Włodzimierz M. Mikulski

THERE EXISTS A BUNDLE FUNCTOR OF JET BUNDLE TYPE OF INFINITE ORDER IN THE FIRST FACTOR

Abstract. It is known a complete description of all bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ of finite order in the first factor. The most known example of such a bundle functor is the r -jet bundle functor J^r on $\mathcal{M}f_m \times \mathcal{M}f$. In the present note an example of a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of (essentially) infinite order in the first factor is constructed.

The concept of bundle functor on the category $\mathcal{M}f$ of manifolds and maps (or on the category $\mathcal{M}f_m$ of m -manifolds and local diffeomorphisms), the concept of natural transformations between bundle functors and the concept of order of bundle functor can be found in [1].

Let $\mathcal{M}f_m \times \mathcal{M}f$ be the product of categories $\mathcal{M}f_m$ and $\mathcal{M}f$. The objects of $\mathcal{M}f_m \times \mathcal{M}f$ are pairs (M, N) , where M is an m -manifold and N is a manifold. The morphisms $(M, N) \rightarrow (\overline{M}, \overline{N})$ are the pairs (φ, f) , where $\varphi : M \rightarrow \overline{M}$ is a local diffeomorphism between m -manifolds and $f : N \rightarrow \overline{N}$ is a map.

Let $\mathcal{FM}$ be the category of fibered manifolds and their fibered maps. A *bundle functor* on $\mathcal{M}f_m \times \mathcal{M}f$ is a covariant functor $F : \mathcal{M}f_m \times \mathcal{M}f \rightarrow \mathcal{FM}$ satisfying the following conditions (see [2]):

(i) (Prolongation) For very m -manifold M and manifold N $F(M, N)$ is a fibered manifold over $M \times N$ with projection $p = p_{M, N} = (\alpha, \beta) : F(M, N) \rightarrow M \times N$, and for every local diffeomorphism $\varphi : M \rightarrow \overline{M}$ of two m -manifolds and every map $f : N \rightarrow \overline{N}$ of manifolds the induced mapping $F(\varphi, f) : F(M, N) \rightarrow F(\overline{M}, \overline{N})$ is a fibered manifold morphism over $\varphi \times f : M \times N \rightarrow \overline{M} \times \overline{N}$.

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(ii) (Locality) If $i = i_1 \times i_2 : U = U_1 \times V_1 \rightarrow M \times N$ is an inclusion with open U_1 and V_1 , then $F(U_1, V_1) = p_{M,N}^{-1}(U)$ and $F(i_1, i_2)$ is the inclusion of $p_{M,N}^{-1}(U)$ into $F(M, N)$.

From now on let r be a natural number.

A very important example of bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ is the holonomic r -jets bundle functor J^r . More precisely, for every pair of a m -manifold M and a manifold N we have the bundle $J^r(M, N) = \{j_x^r(g) \mid g : M \rightarrow N, x \in M\}$ of (holonomic) r -jets of maps $M \rightarrow N$. We have the source projection $\alpha : J^r(M, N) \rightarrow M$ by $\alpha(j_x^r(g)) = x$ and the target projection $\beta : J^r(M, N) \rightarrow N$ by $\beta(j_x^r(g)) = g(x)$. Every pair of a local diffeomorphism $\varphi : M \rightarrow \overline{M}$ and a map $f : N \rightarrow \overline{N}$ induces fibered map $J^r(\varphi, f) : J^r(M, N) \rightarrow J^r(\overline{M}, \overline{N})$ over $\varphi \times f$ by $J^r(\varphi, f)(j_x^r(g)) = j_{\varphi(x)}^r(f \circ g \circ \varphi^{-1})$.

Another example is the bundle functor $\tilde{J}^r$ on $\mathcal{M}f_m \times \mathcal{M}f$ of non-holonomic r -jets in the sense of Ehresman.

To present a general example we recall that a group homomorphism $H : G_m^r \rightarrow \mathcal{NE}(G)$ of the differential group $G_m^r = \text{inv} J_0^r(\mathbb{R}^m, \mathbb{R}^m)_0$ into the group $\mathcal{NE}(G)$ of natural equivalences of a bundle functor G on $\mathcal{M}f$ is called *smooth* if for any manifold N the action $H^N : G_m^r \times G(N) \rightarrow G(N)$ given by $H^N(\xi, v) = H(\xi)_N(v)$ is smooth.

EXAMPLE 1. ([2]) *The associated bundle.* Let G be a bundle functor on $\mathcal{M}f$ and $H : G_m^r \rightarrow \mathcal{NE}(G)$ be a group homomorphism that is smooth in the above sense. Then every $G(N)$ is a G_m^r -space and we put

$$(G, H)(M, N) = P^r(M)[G(N), H^N],$$

where $P^r(M)[G(N), H^N]$ is the associated bundle with $P^r(M)$ = the r -th order frame bundle of an m -manifold M and fibre $G(N)$. We define the source and the target projections $\alpha : (G, H)(M, N) \rightarrow M$ by $\alpha(\langle j_0^r(\gamma), v \rangle) = \gamma(0)$ and $\beta : (G, H)(M, N) \rightarrow N$ by $\beta(\langle j_0^r(\gamma), v \rangle) = y$ for all $j_0^r(\gamma) \in P^r(M)$ and $v \in G_y(N)$, $y \in N$. Every pair of a local diffeomorphism $\varphi : M \rightarrow \overline{M}$ and a map $f : N \rightarrow \overline{N}$ induce a fibered map $(G, H)(\varphi, f) = (G, H)(M, N) \rightarrow (G, H)(\overline{M}, \overline{N})$ by

$$(G, H)(\varphi, f)(\langle j_0^r(\gamma), v \rangle) = \langle j_0^r(\varphi \circ \gamma), G(f)(v) \rangle.$$

The correspondence $(G, H) : \mathcal{M}f_m \times \mathcal{M}f \rightarrow \mathcal{FM}$ is a bundle functor.

Let F be a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$. We say that F is of order less or equal to r in the first factor if from $j_x^r(\varphi_1) = j_x^r(\varphi_2)$ it follows $F_x(\varphi_1, f) = F_x(\varphi_2, f)$ for every local diffeomorphisms $\varphi_1, \varphi_2 : M \rightarrow \overline{M}$ between m -manifolds, every map $f : N \rightarrow \overline{N}$ and every point $x \in M$.

The bundles J^r and $\tilde{J}^r$ are of order r in the first factor. The associated bundle (G, H) is of order less or equal to r in the first factor.

Let F be a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of finite order r in the first factor. Functor F induces a bundle functor G^F on $\mathcal{M}f$ by $G^F(N) = F_0(\mathbb{R}^m, N)$ and $G^F(f) = F_0(id_{\mathbb{R}^n}, f) : G^F(N) \rightarrow G^F(\bar{N})$ for $f : N \rightarrow \bar{N}$ and a (smooth) group homomorphism $H^F : G_m^r \rightarrow \mathcal{NE}(G^F)$ by

$$H^F(j_0^r(\varphi))_N(v) = F(\varphi, id_N)(v)$$

for any manifold N any $v \in G^F(N)$ and any $j_0^r(\varphi) \in G_m^r$.

We have the following classification theorem.

THEOREM 1 ([2]). *Every bundle functor F on $\mathcal{M}f_n \times \mathcal{M}f$ of finite order r in the first factor is equivalent with the associated bundle (G^F, H^F) .*

So, it is interesting the following question. Is there a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ of (essentially) infinite order in the first factor? The following example shows that the answer to the above question is positive.

EXAMPLE 2. *There exists a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of (essentially) infinite order in the first factor.* For any $r = 1, 2, 3, \dots$ we define a vector bundle functor $G^r : \mathcal{M}f \rightarrow \mathcal{VB}$ by $G^r(N) = T(N) \times \mathcal{D}^r$ and $G^r(f) = T(f) \times id_{\mathcal{D}^r} : G^r(N) \rightarrow G^r(\bar{N})$ for any map $f : N \rightarrow \bar{N}$, where $\mathcal{D}^r = J_0^r(\mathbb{R}^m, \mathbb{R})$ is the vector space of r -jets at 0 of maps $\mathbb{R}^m \rightarrow \mathbb{R}$.

Let $d_r = \dim(G_0^r(\mathbb{R}^r))$. We set

$$G(N) = \bigoplus_{k \in \mathbb{N}} (\bigwedge^{d_k} G^k(N)).$$

For every manifold N , $G(N)$ is a finite dimensional manifold because for $k > \dim(N)$ the bundle $\bigwedge^{d_k} G^k(N)$ is zero-bundle. Hence the direct sum in the definition of $G(N)$ is in reality a finite sum. For a mapping $f : N \rightarrow \bar{N}$ the induced mapping $G(f) : G(N) \rightarrow G(\bar{N})$ is defined in the natural way from $\bigwedge^{d_k} G^k(f)$. In this way we obtain vector bundle functor G on $\mathcal{M}f$ of order 1.

The group $G_m^\infty = inv J_0^\infty(\mathbb{R}^m, \mathbb{R}^m)_0$ of invertible ∞ -jets acts on $\mathcal{D}^r$ by the pull-back. Hence for $r = 1, 2, \dots$ we have a group homomorphism $H^r : G_m^\infty \rightarrow \mathcal{NE}_{vb}(G^r)$ into the group of vector bundle natural equivalences of G by $H^r(\xi)_N : G^r(N) \rightarrow G^r(N)$, $H^r(\xi)_N(v, \eta) = (v, \xi \cdot \eta)$, $v \in T(N)$, $\xi \in G_m^\infty$, $\eta \in \mathcal{D}^r$, where the dot is the action. Hence for any $\xi \in G_m^\infty$ and a manifold N we have a vector natural equivalence $H(\xi)_N : G(N) \rightarrow G(N)$ which is defined in natural way from $\bigwedge^{d_k} H^k(\xi)_N : \bigwedge^{d_k} G^k(N) \rightarrow \bigwedge^{d_k} G^k(N)$. This defines a group homomorphism $H : G_m^\infty \rightarrow \mathcal{NE}_{vb}(G)$.

For any manifold N the corresponding action H of G_m^∞ on $G(N)$ can be factorized by a smooth action of $G_m^{l_N}$ on $G(N)$ for some finite l_N . Hence we can proceed similarly as in Example 1 for above G and H and obtain bundle functor (G, H) on $\mathcal{M}f_m \times \mathcal{M}f$. This bundle functor is of (essentially)

infinite order in the first factor because the homomorphism H can not be factorized by a group homomorphism $G_m^l \rightarrow \mathcal{NE}(G)$ with finite l .

References

- [1] I. Kolář et al, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin 1993.
- [2] I. Kolář, W. M. Mikulski, *On the fiber product preserving bundle functors*, Diff. Geom. Appl. 11 (1999), 105–115.

INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
ul. Reymonta 4
30-059 KRAKÓW, POLAND
e-mail: mikulski@im.uj.edu.pl

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