{\mu\nu}$ are components of a two-fold covariant tensor field on M , built algebraically of e and its first derivatives. The square root structure of (1.1) resembles the Born-Infeld electrodynamics [8]. Hence it is called the modified Born-Infeld-type nonlinearity.

Key words and phrases: Euler-Lagrange equations; field of complex linear frames; semisimple Lie group; teleparallelism connection.

In our first paper concerning the field of linear frames [1], we extended the model suggested by Ślawianowski. We constructed a new family of Lagrangians of the self-interacting field e . We classified them and derived their general properties. We proved the existence of a wide class of solutions of the Euler-Lagrange equations for the Lagrangians belonging to the family we constructed. These solutions exist if the manifold M is (locally) diffeomorphic to $\mathbb{R} \times G$, where G is a semisimple Lie group. The field of real linear frames e satisfying the Euler-Lagrange equations can be then identified with a basis for the Lie algebra of left invariant vector fields on $\mathbb{R} \times G$ deformed by some factor of a natural exponential structure.

In [2], [3] one finds a further extension of Ślawianowski's theory. This is the model of the field of real linear frames e interacting with the complex scalar field or the multiplet of complex scalar fields. The latter may be interpreted as a quasiclassical model of matter.

After the self-interaction of the real field e and the mutual interaction of e with the complex scalar fields was discussed, the case where the real field e is replaced by the complex one seems to be the next step of our investigations. And so, in this paper we present a model of the self-interacting field of complex linear frames E on the real manifold M . The model is invariant under the action of $GL(n, \mathbb{C})$.

In Section 4 we modify definitions of basic geometric objects built of the real field e to adapt them to the complex field E . As in the case of the real field e , the Lagrangian $\mathcal{L}[E]$ of E is of the Born-Infeld-type (see (1.1)).

In Section 5 we derive the Euler-Lagrange equations from $\mathcal{L}[E]$. Equations written in generally-covariant form are linear with respect to components of some tensor density H of weight one and its covariant differential ∇H . For the real field e , ∇ is the covariant differentiation corresponding to the teleparallelism connection induced by e . Hence $\nabla H[e]$ is determined uniquely. In contrast to e , the complex field E induces two teleparallelism connections. Consequently, $\nabla H[E]$ can be defined in various ways. Thus we discuss conditions which determine uniquely the form of $\nabla H[E]$.

In Section 6 we construct a class of solutions of the Euler-Lagrange equations derived from $\mathcal{L}[E]$. It is an extension of the family of the solutions for the real field e we mentioned above. We also discuss a spatio-temporal metric being a byproduct of the solutions we constructed.

We finish with physical comments and a discussion of expected physical tests (Section 7).

In a sense, the model presented here is a complexification of tetrad approaches studied in [9], [10] and [11]. It is a well-known fact that complexification very often leads to interesting physical results (e.g. poles of the scattering amplitude analytically extended onto the complex energy plane give

energy levels of bounded states). We expect that complexified tetrad field-theoretic models can shed new light onto the theory of generally-relativistic spinors and the conformal $U(2, 2)$ -symmetry ([12], [13]).

2. Notations and preliminaries

Let M be an n -dimensional real differentiable manifold of class C^∞ .

Let $\mathcal{F}(p, \mathbb{C})$ be the set of all complex functions of class C^1 defined in a neighborhood of a point $p \in M$. The differentiability of $f \in \mathcal{F}(p, \mathbb{C})$ is understood as the differentiability in the real domain. For $X, Y \in T_p M$, the complex tangent vector $X + iY$ at p is a $\mathbb{C}$ -linear mapping $\mathcal{F}(p, \mathbb{C}) \rightarrow \mathbb{C}$ given by

$$(X + iY)f = Xf + iYf.$$

The complex tangent space $T_p(M, \mathbb{C})$ of M at p is the $\mathbb{C}$ -linear space of all $X + iY$. The complex conjugate of $X + iY$ is

$$(2.1) \quad \overline{X + iY} = X - iY.$$

The complex cotangent space $T_p^*(M, \mathbb{C})$ of M at p is the $\mathbb{C}$ -dual space of $T_p(M, \mathbb{C})$. Every $\omega \in T_p^*(M, \mathbb{C})$ can be uniquely written as

$$\omega = \alpha + i\beta,$$

where $\alpha, \beta \in T_p^* M$. The complex conjugate of ω is

$$(2.2) \quad \bar{\omega} = \alpha - i\beta.$$

An ordered basis of $T_p(M, \mathbb{C})$ (resp. $T_p M$) will be called the complex (resp. real) linear frame at p . The principal bundle of complex (resp. real) linear frame over M will be denoted by $F(M, \mathbb{C})$ (resp. $F(M)$). Similarly, by $F^*(M, \mathbb{C})$ and $F^*(M)$ we shall denote the principal bundles of complex and real linear coframes over M , respectively.

REMARK 2.1. Obviously, we shall regard $F(M, \mathbb{C})$ and $F^*(M, \mathbb{C})$ as real manifolds. Similarly, the structure group of the bundles (i.e., $GL(n, \mathbb{C})$) will be treated as a real Lie group.

3. Field of real linear frames

In this section we shall review basic concepts and results that one finds in [9, 10] and [1].

Let $e = (e_A) : M \rightarrow F(M)$, $A = 0, 1, \dots, n-1$, be a cross section of class C^∞ . In other words, e is a field of real linear frames on M . Let $\tilde{e} = (e^A) : M \rightarrow F^*(M)$ be the field of real linear coframes $\mathbb{R}$ -dual to e , i.e., $\langle e^A, e_B \rangle = \delta^A_B$.

REMARK 3.1. As a matter of fact, we assume that M is parallelizable, i.e., there exist smooth cross sections of $F(M)$ defined over M . At the end of this section and in Section 6, we shall discuss a case which satisfies this assumption.

The structure group of the bundles $F(M)$ and $F^*(M)$, i.e., $GL(n, \mathbb{R})$, acts on e and $\tilde{e}$ as follows:

$$\begin{aligned} e &= (e_A) \rightarrow eL = ((eL)_A) = (L^B_A e_B), \\ \tilde{e} &= (e^A) \rightarrow \tilde{e}L = ((\tilde{e}L)^A) = ((L^{-1})^A_B e^B) \end{aligned}$$

for $L = [L^A_B] \in GL(n, \mathbb{R})$, where L^{-1} denotes the inverse matrix of L .

We define a system of real functions $\gamma^A_{BC}[e]$, $A, B, C = 0, 1, \dots, n-1$, of class C^∞ on M by

$$(3.1) \quad [e_B, e_C] = \gamma^A_{BC}[e]e_A.$$

We call $\gamma^A_{BC}[e]$ the structural functions. They are skew-symmetric with respect to the lower indices:

$$(3.2) \quad \gamma^A_{BC}[e] = -\gamma^A_{CB}[e].$$

From $\gamma^A_{BC}[e]$ we build a tensor field $S[e]$ of type $(1, 2)$ and a symmetric covariant tensor field $\gamma[e]$ of degree 2:

$$(3.3) \quad S[e] = \frac{1}{2}\gamma^A_{BC}[e]e_A \otimes e^B \otimes e^C,$$

$$(3.4) \quad \gamma[e] = \gamma_{AB}[e]e^A \otimes e^B, \quad \gamma_{AB}[e] = \gamma^C_{AD}[e]\gamma^D_{BC}[e].$$

In terms of a local coordinate system (x^μ) , $\mu = 0, 1, \dots, n-1$, in a coordinate neighborhood $U \subset M$,

$$(3.5) \quad S[e]|U = S^\lambda_{\mu\nu}[e] \frac{\partial}{\partial x^\lambda} \otimes dx^\mu \otimes dx^\nu, \quad S^\lambda_{\mu\nu}[e] = \frac{1}{2}e_A^\lambda (e^A_{\mu,\nu} - e^A_{\nu,\mu}),$$

and

$$(3.6) \quad \gamma[e]|U = \gamma_{\mu\nu}[e]dx^\mu \otimes dx^\nu, \quad \gamma_{\mu\nu}[e] = 4S^\lambda_{\mu\rho}[e]S^\rho_{\nu\lambda}[e].$$

Here $(\cdot)_{,\nu}$ denotes partial differentiation with respect to x^ν .

Let us denote by $\text{Diff}M$ the group of all diffeomorphisms of M onto itself of class C^∞ . The fields $S[e]$ and $\gamma[e]$ are $\text{Diff}M$ -covariant (generally-covariant) and invariant under the action of $GL(n, \mathbb{R})$, i.e., for any $\varphi \in \text{Diff}M$ and $L \in GL(n, \mathbb{R})$,

$$S[\varphi_*e] = \tilde{\varphi}_*(S[e]), \quad S[eL] = S[e]$$

and

$$(3.7) \quad \gamma[\varphi_*e] = \tilde{\varphi}_*(\gamma[e]), \quad \gamma[eL] = \gamma[e],$$

where $\varphi_*e = ((\varphi_*e)_A) = (\varphi_*e_A)$ and $\tilde{\varphi}_*$ denotes the automorphism of the tensor algebra of all tensor fields on M induced by φ_* .

Since e is a cross section of the bundle $F(M)$, it induces a flat linear connection $\Gamma_{tel}[e]$ in $F(M)$, called the teleparallelism connection. This connection is uniquely determined by the condition

$$(3.8) \quad {}^e\nabla e_A = 0, \quad A = 0, 1, \dots, n-1,$$

where ${}^e\nabla$ denotes the covariant differentiation corresponding to $\Gamma_{tel}[e]$. The components of $\Gamma_{tel}[e]$ with respect to a local coordinate system (x^μ) have the form

$$(3.9) \quad \Gamma_{\mu\nu}^\lambda[e] = e_A^\lambda e_{\mu,\nu}^A.$$

Comparing (3.9) with (3.5) gives

$$(3.10) \quad S_{\mu\nu}^\lambda[e] = \frac{1}{2}(\Gamma_{\mu\nu}^\lambda[e] - \Gamma_{\nu\mu}^\lambda[e]).$$

This means that $S[e]$ is the torsion tensor field of $\Gamma_{tel}[e]$.

The Lagrangian $\mathcal{L}[e]$ of the self-interacting field e will be a differential n -form on M . We require the functional dependence $e \mapsto \mathcal{L}[e]$ to be of first-order and local. We also require $\mathcal{L}[e]$ to be $\text{Diff}M$ -covariant and invariant under the action of $GL(n, \mathbb{R})$:

$$(3.11) \quad \mathcal{L}[\varphi_*e] = \tilde{\varphi}_*(\mathcal{L}[e]), \quad \mathcal{L}[eL] = \mathcal{L}[e]$$

for $\varphi \in \text{Diff}M$ and $L \in GL(n, \mathbb{R})$.

Let us observe that $\gamma[e]$ is a pseudo-Riemannian metric on M , provided it is nondegenerate at each $p \in M$. The metric is quadratic with respect to the first derivatives of e . Hence, the simplest Lagrangian $\mathcal{L}[e]$ satisfying the assumptions made above is the differential n -form corresponding to the scalar density of weight one on M induced by $\gamma[e]$. More precisely, for a local coordinate system (x^μ) in a coordinate neighborhood $U \subset M$,

$$(3.12) \quad \mathcal{L}[e]|U = \sqrt{|\det[\gamma_{\mu\nu}[e]]|} dx^0 \wedge dx^1 \wedge \dots \wedge dx^{n-1}.$$

Transformation rules (3.11) follow immediately from (3.7).

In terms of a local coordinate system (x^μ) , the Euler-Lagrange equations (derived from the Lagrangian $\mathcal{L}[e]$) can be written as

$$(3.13) \quad {}^e\nabla_\nu H_\lambda^{\mu\nu}[e] + 2S^\rho_{\nu\lambda}[e] H_\lambda^{\mu\nu}[e] = 0, \quad \lambda, \mu = 0, 1, \dots, n-1,$$

where ${}^e\nabla$ denotes the covariant differentiation corresponding to $\Gamma_{tel}[e]$ and

$$(3.14) \quad H_\lambda^{\mu\nu}[e] = 2\sqrt{|\det[\gamma_{\alpha\beta}[e]]|}(\gamma^{\mu\rho}[e]S^\nu_{\rho\lambda}[e] - \gamma^{\nu\rho}[e]S^\mu_{\rho\lambda}[e])$$

are the components of a tensor density $H[e]$ of type $(2, 1)$ and weight one on M . $[\gamma^{\mu\nu}[e]]$ denotes the inverse matrix of $[\gamma_{\mu\nu}[e]]$.

It is convenient to turn system (3.13) into the form

$$(3.15) \quad e_C(H_A^{BC}[e]) + \gamma^D_{DC}[e]H_A^{BC}[e] = 0, \quad A, B = 0, 1, \dots, n-1,$$

where

$$(3.16) \quad H_A{}^{BC}[e] = \sqrt{|\det[\gamma_{EF}[e]]|}(\gamma^{BD}[e]\gamma^C{}_{DA}[e] - \gamma^{CD}[e]\gamma^B{}_{DA}[e])$$

are real functions of class C^∞ defined on M , provided $\gamma[e]$ is nondegenerate at each $p \in M$. $[\gamma^{AB}[e]]$ denotes the inverse matrix of $[\gamma_{AB}[e]]$. In contrast to (3.13), equations (3.15) have the global form, i.e., they are valid on the whole of M .

In [10] and [1], one finds a family of solutions of Euler-Lagrange equations (3.15). These solutions are constructed as follows. We assume that

$$M = \mathbb{R} \times G,$$

where G is an $(n-1)$ -dimensional real semisimple Lie group.

REMARK 3.2. Since there exist semisimple Lie groups of dimension 3, the assumption made above applies to the "physical case" where M is a space-time manifold of dimension 4.

Let $\mathcal{E}_0$ be a vector field on $\mathbb{R}$ defined by

$$\mathcal{E}_0 = \frac{\partial}{\partial \tau},$$

where τ is the natural coordinate in $\mathbb{R}$, i.e.,

$$\tau : \mathbb{R} \ni t \rightarrow \tau(t) = t \in \mathbb{R}.$$

Let $(\mathcal{E}_1, \dots, \mathcal{E}_{n-1})$ be a basis for the Lie algebra of left invariant vector fields on G . We identify $\tau, \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{n-1}$ with their natural lifts to $M = \mathbb{R} \times G$ and define a field of real linear frames $e = (e_A)$ on M by

$$(3.17) \quad e_A = \rho e^{\kappa \tau} \mathcal{E}_A, \quad A = 0, 1, \dots, n-1,$$

where ρ and κ are any real constants such that $\rho > 0$, $\kappa \neq 0$. The factor $e^{\kappa \tau}$ denotes the exponential function $\mathbb{R} \rightarrow \mathbb{R}$. By a straightforward calculation, we show that $e = (e_A)$ (given by (3.17)) is a solution of Euler-Lagrange equations (3.15) (see [1, pp. 95–97]).

4. Lagrangian of field of complex linear frames

Let $E = (E_A) : M \rightarrow F(M, \mathbb{C})$, $A = 0, 1, \dots, n-1$, be a cross section of class C^∞ (see Remarks 2.1 and 3.1). In other words, E is a field of complex linear frames on M . Let $\tilde{E} = (E^A) : M \rightarrow F^*(M, \mathbb{C})$ be the field of complex linear coframes $\mathbb{C}$ -dual to E , i.e., $\langle E^A, E_B \rangle = \delta^A_B$.

The structure group of the bundles $F(M, \mathbb{C})$ and $F^*(M, \mathbb{C})$, i.e., $GL(n, \mathbb{C})$, acts on E and $\tilde{E}$ as follows:

$$(4.1) \quad E = (E_A) \rightarrow EL = ((EL)_A) = (L^B{}_A E_B),$$

$$(4.2) \quad \tilde{E} = (E^A) \rightarrow \tilde{E}L = ((\tilde{E}L)^A) = ((L^{-1})^A{}_B E^B)$$

for $L = [L^A{}_B] \in GL(n, \mathbb{C})$.

To build the $GL(n, \mathbb{C})$ -invariant Lagrangian of the complex field $\mathbf{E}$, similar to that of the real field e , we have to replace the definition of the structural functions given by (3.1) with another one. Namely, we define a system of complex functions $\gamma^A_{B\bar{C}}[\mathbf{E}]$, $A, B, \bar{C} = 0, 1, \dots, n-1$, of class C^∞ on M by

$$(4.3) \quad [E_B, \overline{E_C}] = \gamma^A_{B\bar{C}}[\mathbf{E}] E_A, \quad C = \bar{C}.$$

Transformation rule (4.1) implies that

$$(4.4) \quad \gamma^A_{B\bar{C}}[\mathbf{E}L] = (L^{-1})^A_D L^F_B \overline{L^G_C} \gamma^D_{F\bar{G}}[\mathbf{E}]$$

for $L \in GL(n, \mathbb{C})$, where e.g. the value of G in $\overline{L^G_C}$ is equal to $\bar{G}$ in $\gamma^D_{F\bar{G}}[\mathbf{E}]$ for all values of $\bar{G}$.

From $\gamma^A_{B\bar{C}}[\mathbf{E}]$ we build a complex tensor field $S[\mathbf{E}]$ of type $(1, 2)$ on M

$$(4.5) \quad S[\mathbf{E}] = \frac{1}{2} \gamma^A_{B\bar{C}}[\mathbf{E}] E_A \otimes E^B \otimes \overline{E^C}$$

(see (2.2)). The field $S[\mathbf{E}]$ is $\text{Diff}M$ -covariant:

$$(4.6) \quad S[\varphi_* \mathbf{E}] = \tilde{\varphi}_*(S[\mathbf{E}]), \quad \varphi \in \text{Diff}M.$$

Combining (4.1), (4.2) and (4.4) yields

$$(4.7) \quad S[\mathbf{E}L] = S[\mathbf{E}] \quad \text{for } L \in GL(n, \mathbb{C}).$$

Thus $S[\mathbf{E}]$ is invariant under the action of $GL(n, \mathbb{C})$. In terms of a local coordinate system (x^μ) in a coordinate neighborhood $U \subset M$,

$$(4.8) \quad \begin{aligned} S[\mathbf{E}][U] &= S^\lambda_{\mu\nu}[\mathbf{E}] \frac{\partial}{\partial x^\lambda} \otimes dx^\mu \otimes dx^\nu, \\ S^\lambda_{\mu\nu}[\mathbf{E}] &= \frac{1}{2} (E_A^\lambda E^A_{\mu,\nu} - \overline{E_A^\lambda E^A_{\nu,\mu}}). \end{aligned}$$

The functions $\gamma^A_{BC}[e]$ and the components $S^\lambda_{\mu\nu}[e]$ of $S[e]$ are skew-symmetric with respect to the lower indices (see (3.2) and (3.5)), while the components $S^\lambda_{\mu\nu}[\mathbf{E}]$ are skew-hermitian with respect to the lower indices:

$$(4.9) \quad \overline{S^\lambda_{\mu\nu}[\mathbf{E}]} = -S^\lambda_{\nu\mu}[\mathbf{E}].$$

Since $\mathbf{E} = (E_A)$ is the field of complex linear frames, so is its complex conjugate $\overline{\mathbf{E}} = (\overline{E_A})$ (see (2.1)). Hence $\mathbf{E}$ induces two teleparallelism connections: $\Gamma_{tel}[\mathbf{E}]$ and $\Gamma_{tel}[\overline{\mathbf{E}}]$. These connections are uniquely determined by the conditions

$$(4.10) \quad {}^E\nabla E_A = 0 \quad \text{and} \quad \overline{E}\nabla(\overline{E_A}) = 0, \quad A = 0, 1, \dots, n-1,$$

where ${}^E\nabla$ (resp. $\overline{E}\nabla$) denotes the covariant differentiation corresponding to $\Gamma_{tel}[\mathbf{E}]$ (resp. $\Gamma_{tel}[\overline{\mathbf{E}}]$). The components of $\Gamma_{tel}[\mathbf{E}]$ and $\Gamma_{tel}[\overline{\mathbf{E}}]$ with respect to a local coordinate system (x^μ) have the form

$$(4.11) \quad \Gamma^\lambda_{\mu\nu}[\mathbf{E}] = E_A^\lambda E^A_{\mu,\nu}, \quad \Gamma^\lambda_{\mu\nu}[\overline{\mathbf{E}}] = \overline{\Gamma^\lambda_{\mu\nu}[\mathbf{E}]}.$$

Hence

$$(4.12) \quad S^\lambda_{\mu\nu}[\mathbf{E}] = \frac{1}{2}(\Gamma^\lambda_{\mu\nu}[\mathbf{E}] - \overline{\Gamma^\lambda_{\nu\mu}[\mathbf{E}]})$$

(cf. (3.9) and (3.10)).

We define a complex tensor field $\gamma[\mathbf{E}]$ of type $(0, 2)$ on M by

$$(4.13) \quad \begin{aligned} \gamma[\mathbf{E}] &= \gamma_{AB}[\mathbf{E}] \overline{E^A} \otimes E^B, \\ \gamma_{AB}[\mathbf{E}] &= \gamma^C_{AD}[\mathbf{E}] \gamma^D_{BC}[\mathbf{E}]. \end{aligned}$$

The field $\gamma[\mathbf{E}]$ is quadratic with respect to the first derivatives of $\mathbf{E}$. In terms of a local coordinate system (x^μ) in a coordinate neighborhood $U \subset M$,

$$(4.14) \quad \gamma[\mathbf{E}]|U = \gamma_{\mu\nu}[\mathbf{E}] dx^\mu \otimes dx^\nu, \quad \gamma_{\mu\nu}[\mathbf{E}] = 4S^\lambda_{\mu\rho}[\mathbf{E}] S^\rho_{\nu\lambda}[\mathbf{E}].$$

Hence, by virtue of (4.9),

$$(4.15) \quad \gamma_{\mu\nu}[\mathbf{E}] = -4S^\lambda_{\rho\mu}[\mathbf{E}] S^\rho_{\nu\lambda}[\mathbf{E}].$$

From (4.6), (4.7) and (4.15), it follows immediately that $\gamma[\mathbf{E}]$ is Diff M -covariant and

$$(4.16) \quad \gamma[\mathbf{E}L] = \gamma[\mathbf{E}] \quad \text{for } L \in GL(n, \mathbb{C}).$$

By virtue of (4.13) and (4.14), the matrices $[\gamma_{AB}[\mathbf{E}]]$ and $[\gamma_{\mu\nu}[\mathbf{E}]]$ are hermitian:

$$(4.17) \quad \overline{\gamma_{AB}[\mathbf{E}]} = \gamma_{BA}[\mathbf{E}], \quad \overline{\gamma_{\mu\nu}[\mathbf{E}]} = \gamma_{\nu\mu}[\mathbf{E}].$$

We expect the hermitian tensor field $\gamma[\mathbf{E}]$ to play the role that the real symmetric tensor field $\gamma[e]$ plays in the case of the field of real linear frames.

Let us denote by $\mathcal{L}[\mathbf{E}]$ a Lagrangian of the self-interacting field of complex linear frames $\mathbf{E}$. To build $\mathcal{L}[\mathbf{E}]$, we shall use the prescription that we applied to the construction of the Lagrangian $\mathcal{L}[e]$ for the real field e . Namely, $\mathcal{L}[\mathbf{E}]$ will be the real differential n -form on M corresponding to the scalar density of weight one induced by $\gamma[\mathbf{E}]$. In terms of a local coordinate system (x^μ) in a coordinate neighborhood $U \subset M$,

$$(4.18) \quad \mathcal{L}[\mathbf{E}]|U = \sqrt{|\det[\gamma_{\mu\nu}[\mathbf{E}]]|} dx^0 \wedge dx^1 \wedge \dots \wedge dx^{n-1}.$$

REMARK 4.1. Since the matrix $[\gamma_{\mu\nu}[\mathbf{E}]]$ is hermitian, the determinant in (4.18) is real. Thus the absolute value under the square root is taken in the sense of real numbers.

The Lagrangian $\mathcal{L}[\mathbf{E}]$ is of first-order and local. Transformation rules (4.6), (4.7) and formula (4.15) imply that

$$(4.19) \quad \mathcal{L}[\varphi_* E] = \tilde{\varphi}_*(\mathcal{L}[E]), \quad \mathcal{L}[EL] = \mathcal{L}[E]$$

for $\varphi \in \text{Diff } M$ and $L \in GL(n, \mathbb{C})$.

For the sake of some further applications we define an auxiliary system of complex functions $\xi^{\bar{A}}_{BC}[E]$, $\bar{A}, B, C = 0, 1, \dots, n-1$, on M by

$$(4.20) \quad [E_B, E_C] = \xi^{\bar{A}}_{BC}[E] \bar{E}_A.$$

From $\xi^{\bar{A}}_{BC}[E]$ we build a complex tensor field $K[E]$ of type (1,2) on M :

$$(4.21) \quad K[E] = \frac{1}{2} \xi^{\bar{A}}_{BC}[E] \bar{E}_A \otimes E^B \otimes E^C.$$

In terms of a local coordinate system (x^μ) in a coordinate neighborhood $U \subset M$,

$$(4.22) \quad \begin{aligned} K[E]|_U &= K^\lambda_{\mu\nu}[E] \frac{\partial}{\partial x^\lambda} \otimes dx^\mu \otimes dx^\nu, \\ K^\lambda_{\mu\nu}[E] &= \frac{1}{2} (\Gamma^\lambda_{\mu\nu}[E] - \Gamma^\lambda_{\nu\mu}[E]) \end{aligned}$$

(cf. (3.10) and (4.12)).

5. Euler-Lagrange equations for complex field E

From now on, we shall denote $\gamma^A_{B\bar{C}}[E]$, $\Gamma^\lambda_{\mu\nu}[E]$, $\gamma_{\mu\nu}[E]$, ... by $\gamma^A_{B\bar{C}}$, $\Gamma^\lambda_{\mu\nu}$, $\gamma_{\mu\nu}$, ... unless otherwise stated.

Let us fix a coordinate neighborhood $U \subset M$ with a local coordinate system (x^μ) and put

$$(5.1) \quad \det \gamma = \det[\gamma_{\mu\nu}].$$

We take the complex one-forms E^A as the independent dynamical variables. Treating the Lagrangian density $\sqrt{|\det \gamma|}$ as the function of the $n^2 + n^3$ complex arguments E^A_α and $E^A_{\alpha\beta}$, we set

$$\begin{aligned} H_A^{\alpha\beta} &= \frac{\partial \sqrt{|\det \gamma|}}{\partial E^A_{\alpha\beta}}, & \mathcal{H}^{\alpha\beta}_{\bar{A}} &= \frac{\partial \sqrt{|\det \gamma|}}{\partial (\bar{E}^A_{\alpha})_{\beta}}, \\ J_A^\alpha &= -\frac{\partial \sqrt{|\det \gamma|}}{\partial E^A_\alpha}, & \mathcal{J}^\alpha_{\bar{A}} &= -\frac{\partial \sqrt{|\det \gamma|}}{\partial \bar{E}^A_{\alpha}}, \\ A, \bar{A}, \alpha, \beta &= 0, 1, \dots, n-1. \end{aligned}$$

The Euler-Lagrange equations derived from the Lagrangian $\mathcal{L}[E]$ are then given by

$$(5.2) \quad (H_A^{\alpha\beta})_{,\beta} + J_A^\alpha = 0, \quad A, \alpha = 0, 1, \dots, n-1,$$

$$(5.3) \quad (\mathcal{H}^{\alpha\beta}_{\bar{A}})_{,\beta} + \mathcal{J}^\alpha_{\bar{A}} = 0, \quad \bar{A}, \alpha = 0, 1, \dots, n-1.$$

A straightforward calculation leads to

$$(5.4) \quad J_A^\alpha = \sqrt{|\det \gamma|} E_A^\lambda \gamma^{\mu\nu} (\Gamma^\alpha_{\mu\rho} \bar{S}^\rho_{\nu\lambda} - \Gamma^\alpha_{\rho\nu} S^\rho_{\mu\lambda}),$$

$$(5.5) \quad H_A^{\alpha\beta} = E_A^\lambda H_\lambda^{\alpha\beta},$$

where

$$(5.6) \quad H_\lambda^{\alpha\beta} = \sqrt{|\det \gamma|} (\gamma^{\alpha\mu} \overline{S^\beta_{\mu\lambda}} - \gamma^{\mu\beta} S^\alpha_{\mu\lambda})$$

and $[\gamma^{\mu\nu}]$ denotes the inverse matrix of $[\gamma_{\mu\nu}]$. Calculating $\mathcal{H}^{\alpha\beta}_{\overline{A}}$ and $\mathcal{J}^{\alpha}_{\overline{A}}$ one finds that subsystem (5.3) of the Euler-Lagrange equations is equivalent to (5.2). Hence, it is sufficient to consider the system consisting of equations (5.2).

Let us observe that $H_\lambda^{\alpha\beta}$ are the components of a complex tensor density of type (2,1) and weight one on M . We shall denote it by $H[E]$.

Let us put

$$(5.7) \quad \text{Det} \gamma = \det[\gamma_{\overline{A}B}].$$

The Lagrangian density $\sqrt{|\det \gamma|}$ and $\sqrt{|\text{Det} \gamma|}$ are related by

$$(5.8) \quad \sqrt{|\det \gamma|} = \sqrt{|\text{Det} \gamma|} \sqrt{\det[E^F_\kappa] \det[E^G_\sigma]}.$$

REMARK 5.1. Since the matrix $[\gamma_{\overline{A}B}]$ is hermitian, the absolute value under the square root on the right-hand side of (5.8) is taken in the sense of real numbers.

Equation (5.8) and the fact that $H_\lambda^{\alpha\beta}$ are the components of the tensor density of weight one imply that the formula

$$(5.9) \quad H_\lambda^{\alpha\beta} = \sqrt{\det[E^F_\kappa] \det[E^G_\sigma]} E^A_\lambda E_B^\alpha \overline{E_C^\beta} H_A^{B\overline{C}}$$

determines uniquely a system of complex functions $H_A^{B\overline{C}}$, $A, B, \overline{C} = 0, 1, \dots, n-1$, defined on the whole of M . In fact, a straightforward calculation turns (5.9) into the form

$$(5.10) \quad H_A^{B\overline{C}} = \frac{1}{2} \sqrt{|\text{Det} \gamma|} (\gamma^{B\overline{F}} \overline{\gamma^C_{F\overline{A}}} - \langle \overline{E^G}, E_A \rangle \gamma^{F\overline{C}} \gamma^B_{F\overline{G}}),$$

where $[\gamma^{A\overline{B}}]$ is the inverse matrix of $[\gamma_{\overline{A}B}]$ and $\langle \overline{E^G}, E_A \rangle$ denotes the value of the one-form $\overline{E^G}$ on the vector field E_A . Hence $H_A^{B\overline{C}}$ are the complex functions of class C^∞ defined on M , provided $\text{Det} \gamma \neq 0$ for each $p \in M$.

To turn Euler-Lagrange equations (5.2) into a tensorial (generally-covariant) form, we need a covariant differential $\nabla H[E]$ of $H[E]$. Since E induces the two teleparallelism connections $(\Gamma_{tel}[E])$ and $(\Gamma_{tel}[\overline{E}])$, one can define $\nabla H[E]$ in various ways, e.g.

$$(5.11) \quad \begin{aligned} {}^E \nabla_\rho H_\lambda^{\alpha\beta} &= H_\lambda^{\alpha\beta}_{,\rho} - \Gamma^\omega_{\omega\rho} H_\lambda^{\alpha\beta} - \Gamma^\omega_{\lambda\rho} H_\omega^{\alpha\beta} + \Gamma^\alpha_{\omega\rho} H_\lambda^{\omega\beta} + \Gamma^\beta_{\omega\rho} H_\lambda^{\alpha\omega} \end{aligned}$$

or

$$(5.12) \quad \overline{E} \nabla_\rho H_\lambda^{\alpha\beta} = H_\lambda^{\alpha\beta}_{,\rho} - \overline{\Gamma^\omega_{\omega\rho}} H_\lambda^{\alpha\beta} - \overline{\Gamma^\omega_{\lambda\rho}} H_\omega^{\alpha\beta} + \dots$$

(see (4.10) and (4.11)). If one replaces (5.11) or (5.12) with a formula where $\Gamma^\omega_{\lambda\rho}$ and $\overline{\Gamma^\omega_{\lambda\rho}}$ occur simultaneously, e.g.

$$\nabla_\rho H_\lambda^{\alpha\beta} = H_{\lambda,\rho}^{\alpha\beta} - \overline{\Gamma^\omega_{\omega\rho}} H_\lambda^{\alpha\beta} - \Gamma^\omega_{\lambda\rho} H_\omega^{\alpha\beta} + \overline{\Gamma^\alpha_{\omega\rho}} H_\lambda^{\omega\beta} + \Gamma^\beta_{\omega\rho} H_\lambda^{\alpha\omega},$$

one obtains the expression, which also defines $\nabla H[E]$. To choose the "appropriate" covariant differential $\nabla H[E]$ of $H[E]$, we introduce an additional assumption. Namely, we require the Euler-Lagrange equations written in the global form (i.e., a form valid on the whole of M) to be built from the functions $H_A^{B\overline{C}}$ and their derivatives $E_D(H_A^{B\overline{C}})$ or $\overline{E_D}(H_A^{B\overline{C}})$ (cf. (3.15)). The Euler-Lagrange equations written in tensorial form will be built from the components $H_\lambda^{\alpha\beta}$ and $\nabla_\rho H_\lambda^{\alpha\beta}$ of $H[E]$ and $\nabla H[E]$ (cf. (3.13)). Obviously, they will be equivalent to the global ones. Hence transformation rule (5.9), connecting $H_A^{B\overline{C}}$ with $H_\lambda^{\alpha\beta}$, and that connecting $\frac{\partial}{\partial x^\rho}(H_A^{B\overline{C}})$ with $\nabla_\rho H_\lambda^{\alpha\beta}$ have to coincide. This means that

$$(5.13) \quad \nabla_\rho H_\lambda^{\alpha\beta} = \sqrt{\det[E^F_\kappa] \det[E^G_\sigma]} E^A_\lambda E_B^\alpha \overline{E_C}^\beta \frac{\partial}{\partial x^\rho}(H_A^{B\overline{C}}).$$

Condition (5.13) determines uniquely $\nabla H[E]$. In fact, differentiating (5.8) with respect to x^ρ and using (4.11) yields

$$(5.14) \quad \begin{aligned} \frac{\partial}{\partial x^\rho}(\sqrt{|\det \gamma|}) - \frac{1}{2}(\Gamma^\omega_{\omega\rho} + \overline{\Gamma^\omega_{\omega\rho}})\sqrt{|\det \gamma|} = \\ = \sqrt{\det[E^F_\kappa] \det[E^G_\sigma]} \frac{\partial}{\partial x^\rho}(\sqrt{|\det \gamma|}). \end{aligned}$$

Using the last equation, (5.9) and (4.11) one finds that $\nabla H[E]$ whose components are given by

$$(5.15) \quad \begin{aligned} \nabla_\rho H_\lambda^{\alpha\beta} = H_{\lambda,\rho}^{\alpha\beta} \\ - \frac{1}{2}(\Gamma^\omega_{\omega\rho} + \overline{\Gamma^\omega_{\omega\rho}})H_\lambda^{\alpha\beta} - \Gamma^\omega_{\lambda\rho}H_\omega^{\alpha\beta} + \Gamma^\alpha_{\omega\rho}H_\lambda^{\omega\beta} + \overline{\Gamma^\beta_{\omega\rho}}H_\lambda^{\alpha\omega} \end{aligned}$$

is the only covariant differential of $H[E]$ satisfying (5.13).

Substituting (5.4) and (5.5) into (5.2) and using (5.15), (5.6), (4.12) and (4.22) leads to the Euler-Lagrange equations written in tensorial form:

$$(5.16) \quad \nabla_\beta H_\lambda^{\alpha\beta} + S^\omega_{\omega\beta} H_\lambda^{\alpha\beta} + \overline{K^\omega_{\omega\beta}} H_\lambda^{\alpha\beta} = 0, \quad \alpha, \lambda = 0, 1, \dots, n-1.$$

Substituting (5.13) and (5.9) into (5.16) and using (4.20) turns (5.16) into a system of equations, which is valid on the whole of M :

$$(5.17) \quad \overline{E_C}(H_A^{B\overline{C}}) + \gamma^D_{D\overline{C}} H_A^{B\overline{C}} - \frac{1}{2}(\overline{E^F}, E_D) H_A^{B\overline{C}} \overline{E_C}(\langle E^D, \overline{E_F} \rangle) = 0, \\ A, B = 0, 1, \dots, n-1.$$

REMARK 5.2. If the field $E = (E_A)$ is real, then the functions $H_A^{B\overline{C}}$ defined by (5.10) become the functions H_A^{BC} given by (3.16) with $C = \overline{C}$

and equations (5.17) become equations (3.15). Hence the model of the self-interacting complex field E we present seems to be the natural extension of the theory of the self-interacting real field e described in [9, 10] or [1].

6. Solutions Euler-Lagrange equations for complex field E

In this section we shall construct a family of solutions of Euler-Lagrange equations (5.17) that are an extension of the family given by (3.17).

We shall use the following notation: Greek capital indices (i.e., $\bar{\Lambda}, \Sigma, \dots$) range from 1 to $n-1$, whereas Latin ones (i.e., $\bar{A}, B, \dots$) run from 0 to $n-1$.

We assume that

$$M = \mathbb{R} \times G,$$

where G is an $(n-1)$ -dimensional real semisimple Lie group (see Remark 3.2). We define a vector field $\mathcal{E}_0$ on $\mathbb{R}$ by

$$(6.1) \quad \mathcal{E}_0 = \frac{\partial}{\partial \tau},$$

where τ is the natural coordinate in $\mathbb{R}$, i.e.,

$$\tau : \mathbb{R} \ni t \rightarrow \tau(t) = t \in \mathbb{R}.$$

Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G and $(\mathcal{E}_\Lambda), \Lambda = 1, \dots, n-1$, any basis for $\mathfrak{g}$. We have

$$(6.2) \quad [\mathcal{E}_\Lambda, \mathcal{E}_\Sigma] = C^\Delta_{\Lambda\Sigma} \mathcal{E}_\Delta,$$

where $C^\Delta_{\Lambda\Sigma}$ are the structure constants of $\mathfrak{g}$. The components of the Killing-Cartan form of $\mathfrak{g}$ with respect to the basis $(\mathcal{E}_\Lambda)$ are given by

$$(6.3) \quad C_{\Lambda\Sigma} = C^\Delta_{\Lambda\Pi} C^\Pi_{\Sigma\Delta}.$$

Since $\mathfrak{g}$ is semisimple,

$$(6.4) \quad \det[C_{\Lambda\Sigma}] \neq 0,$$

$$(6.5) \quad C^\Lambda_{\Lambda\Sigma} = 0.$$

The structure constants and the components of the Killing-Cartan form are real. Hence they can be interchangeably numbered by unbarred and barred indices: $C^\Delta_{\Lambda\Sigma}$, $C^\Delta_{\Lambda\bar{\Sigma}}$, $C^\Delta_{\bar{\Lambda}\Sigma}$, $C_{\Lambda\Sigma}$, $C_{\bar{\Lambda}\Sigma}$, etc.

We shall identify $\tau, \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{n-1}$ with their natural lifts to $M = \mathbb{R} \times G$.

THEOREM 6.1. *Let*

$$\lambda : M \rightarrow \mathbb{C}$$

be given by

$$(6.6) \quad \begin{aligned} \lambda(t, g) &= \rho \exp((\kappa + i\omega)t) \quad \text{for } (t, g) \in \mathbb{R} \times G = M, \text{ i.e.,} \\ \lambda &= \rho \exp((\kappa + i\omega)\tau), \end{aligned}$$

where $\exp$ denotes the exponential function $\mathbb{C} \rightarrow \mathbb{C}$ and ρ, κ and ω are any real constants such that

$$(6.7) \quad \kappa^2 + \omega^2 > 0 \quad \text{and} \quad \rho > 0.$$

Define a field of complex linear frames $E = (E_A)$ on M by

$$(6.8) \quad E_A = \lambda \mathcal{E}_A, \quad A = 0, 1, \dots, n-1.$$

Then $E = (E_A)$ is a solution of Euler-Lagrange equations (5.17).

Proof. From the definition of the product of two manifolds, it follows that

$$(6.9) \quad [\mathcal{E}_0, \mathcal{E}_\Lambda] = 0, \quad \mathcal{E}_\Lambda \lambda = \mathcal{E}_\Lambda \bar{\lambda} = 0.$$

By virtue of (6.6),

$$(6.10) \quad \mathcal{E}_0 \lambda = (\kappa + i\omega)\lambda, \quad \mathcal{E}_0 \bar{\lambda} = (\kappa - i\omega)\bar{\lambda}.$$

Substituting (6.8) into (4.3) and using (6.2), (6.9) and (6.10) yields

$$(6.11) \quad \begin{aligned} \gamma^0_{0\bar{0}} &= -2i\omega\bar{\lambda}, & \gamma^{\Delta}_{0\bar{0}} &= 0, & \gamma^A_{0\bar{\Lambda}} &= (\kappa - i\omega)\bar{\lambda}\delta^A_{\bar{\Lambda}}, \\ \gamma^A_{\Lambda\bar{0}} &= -(\kappa + i\omega)\bar{\lambda}\delta^A_{\Lambda}, & \gamma^0_{\Lambda\bar{\Sigma}} &= 0, & \gamma^{\Delta}_{\Lambda\bar{\Sigma}} &= \bar{\lambda}C^{\Delta}_{\Lambda\bar{\Sigma}}, \end{aligned}$$

where $\gamma^A_{B\bar{C}} = \gamma^A_{B\bar{C}}[E]$.

The components $\gamma_{AB}[E]$ of $\gamma[E]$ (defined by (4.13)) are then given by

$$(6.12) \quad \begin{aligned} \gamma_{00} &= (4\omega^2 + (n-1)(\kappa^2 + \omega^2))\lambda\bar{\lambda}, \\ \gamma_{0\Lambda} &= 0 = \gamma_{\Lambda 0} \quad \text{and} \quad \gamma_{\Lambda\Sigma} = \lambda\bar{\lambda}C_{\Lambda\Sigma}. \end{aligned}$$

Hence

$$(6.13) \quad \text{Det} \gamma = (4\omega^2 + (n-1)(\kappa^2 + \omega^2)) \det[C_{\Lambda\Sigma}](\lambda\bar{\lambda})^n$$

(see (5.7)). Equation (6.6) and conditions (6.4) and (6.7) imply that

$$\text{Det} \gamma \neq 0 \quad \text{for each } p \in M.$$

Thus the components $\gamma^{A\bar{B}}[E]$ of the inverse matrix of $[\gamma_{AB}[E]]$ are the functions defined on the whole of M . By virtue of (6.12),

$$(6.14) \quad \begin{aligned} \gamma^{0\bar{0}} &= \frac{1}{(4\omega^2 + (n-1)(\kappa^2 + \omega^2))\lambda\bar{\lambda}}, \\ \gamma^{0\bar{\Lambda}} &= 0 = \gamma^{\Lambda\bar{0}}, & \gamma^{\Lambda\bar{\Sigma}} &= \frac{1}{\lambda\bar{\lambda}}C^{\Lambda\bar{\Sigma}}, \end{aligned}$$

where $[C^{\Lambda\bar{\Sigma}}]$ denotes the inverse matrix of $[C_{\Lambda\Sigma}]$. We have

$$(6.15) \quad \langle \overline{E^F}, E_D \rangle = \delta^{\bar{F}}_D \frac{\lambda}{\bar{\lambda}}.$$

After using second formula (6.9) and formulae (6.11), (6.5), (6.15) and (6.10), the system of equations (5.17) becomes

$$(6.16) \quad \mathcal{E}_0(H_A^{B\bar{0}}) - ((n-1)\kappa + i\omega)H_A^{B\bar{0}} = 0, \quad \bar{0} = 0, \\ A, B = 0, 1, \dots, n-1.$$

Substituting (6.13), (6.14), (6.11) and (6.15) into (5.10), we calculate $H_A^{B\bar{0}}$. Using the expression for $H_A^{B\bar{0}}$, we obtain, turns (6.16) into the one ordinary differential equation

$$(6.17) \quad \mathcal{E}_0 \left((\lambda\bar{\lambda})^{\frac{n}{2}} \frac{1}{\lambda} \right) - ((n-1)\kappa + i\omega)(\lambda\bar{\lambda})^{\frac{n}{2}} \frac{1}{\lambda} = 0.$$

By a simple calculation, we verify that λ defined by (6.6) satisfies identically equation (6.17). ■

REMARK 6.1. If we set $\omega = 0$ in (6.6), then E_A (given by (6.8)) become the real vector fields e_A defined by (3.17). Hence $\mathbf{E} = (E_A)$ becomes the solution of Euler-Lagrange equations (3.15) for the field of real linear frames.

THEOREM 6.2. Let $\lambda : M \rightarrow \mathbb{C}$ be given by (6.6), where ρ, κ and ω satisfy (6.7). Let

$$[L^B_A] \in GL(n, \mathbb{C}).$$

Define a field of complex linear frames $\mathbf{E} = (E_A)$ on M by

$$(6.18) \quad E_A = \lambda L^B_A \mathcal{E}_B, \quad A = 0, 1, \dots, n-1.$$

Then $\mathbf{E} = (E_A)$ is a solution of Euler-Lagrange equations (5.17).

Proof. Our theorem follows immediately from Theorem 6.1 and second transformation rule (4.19). ■

REMARK 6.2. The family of solutions of the Euler-Lagrange equations corresponding to Theorem 6.2 (formula (6.18)) is an extension of that given by (6.8) and (6.6) (Theorem 6.1). Thus it is sufficient to discuss the extended family (Theorem 6.2).

The field $\gamma[\mathbf{E}]$, i.e., the mathematical quantity applied to the construction of the Lagrangian $\mathcal{L}[\mathbf{E}]$ (see (4.18)), can be interpreted in physical terms if $\mathbf{E}$ is a solution of the form (6.18) (Theorem 6.2). Namely, for these solutions, the hermitian matrix $[\gamma_{\mu\nu}[\mathbf{E}]]$ of the components of $\gamma[\mathbf{E}]$ (see (4.14)) is real and nondegenerate at each $p \in M$. (It follows from transformation rule (4.16) and from the fact that the non-singular matrix $[\gamma_{AB}[\mathbf{E}]]$ - formulae (6.12) and (6.13) - and the vector fields $\mathcal{E}_A$ - equation (6.18) - are real.) This means that $\gamma[\mathbf{E}]$ is a pseudo-Riemannian metric. Moreover, in the physical case where M is a space-time manifold of dimension $n = 4$, the signature of $\gamma[\mathbf{E}]$, for every solution $\mathbf{E}$ corresponding to Theorem 6.2, is normal-hyperbolic: $(+ - - -)$ or $(+ + + -)$. And so, $\gamma[\mathbf{E}]$ can be interpreted as a spatio-temporal metric. In contrast to the tetrad formulation of

the Einstein gravitation, where the normal-hyperbolic signature of the metric is the "absolute" parameter introduced into the theory, for our model, this signature is an intrinsic feature of the very solutions presented above, i.e., it results simply from their shape. Indeed, by virtue of (6.12), the signature of $\gamma[E]$ is $(+ \times \dots \times)$, where $(\times \dots \times)$ denotes the signature of the Killing-Cartan form of an $(n-1)$ -dimensional real semisimple Lie algebra $\mathfrak{g}$. If $n=4$, $\mathfrak{g}$ is isomorphic to $su(2)$ or $sl(2, \mathbb{R})$ and so $(\times \dots \times)$ is $(- - -)$ or $(+ + -)$, respectively. Hence $\gamma[E]$ is normal-hyperbolic.

7. Some physical comments. Motivation and expected physical tests

The above treatment was rather formal and concentrated about the mathematical structures. Relatively few physical comments were quoted. Nevertheless, our motivation was rather physical; likewise traditional formulations of alternative gravitational theories based on tetrad fields (more generally n -leg fields) were inspired by ideas of rather physical nature. First of all (it was the Einstein motivation in his teleparallelism studies) the fact that one works locally in pseudo-orthonormal frames implies that some links with specially-relativistic concepts are more visible. In generalized metric - teleparallelism models ([6], [7]) some unpleasant singularities seem to be avoidable. Tetrads are necessary for the very description of spinors in curved manifolds. They are also used in the $SL(2, \mathbb{C})$ -gauge theories of gravitation [4]. The models [9, 10, 11, 1, 2, 3] with the internal $GL(4, \mathbb{R})$ -symmetry ($GL(n, \mathbb{R})$ -symmetry) are based on the very essential generalized Born-Infeld-type nonlinearity. They also possess some interesting aspects concerning cosmological problems and they may shed new light onto the cosmological expansion and the Universe evolution [10]. Their predictions are, at least principally, testifiable.

Our model of the complexified tetrad (more generally n -leg) field is interesting from this point of view. For example, the solutions of the field equations we found have an oscillatory nature (equations (6.6) and (6.8) for $\omega \neq 0$). Therefore, we expect that some new possibilities of the oscillatory cosmological scenario may be derivable from the model and, at least in principle, verifiable on the basis of astrophysical observation data.

Among other things, our expectations are motivated by the fact that a correspondence between the model we present and the Einstein theory may be investigated. Namely, within our framework, the concept of spatio-temporal metric occurs, however, not as the primary notion, but as a byproduct of the complexified tetrad field. There are some indications that our approach is convergent with the Einstein theory. For example, in the model we present, the normal-hyperbolic signature of the metric is not the ab-

solute parameter, but it is a consequence of the field equations (see Remark 6.2).

There is also a microscopic motivation of our approach, concerning the coupling between quantum and gravitational phenomena. By the way, this two-scale coupling is rather typical for modern studies on elementary particles and alternative theories of gravitation. Namely, there is an essential difficulty in the standard approach to spinor structures. The point is that, unlike the tensor bundles describing bosonic fields, the spinor bundles, one uses in commonly accepted theories, are non-canonical, so to speak, "non-soldered" to the space-time manifold. The complexified bundle of linear frames (denoted in our paper by $F(M, \mathbb{C})$) is natural and provides a proper framework for discussing, e.g. $U(2, 2)$ -based models ([12], [13]), where the gauge group underlying gravitation is the covering group of the Lorentz-conformal group. The corresponding description of $U(2, 2)$ -ruled spinors may be a proper way to overcome the problem of the lack of natural spinor bundles in standard $SL(2, \mathbb{C})$ -based models. This may change our views concerning the microphysical fields of fundamental fermions ([12], [13]). To be complete, such an analysis has to start from a systematically developed framework of the complexified principal bundle $F(M, \mathbb{C})$ of linear frames (tetrads) with $GL(n, \mathbb{C})$ ($GL(4, \mathbb{C})$) as the structure group which is the subject of our paper. There are also some indications that the $GL(4, \mathbb{C})$ -gauge models of gravitation may lead to results convergent with that described by the more traditional approaches and, at the same time, they may provide possibilities of predicting new phenomena in both the large scale gravitational sector and the realm of microphysics. This could also contribute to attempts of the gravity quantization.

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