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AN ALTERNATIVE PROOF OF THE TYCHONOFF THEOREM

Abstract. An alternative proof of the Tychonoff Theorem, is presented via the Riesz homomorphism.

The Tychonoff theorem states that the product of a family of compact Hausdorff spaces is compact. In its original form, it was proven by Tychonoff [9] for compact intervals. The general form is due to Čech [1]. Kakutani has conjectured that the axiom of choice is equivalent to the Tychonoff theorem and this conjecture has been proven by Kelley [5]. The Tychonoff theorem for compact Hausdorff spaces is equivalent to the Boolean algebra prime ideal theorem (see [8]). In the literature there are many different proofs of the Tychonoff's theorem (see [4] and [6] for more details). We refer to [2] and [10] for recent proofs. In this paper we present a new proof of the Tychonoff theorem via Riesz homomorphisms for compact Hausdorff spaces.

Given a topological space, X , let $C(X)$ denote the vector space of all real valued continuous functions and $C_b(X) = \{f \in C(X) : f \text{ is bounded}\}$. For each $f, g \in C(X)$, the pointwise supremum of f and g is denoted by $f \vee g$. A linear map $T : C_b(X) \rightarrow R$ is called *Riesz homomorphism* if $T(f \vee g) = T(f) \vee T(g)$ for each $f, g \in C_b(X)$. The set of all Riesz homomorphisms from $C_b(X)$ into R is denoted by $Hom(C_b(X), R)$. Let $1 : C_b(X) \rightarrow R$ be defined by $1(f) = 1$ and

$$\beta X = \{T \in Hom(C_b(X), R) : T(1) = 1\}.$$

For given $x \in X$, we define

$$\pi_x : C_b(X) \rightarrow R, \quad e_X : X \rightarrow \beta X$$

by

$$\pi_x(f) = f(x), \quad e_X(x) = \pi_x.$$

The proof of the following theorem is routine and closely allied versions can be found in [3] and [7].

THEOREM 1. *Let X be a topological space. Then βX is a compact Hausdorff space under the Hull-Kernel topology*

$$\mathcal{H} = \{\{T \in \beta X : T(A) \neq \{0\}\} : A \subset C_b(X)\}.$$

Moreover, if X is a completely regular Hausdorff space then

a. X is densely imbedded as a subspace of βX under the map e_X , i.e., X and $e_X(X)$ are homeomorphic and $\text{cle}_X(X) = \beta X$.

b. $\phi : C_b(X) \rightarrow C(\beta X)$, $\phi(f)(T) = T(f)$, is a Riesz isomorphism and we write $f^* = \phi(f)$.

We can now proceed to prove the Tychonoff theorem.

THEOREM 2 (Tychonoff theorem). *The product $X_\infty = \prod_{i \in I} X_i$ of a family $(X_i)_{i \in I}$ of compact Hausdorff spaces is a compact Hausdorff space.*

Proof. It is clear that X_∞ is completely regular Hausdorff space. Let $T \in \beta X_\infty$. It is enough to show that $T \in e_{X_\infty}(X_\infty)$. For each $j \in I$ define the map

$$c_j : C(X_j) \rightarrow C_b(X_\infty), \quad c_j(f)((x_i)_{i \in I}) = f(x_j).$$

Since X_j is compact, for each $j \in I$, by the above theorem (b) there exists $a_j \in X_j$ such that $T \circ c_j = \pi_{a_j}$. Let $a_\infty = (a_i)_{i \in I}$. Since X_∞ is homeomorphic to a dense subspace of βX_∞ under the map e_{X_∞} , there exists a net (y_α) in X_∞ with $\pi_{y_\alpha} \rightarrow T$ in βX_∞ . For each $f \in C(X_j)$, $f(a_j) = \pi_{a_j}(f) = T(c_j f)$ and

$$f(x_j^\alpha) = c_j f(y_\alpha) = (c_j f)^*(\pi_{y_\alpha}) \rightarrow (c_j f)^*(T) = T(c_j f) = f(a_j)$$

where $y_\alpha = (x_i^\alpha)$. This shows that $x_j^\alpha \rightarrow a_j$ in X_j for each j , so $y_\alpha \rightarrow a_\infty$ in X_∞ . Then $\pi_{y_\alpha} \rightarrow \pi_{a_\infty}$ in βX_∞ , hence $\pi_{a_\infty} = T$, since βX_∞ is Hausdorff.

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Received May 27, 2004; revised version December 21, 2004.