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STRONG MAXIMUM PRINCIPLES FOR IMPLICIT PARABOLIC FUNCTIONAL—DIFFERENTIAL PROBLEMS TOGETHER WITH NONSTANDARD INEQUALITIES WITH SUMS

Abstract. The aim of the paper is to give strong maximum principles for implicit parabolic functional - differential problems together with nonstandard inequalities with sums in relatively arbitrary $(n + 1)$ —dimensional time—space sets more general than the cylindrical domain. The results obtained can be applied in the theory of diffusion and in the theory of heat conduction.

1. Introduction

In this paper we consider implicit diagonal systems of non-linear parabolic functional—differential inequalities of the form

$$(1.1) \quad F^i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), u_{xx}^i(t, x), u) \\ \geq F^i(t, x, v(t, x), v_t^i(t, x), v_x^i(t, x), v_{xx}^i(t, x), v) \quad (i = 1, \dots, m)$$

for $(t, x) = (t, x_1, \dots, x_n) \in D$, where $D \subset (t_0, t_0 + T] \times \mathbb{R}^n$ is one of five relatively arbitrary sets more general than the cylindrical domain $(t_0, t_0 + T] \times D_0 \subset \mathbb{R}^{n+1}$. The symbol w ($= u$ or v) denotes the mapping

$$w : \tilde{D} \ni (t, x) \longrightarrow w(t, x) = (w^1(t, x), \dots, w^m(t, x)) \in \mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty, t_0 + T] \times \mathbb{R}^n$ such that $\bar{D} \subset \tilde{D}$; F^i ($i = 1, \dots, m$) are functionals of w ; $w_x^i(t, x) = \text{grad}_x w^i(t, x)$ ($i = 1, \dots, m$) and $w_{xx}^i(t, x)$ ($i = 1, \dots, m$) denote the matrices of second order derivatives with respect to x of $w^i(t, x)$ ($i = 1, \dots, m$). We give a theorem on strong maximum principles for problems with inequalities of types

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(1.1) and with the nonstandard inequalities

$$[u^j(t_0, x) - K^j] + \sum_{i \in I_*} h_i(x)[u^j(T_i, x) - K^j] \leq 0 \text{ for } x \in S_{t_0} \quad (j = 1, \dots, m),$$

where $K = (K^1, \dots, K^m)$ is a constant function, I_* is a subset of a countable set I of natural indices, $t_0 < T_i \leq t_0 + T$ ($i \in I$), $h_i : S_{t_0} \rightarrow (-\infty, 0]$ ($i \in I_*$) are some functions and

$$S_{t_0} := \text{int}\{x \in \mathbb{R}^n : (t_0, x) \in \bar{D}\}.$$

The results obtained are a generalization of some results given by J. Chabrowski [7], by R. Redheffer and W. Walter [8], by J. Szarski [10], by P. Besala [1], by W. Walter [12], by N. Yoshida [13], by the author [3], [4] and [6], and base on those results. To prove the results of this paper we use the theorem on a strong maximum principle from [6].

Partial differential problems together with nonstandard ("nonlocal") conditions were also studied in papers [2] and [5]. Since, in the theory of differential equations the name "the nonlocal condition" is sometimes confused then the name "the nonlocal condition" (used for example in [7] and [4]) has been changed to the name "the nonstandard condition" in paper [2] and next consequently used in the changed terminology in [5].

2. Preliminaries

The notation and definitions given in this section are valid throughout this paper. Some of them are similar to those applied by J. Szarski [9]–[11], by R. Redheffer and W. Walter [8], by P. Besala [1], by N. Yoshida [13], by J. Chabrowski [7] and by the author [4], [6].

We use the following notation: $\mathbb{R} = (-\infty, \infty)$, $\mathbb{N} = \{1, 2, \dots\}$, $x = (x_1, \dots, x_n)$ ($n \in \mathbb{N}$).

For any vectors $z = (z^1, \dots, z^m) \in \mathbb{R}^m$, $\tilde{z} = (\tilde{z}^1, \dots, \tilde{z}^m) \in \mathbb{R}^m$ we write

$$z \leq \tilde{z} \text{ if } z^i \leq \tilde{z}^i \quad (i = 1, \dots, m).$$

Let t_0 be a real finite number and let $0 < T < \infty$. A set $D \subset \{(t, x) : t > t_0, x \in \mathbb{R}^n\}$ (bounded or unbounded) is called a *set of type (P)* if:

1. The projection of the interior of D on the t -axis is the interval $(t_0, t_0 + T)$.
2. For every $(\tilde{t}, \tilde{x}) \in D$ there is a positive number r such that

$$\{(t, x) : (t - \tilde{t})^2 + \sum_{i=1}^n (x_i - \tilde{x}_i)^2 < r, \quad t < \tilde{t}\} \subset D.$$

3. All the boundary points $(\tilde{t}, \tilde{x})$ of D for which there is a positive number r such that

$$\{(t, x) : (t - \tilde{t})^2 + \sum_{i=1}^n (x_i - \tilde{x}_i)^2 < r, \quad t \leq \tilde{t}\} \subset D$$

belong to D .

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} \text{int}\{x \in \mathbb{R}^n : (t_0, x) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in \mathbb{R}^n : (t, x) \in D\} & \text{for } t \neq t_0 \end{cases}$$

and

$$\sigma_t = \begin{cases} \text{int}[\bar{D} \cap (\{t_0\} \times \mathbb{R}^n)] & \text{for } t = t_0, \\ D \cap (\{t\} \times \mathbb{R}^n) & \text{for } t \neq t_0. \end{cases}$$

Let $\tilde{D}$ be a set contained in $(-\infty, t_0 + T] \times \mathbb{R}^n$ such that $\bar{D} \subset \tilde{D}$. We introduce the following sets:

$$\partial_p D := \tilde{D} \setminus D \quad \text{and} \quad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t}, \tilde{x}) \in D$, we denote by $S^-(\tilde{t}, \tilde{x})$ the set of points $(t, x) \in D$ that can be joined to $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t -coordinate is weakly increasing from (t, x) to $(\tilde{t}, \tilde{x})$.

Let $Z_m(\tilde{D})$ denote the space of mappings

$$w : \tilde{D} \ni (t, x) \longrightarrow w(t, x) = (w^1(t, x), \dots, w^m(t, x)) \in \mathbb{R}^m$$

continuous in $\tilde{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z_m(\tilde{D})$ we define the functional

$$[w]_t = \max_{i=1, \dots, m} \sup\{0, w^i(\tilde{t}, x) : (\tilde{t}, x) \in \tilde{D}, \tilde{t} \leq t\}, \quad \text{where } t \leq t_0 + T.$$

By $M_{n \times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$.

A mapping $w \in Z_m(\tilde{D})$ is called *regular* in D if $w_t^i, w_x^i = \text{grad}_x w^i$, $w_{xx}^i = [w_{x_j x_k}^i]_{n \times n}$ ($i = 1, \dots, m$) are continuous in D .

Let the mappings

$$\begin{aligned} F^i : D \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \times Z_m(\tilde{D}) \ni (t, x, z, p, q, r, w) \rightarrow \\ \rightarrow F^i(t, x, z, p, q, r, w) \in \mathbb{R} \quad (i = 1, \dots, m) \end{aligned}$$

be given and let for an arbitrary regular in D function $w \in Z_m(\tilde{D})$

$$\begin{aligned} F^i[t, x, w] := F^i(t, x, w(t, x), w_t^i(t, x), w_x^i(t, x), w_{xx}^i(t, x), w), \\ (t, x) \in D \quad (i = 1, \dots, m). \end{aligned}$$

Each two regular in D mappings $u, v \in Z_m(\tilde{D})$ are said to be *solutions* of the system

$$(2.1) \quad F^i[t, x, u] \geq F^i[t, x, v] \quad (i = 1, \dots, m)$$

in D if they satisfy (2.1) for all $(t, x) \in D$.

For every set $A \subset \tilde{D}$ and for each function $w \in Z_m(\tilde{D})$ we apply the notation:

$$\max_{(t,x) \in A} w(t, x) := \left(\max_{(t,x) \in A} w^1(t, x), \dots, \max_{(t,x) \in A} w^m(t, x) \right).$$

Let $I = \mathbb{N}$ or I be a finite set of mutually different natural numbers. Define the set

$$Z = \bigcup_{i \in I} \sigma_{T_i},$$

where, in the case if $I = \mathbb{N}$, the following conditions are satisfied:

- (i) $t_0 < T_i \leq t_0 + T$ for $i \in I$ and $T_i \neq T_j$ for $i, j \in I$, $i \neq j$;
- (ii) $T_0 := \inf_{i \in I} T_i > t_0$;
- (iii) $S_{t_0} \subset S_{T_i}$ for $i \in I$;
- (iv) $S_{t_0} \subset S_t$ for every $t \in [T_0, t_0 + T]$,

and, in the case if I is a finite set of mutually different natural numbers, conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called a *set of type* $(P_{Z\Gamma})$ if:

- (a) $Z \neq \emptyset$,
- (b) $\Gamma \cap \bar{\sigma}_{t_0} \neq \emptyset$.

Let Z_* denote a non - empty subset of Z . We define the following set:

$$I_* = \{i \in I : \sigma_{T_i} \subset Z_*\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{Z\Gamma})$ is called a *set of type* (P_{ZB}) .

It is easy to see that if D is a set of type (P_{ZB}) then D satisfies condition (b) of the definition of a set of type $(P_{Z\Gamma})$. Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded proper subset] of $\mathbb{R}^n$ then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_{ZB}) [$(P_{Z\Gamma})$, respectively].

3. Strong maximum principles with nonstandard inequalities in sets of types $(P_{Z\Gamma})$ or (P_{ZB})

Now we shall prove the following theorem on strong maximum principles with nonstandard inequalities in sets of types $(P_{Z\Gamma})$ and (P_{ZB}) :

THEOREM 3.1. *Assume that:*

- (1) D is a set of type $(P_{Z\Gamma})$ or (P_{ZB}) .

(2)₁ The mappings F^i ($i = 1, \dots, m$) are weakly increasing with respect to $z^1, \dots, z^{i-1}, z^{i+1}, \dots, z^m$ ($i = 1, \dots, m$). Moreover, there is a constant $L > 0$ such that

$$F^i(t, x, z, p, q, r, w) - F^i(t, x, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{w}) \\ \leq L \left(\max_{k=1, \dots, m} |z^k - \tilde{z}^k| + |x| \sum_{j=1}^n |q^j - \tilde{q}^j| + |x|^2 \sum_{j,k=1}^n |r_{jk} - \tilde{r}_{jk}| + [w - \tilde{w}]_t \right)$$

for all $(t, x) \in D$, $z, \tilde{z} \in \mathbb{R}^m$, $p \in \mathbb{R}$, $q, \tilde{q} \in \mathbb{R}^n$, $r, \tilde{r} \in M_{n \times n}(\mathbb{R})$, $w, \tilde{w} \in Z_m(\tilde{D})$, where $\sup_{(t,x) \in \tilde{D}} (w(t, x) - \tilde{w}(t, x)) < \infty$ ($i = 1, \dots, m$).

(2)₂ There are constants $C_i > 0$ ($i = 1, 2$) such that

$$F^i(t, x, z, p, q, r, w) - F^i(t, x, z, \tilde{p}, q, r, w) < C_1(\tilde{p} - p) \quad (i = 1, \dots, m)$$

for all $(t, x) \in D$, $z \in \mathbb{R}^m$, $p > \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$ and

$$F^i(t, x, z, p, q, r, w) - F^i(t, x, z, \tilde{p}, q, r, w) < C_2(\tilde{p} - p) \quad (i = 1, \dots, m)$$

for all $(t, x) \in D$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$.

(3) The mapping u belongs to $Z_m(\tilde{D})$ and the maximum of u on Γ is attained. Moreover,

$$(3.1) \quad K = (K^1, \dots, K^m) := \max_{(t,x) \in \Gamma} u(t, x).$$

(4) The inequalities

$$(3.2) \quad [u^j(t_0, x) - K^j] + \sum_{i \in I_*} h_i(x) [u^j(T_i, x) - K^j] \leq 0$$

for $x \in S_{t_0}$ ($j = 1, \dots, m$) are satisfied, where $h_i : S_{t_0} \rightarrow (-\infty, 0]$ ($i \in I_*$) are given functions such that $-1 \leq \sum_{i \in I_*} h_i(x) \leq 0$ for $x \in S_{t_0}$ and, additionally, if $\text{card } I_* = \aleph_0$, then the series $\sum_{i \in I_*} h_i(x) u^j(T_i, x)$ ($j = 1, \dots, m$) are convergent for $x \in S_{t_0}$.

(5) The maximum of u in $\tilde{D}$ is attained. Moreover

$$(3.3) \quad M = (M^1, \dots, M^m) := \max_{(t,x) \in \tilde{D}} u(t, x).$$

(6) The mappings u and M are solutions of the system

$$F^i[t, x, u] \geq F^i[t, x, M] \quad (i = 1, \dots, m)$$

in D .

(7) The mappings F^i ($i = 1, \dots, m$) are parabolic with respect to u in D and uniformly parabolic with respect to M in any compact subset of D (see [3] or [6]).

Then

$$(3.4) \quad \max_{(t,x) \in \tilde{D}} u(t, x) = \max_{t,x \in \Gamma} u(t, x).$$

Moreover, if there is a point $(\tilde{t}, \tilde{x}) \in D$ such that $u(\tilde{t}, \tilde{x}) = \max_{(t,x) \in \tilde{D}} u(t, x)$ then

$$u(t, x) = \max_{(t,x) \in \Gamma} u(t, x) \text{ for } (t, x) \in S^-(\tilde{t}, \tilde{x}).$$

Proof. We shall prove Theorem 3.1 for a set of type $(P_{Z\Gamma})$ only since the proof for a set of type (P_{ZB}) is analogous.

Since each set of type $(P_{Z\Gamma})$ is a set of type (P_Γ) from [6], it follows that in the case where $\sum_{i \in I_*} h_i(x) = 0$ for $x \in S_{t_0}$, Theorem 3.1 is a consequence of Theorem 4.1 of [6]. Therefore, we shall give the proof of Theorem 3.1 only in the case where

$$(3.5) \quad -1 \leq \sum_{i \in I_*} h_i(x) < 0 \text{ for } x \in S_{t_0}.$$

Assume, so, that (3.5) holds and, since we shall argue by contradiction, suppose

$$(3.6) \quad M \neq K.$$

From (3.1) and (3.3) we have

$$(3.7) \quad K \leq M.$$

Consequently,

$$(3.8) \quad K < M.$$

Observe, from assumption (5), that

$$(3.9) \quad \text{There is } (t^*, x^*) \in \tilde{D} \text{ such that } u(t^*, x^*) = M := \max_{(t,x) \in \tilde{D}} u(t, x).$$

By (3.9), by assumption (3) and by (3.8), we have

$$(3.10) \quad (t^*, x^*) \in \tilde{D} \setminus \Gamma = D \cup \sigma_{t_0}.$$

An argument analogous to the proof of Theorem 4.1 from [6] yields

$$(t^*, x^*) \notin D.$$

Hence

$$(3.11) \quad (t^*, x^*) \in \sigma_{t_0}.$$

On the other hand, by the definition of sets I and I_* , we must consider the following cases:

(A) I_* is a finite set, i.e., without loss generality there is a number $p \in \mathbb{N}$ such that $I_* = \{1, \dots, p\}$.

(B) $\text{card } I_* = \aleph_0$.

First we shall consider case (A). By (3.2) and by the inequalities $u(T_i, x^*) < u(t_0, x^*)$ ($i = 1, \dots, p$), which are consequences of (3.9) and (3.11) and of

(a)(i) and (a)(iii) of the definition of a set of type $(P_{Z\Gamma})$, we have

$$\begin{aligned} 0 &\geq [u^j(t_0, x^*) - K^j] + \sum_{i=1}^p h_i(x^*)[u^j(T_i, x^*) - K^j] \\ &\geq [u^j(t_0, x^*) - K^j] \cdot [1 + \sum_{i=1}^p h_i(x^*)] \quad (j = 1, \dots, m). \end{aligned}$$

Hence

$$(3.12) \quad u(t_0, x^*) \leq K \text{ if } 1 + \sum_{i=1}^p h_i(x^*) > 0.$$

Then, from (3.8) and (3.11), we obtain a contradiction of (3.12) with (3.9). Assume now that $\sum_{i=1}^p h_i(x^*) = -1$. Since for every $j \in \{1, \dots, m\}$ there is $\ell_j \in \{1, \dots, p\}$ such that

$$u^j(T_{\ell_j}, x^*) = \max_{i=1, \dots, p} u^j(T_i, x^*),$$

we obtain, by (3.2),

$$\begin{aligned} u^j(t_0, x^*) - u^j(T_{\ell_j}, x^*) &= [u^j(t_0, x^*) - K^j] - [u^j(T_{\ell_j}, x^*) - K^j] \\ &= [u^j(t_0, x^*) - K^j] + \sum_{i=1}^p h_i(x^*)[u^j(T_{\ell_j}, x^*) - K^j] \\ &\leq [u^j(t_0, x^*) - K^j] + \sum_{i=1}^p h_i(x^*)[u^j(T_i, x^*) - K^j] \\ &\leq 0 \quad (j = 1, \dots, m). \end{aligned}$$

Hence

$$(3.13) \quad u^j(t_0, x^*) \leq u^j(T_{\ell_j}, x^*) \quad (j = 1, \dots, m) \text{ if } \sum_{i=1}^p h_i(x^*) = -1.$$

Since, by (a) (i) of the definition of a set type $(P_{Z\Gamma})$, $T_{\ell_j} > t_0$ ($j = 1, \dots, m$), we see from (3.11) that (3.13) contradicts (3.9). This completes the proof of (3.4) if I_* is a finite set.

It remains to investigate case (B). Analogously to the proof of (3.4) in case (A), by assumption (4) and by the inequalities $u(T_i, x^*) < u(t_0, x^*)$ ($i \in I_*$) we have

$$\begin{aligned} 0 &\geq [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*)[u^j(T_i, x^*) - K^j] \\ &\geq [u^j(t_0, x^*) - K^j] \cdot [1 + \sum_{i \in I_*} h_i(x^*)] \quad (j = 1, \dots, m). \end{aligned}$$

Hence

$$(3.14) \quad u(t_0, x^*) \leq K \text{ if } 1 + \sum_{i \in I_*} h_i(x^*) > 0.$$

Then, from (3.8) and (3.11), we obtain a contradiction of (3.14) with (3.9). Assume now that $\sum_{i \in I_*} h_i(x^*) = -1$ and let

$$T_0^* := \inf_{i \in I_*} T_i.$$

Since $u \in C(\bar{D})$ and since, by (a) (iv) of the definition of a set of type $(P_{Z\Gamma})$, $x^* \in S_t$ for every $t \in [T_0, t_0 + T]$ if $I = \mathbb{N}$, it follows that for every $j \in \{1, \dots, m\}$ there is $\hat{t}_j \in [T_0^*, t_0 + T]$ such that

$$u^j(\hat{t}_j, x^*) = \max_{t \in [T_0^*, t_0 + T]} u^j(t, x^*).$$

Consequently, by assumption (4), we obtain

$$\begin{aligned} u^j(t_0, x^*) - u^j(\hat{t}_j, x^*) &= [u^j(t_0, x^*) - K^j] - [u^j(\hat{t}_j, x^*) - K^j] \\ &= [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*) [u^j(\hat{t}_j, x^*) - K^j] \\ &\leq [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*) [u^j(T_i, x^*) - K^j] \\ &\leq 0 \quad (j = 1, \dots, m). \end{aligned}$$

Hence

$$(3.15) \quad u^j(t_0, x^*) \leq u^j(\hat{t}_j, x^*) \quad (j = 1, \dots, m) \text{ if } \sum_{i \in I_*} h_i(x^*) = -1.$$

Since, by (a)(ii) of the definition of a set of type $(P_{Z\Gamma})$, $\hat{t}_j > t_0$ ($j = 1, \dots, m$), we see from (3.11) that (3.15) contradicts (3.9). This completes the proof of (3.4).

The second part of Theorem 3.1 is a consequence of (3.4) and Lemma 3.1 from [6]. Therefore, the proof of Theorem 3.1 is complete.

4. Remarks

REMARK 4.1. Observe, by the proof of Theorem 3.1 from this paper and by the proof of Theorem 4.1 from [6], that if the functions h_i ($i \in I_*$) from assumption (4) of Theorem 3.1 satisfy the condition

$$\left[\sum_{i \in I_*} h_i(x) = 0 \right] \quad -1 < \sum_{i \in I_*} h_i(x) \leq 0 \quad \text{for } x \in S_{t_0},$$

then it is sufficient to assume in these theorems that $[D$ is an unbounded set of type (P) satisfying condition (b) of the definition of a set of type $(P_{Z\Gamma})$ or D is a bounded set of type (P) , i.e., according to the terminology

of [6], D is a set of type (P_Γ) or (P_B) , respectively] D is an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii) and (b) of the definition of a set type $(P_{Z\Gamma})$ or D is a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii) of the definition of a set of type $(P_{Z\Gamma})$. Moreover, if I_* is a finite set and $-1 \leq \sum_{i \in I_*} h_i(x) \leq 0$ for $x \in S_{t_0}$, then it is sufficient to assume in Theorem 3.1 that D is an unbounded set of type (P) satisfying conditions (a)(i) and (a)(iii) and (b), or D is a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii).

REMARK 4.2. If D is a set of type (P_{ZB}) and if $\tilde{D} = \bar{D}$ then the first part of assumption (3) of Theorem 3.1 relative to the maximum of u and the first part of assumption (5) of this theorem are trivially satisfied since $u \in C(\bar{D})$ and Γ is a bounded and closed set in this case.

REMARK 4.3. If the mappings F^i ($i = 1, \dots, m$) do not depend on the functional argument w then Theorem 3.1 reduces to the theorem on parabolic differential inequalities of the form

$$\begin{aligned} F^i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), u_{xx}^i(t, x)) \\ \geq F^i(t, x, M, 0, 0, 0), \quad (t, x) \in D \quad (i = 1, \dots, m) \end{aligned}$$

and in this case we can put $\tilde{D} = \bar{D}$.

REMARK 4.4. Nonstandard conditions have physical applications. Namely, nonstandard conditions together with boundary conditions often give better descriptions of the diffusion phenomena than the initial conditions together with the boundary conditions from the nonstandard problems. For example, in a gaseous diffusion the measurement of a small amount of a gas in the form of the following sum

$$u(t_0, x) + \sum_{i \in I_*} h_i(x) u(T_i, x)$$

(h_i ($i \in I_*$) are known functions) can be more precise than the measurement of a small amount of this gas at the initial instant t_0 .

If $I_* = \{1\}$, $T_1 = t_0 + T$ and $h_1(x) = -h(x)$ ($0 < h(x) \leq 1$) for $x \in S_{t_0}$, then the nonstandard condition

$$u(t_0, x) + \sum_{i \in I_*} h_i(x) u(T_i, x) = 0 \quad \text{for } x \in S_{t_0}$$

reduces to the following nonstandard condition

$$u(t_0, x) = h(x) u(t_0 + T, x) \quad \text{for } x \in S_{t_0}$$

and this condition can be used to the description of heat effects in atomic reactors. Namely, for the given safe temperature $u(t_0, x)$ in an atomic reactor, the reaction, at the instant $t_0 + T$, is the safest for $1 \simeq h(x) \leq 1$, $x \in S_{t_0}$,

and the reaction, at the instant $t_0 + T$, can be the most dangerous for $0 < h(x) \simeq 0$, $x \in S_{t_0}$.

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