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A short note on the common fixed points of the Geraghty contraction of type $E_{S,T}$

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Abstract: In this paper, we propose a new contraction type for two self-mappings and investigate the necessary conditions for the existence and uniqueness of coincidence points and common fixed points, respectively. We put some examples to illustrate our main results.

Keywords: Geraghty type contraction, E -contraction, Geraghty contraction of type $E_{S,T}$

MSC: 47H10, 54H25

1 Introduction and preliminaries

Let (M, d) be a complete metric space. A map $T : M \rightarrow M$ is a contraction if there exists a constant $\lambda \in (0, 1)$, such that

$$d(T\xi, Tv) \leq \lambda d(\xi, v), \quad \text{for each } \xi, v \in M. \quad (1.1)$$

An operator $T : M \rightarrow M$ is called a Geraghty contraction [1] if there exists a function $\varphi : [0, \infty) \rightarrow [0, 1)$ which satisfies the condition

$$\lim_{n \rightarrow \infty} \varphi(t_n) = 1 \text{ implies that } \lim_{n \rightarrow \infty} t_n = 0, \quad (1.2)$$

such that

$$d(T\xi, Tv) \leq \varphi(d(\xi, v))d(\xi, v), \quad \text{for all } \xi, v \in M. \quad (1.3)$$

In the following we denote the class of functions $\varphi : [0, \infty) \rightarrow [0, 1)$ which satisfies the condition (1.2) by Φ .

The concept of Geraghty contraction extends the famous Banach contraction principle [2].

Theorem 1. [1] Suppose that T is a self-mapping on a complete metric space (M, d) . If T is a Geraghty contraction, then it has a unique fixed point $\xi^* \in M$. Moreover, for any initial point $\xi_0 \in M$, the iterative sequence $\{T^n \xi_0\}_{n=1}^\infty$ converges to ξ^* .

This interesting result has attracted the attention of a great number of authors [3–7]. Among them, we recall the notion of Geraghty contraction of type E which was defined by Fulga and Proca [4]: A self-mapping T on a complete metric space (M, d) is called Geraghty contraction of type E if there exists a function $\varphi \in \Phi$, such that

$$d(T\xi, Tv) \leq \varphi(E(\xi, v))E(\xi, v), \quad \text{for all } \xi, v \in M, \quad (1.4)$$

where

$$E(\xi, v) = d(\xi, v) + |d(\xi, T\xi) - d(v, Tv)|.$$

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Theorem 2. [4] Suppose that T is a self-mapping on a complete metric space (M, d) . If T is a Geraghty contraction of type E , then it has a unique fixed point $\xi^* \in M$. Furthermore, for any initial point $\xi_0 \in M$, the iterative sequence $\{T^n \xi\}_{n=1}^\infty$ converges to ξ^* .

From now on we use the symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ for the reals, positive reals and natural numbers, accordingly. Moreover, we employ the symbols $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\} = [0, \infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The main aim of this paper is to prove the existence and uniqueness of a common fixed point for given two self-mappings T, S on complete metric space X . We consider an illustrative example to indicate the validity of the main results.

2 Main results

We start this section by giving the following definition:

Definition 3. Suppose that T and S are two self-mappings on a metric space (M, d) . Suppose also that there is a $\varphi \in \Phi$, such that the inequality

$$d(T\xi, Sv) \leq \varphi(E_{T,S}(\xi, v)) E_{T,S}(\xi, v), \text{ for all } \xi, v \in M, \quad (2.1)$$

is satisfied, where

$$E_{T,S}(\xi, v) = d(\xi, v) + |d(\xi, T\xi) - d(v, Sv)|. \quad (2.2)$$

Then, we say that the mappings T and S satisfy the Geraghty contraction of type $E_{S,T}$.

Lemma 4. Suppose that T and S are self-mappings on a complete metric space (M, d) . If the pair T, S forms a Geraghty contraction of type $E_{S,T}$, then we have the following statement:
 $E_{T,S}(\xi, v) = 0$ if and only if $\xi = v$ is a coincidence point of T and S .

Proof. Let $E_{T,S}(\xi, v) = 0$, then we have

$$0 \leq d(T\xi, Sv) \leq \varphi(E_{T,S}(\xi, v)) E_{T,S}(\xi, v) \leq E_{T,S}(\xi, v) = 0$$

since $\varphi : [0, \infty) \rightarrow [0, 1)$. Consequently, we find $d(T\xi, Sv) = 0$, and hence

$$T\xi = Sv. \quad (2.3)$$

On the other hand, by definition, we have

$$0 = E_{T,S}(\xi, v) = d(\xi, v) + |d(\xi, T\xi) - d(v, Sv)| \geq d(\xi, v) \geq 0$$

which yields that

$$d(\xi, v) = 0 \Rightarrow \xi = v. \quad (2.4)$$

By combining the expressions (2.3) and (2.4), we conclude that $T\xi = S\xi$.

For the inverse implication, we suppose that $\xi = v$ is a coincidence point of S and T , that is, $d(T\xi, S\xi) = 0$. Then, we find that

$$E_{T,S}(\xi, \xi) = E_{T,S}(\xi, v) = d(\xi, v) + |d(\xi, T\xi) - d(v, Sv)| = |d(\xi, T\xi) - d(\xi, S\xi)| = 0.$$

□

Employing the ideas from [4] and [7] we shall obtain a generalization of Theorem 2.

Theorem 5. Suppose that T and S are self-mappings on a complete metric space (M, d) . If the pair T, S forms a Geraghty contraction of type $E_{S,T}$, then the pair of mappings T, S has a unique common fixed point ξ^* , that is, $T\xi^* = \xi^* = S\xi^*$.

Proof. As a first step, by starting with an arbitrary point $\xi_0 \in M$, we construct a sequence $\{\xi_n\}$ in X as follows:

$$\xi_1 = S\xi_0, \xi_2 = T\xi_1, \xi_{2n+1} = S\xi_{2n} \text{ and } \xi_{2n+2} = T\xi_{2n+1}, \text{ for all } n \in \mathbb{N}_0. \quad (2.5)$$

Without loss of generality, we assume that

$$\xi_n \neq \xi_{n+1} \text{ for each } n \in \mathbb{N}_0.$$

Indeed, if $\xi_{n_0} = \xi_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then we find that $\xi^* = \xi_{n_0}$ is a common fixed point for S and T . This constitutes the proof. It is easy to conclude this fact by considering two cases. The first case is to assume that n_0 is odd, that is, $n_0 = 2k - 1$. In this case we have $\xi_{2k-1} = \xi_{2k} = T\xi_{2k-1}$, that is, ξ_{2k-1} is a fixed point of T . Now, we shall prove that $\xi_{2k-1} = \xi_{2k} = T\xi_{2k-1} = S\xi_{2k}$. Suppose on the contrary that $d(T\xi_{2k-1}, S\xi_{2k}) > 0$. By letting $x = \xi_{2k-1}$ and $y = \xi_{2k}$ in (2.1), and keeping in mind that $\varphi \in \Phi$, we get

$$\begin{aligned} d(\xi_{2k}, \xi_{2k+1}) &= d(T\xi_{2k-1}, S\xi_{2k}) \leq \varphi(E_{T,S}(\xi_{2k-1}, \xi_{2k})) E_{T,S}(\xi_{2k-1}, \xi_{2k}) \\ &\leq \varphi(d(\xi_{2k}, \xi_{2k+1})) d(\xi_{2k}, \xi_{2k+1}) < d(\xi_{2k}, \xi_{2k+1}), \end{aligned}$$

a contradiction. Hence, $d(T\xi_{2k-1}, S\xi_{2k}) = 0$ and $\xi_{2k-1} = \xi_{2k} = T\xi_{2k-1} = S\xi_{2k}$. In other words, by letting $\xi_{2k-1} = \xi_{2k} = \xi^*$ we get that ξ^* is a common fixed point of S and T . Notice that the case $n_0 = 2k$ can be derived by following the steps of the first case.

As a next step, we shall show that the sequence $\{d(\xi_{n+1}, \xi_n)\}$ is monotone. For this we examine the following two cases:

Case (i). Let $x = \xi_{2n+1}$, $y = \xi_{2n}$. Then, by the inequality (2.1), we get

$$\begin{aligned} 0 &< d(\xi_{2n+2}, \xi_{2n+1}) = d(T\xi_{2n+1}, S\xi_{2n}) \\ &\leq \varphi(E_{T,S}(\xi_{2n+1}, \xi_{2n})) E_{T,S}(\xi_{2n+1}, \xi_{2n}) \\ &= \varphi(d(\xi_{2n+1}, \xi_{2n}) + |d(\xi_{2n+1}, T\xi_{2n+1}) - d(\xi_{2n}, S\xi_{2n})|) [d(\xi_{2n+1}, \xi_{2n}) + |d(\xi_{2n+1}, T\xi_{2n+1}) - d(\xi_{2n}, S\xi_{2n})|] \\ &= \varphi(d(\xi_{2n+1}, \xi_{2n}) + |d(\xi_{2n+1}, \xi_{2n+2}) - d(\xi_{2n}, \xi_{2n+1})|) [d(\xi_{2n+1}, \xi_{2n}) + |d(\xi_{2n+1}, \xi_{2n+2}) - d(\xi_{2n}, \xi_{2n+1})|]. \end{aligned}$$

By letting $d_{2n} = d(\xi_{2n}, \xi_{2n+1})$ the inequality above becomes

$$0 < d_{2n+1} \leq \varphi(d_{2n} + |d_{2n+1} - d_{2n}|) (d_{2n} + |d_{2n+1} - d_{2n}|). \quad (2.6)$$

If $d_{2n+1} \geq d_{2n}$ from the property of the function φ it follows that

$$0 < d_{2n+1} \leq \varphi(d_{2n+1}) d_{2n+1} < d_{2n+1},$$

which is a contradiction. Hence, we conclude that $d_{2n} > d_{2n+1}$, for each $n \in \mathbb{N}_0$.

Case (ii). By taking $x = \xi_{2n-1}$ and $y = \xi_{2n}$ in (2.1), we find that

$$\begin{aligned} 0 &< d(\xi_{2n}, \xi_{2n+1}) = d(T\xi_{2n-1}, S\xi_{2n}) \\ &\leq \varphi(E_{T,S}(\xi_{2n-1}, \xi_{2n})) E_{T,S}(\xi_{2n-1}, \xi_{2n}) \\ &= \varphi(d(\xi_{2n-1}, \xi_{2n}) + |d(\xi_{2n-1}, T\xi_{2n-1}) - d(\xi_{2n}, S\xi_{2n})|) [d(\xi_{2n-1}, \xi_{2n}) + |d(\xi_{2n-1}, T\xi_{2n-1}) - d(\xi_{2n}, S\xi_{2n})|] \\ &= \varphi(d(\xi_{2n-1}, \xi_{2n}) + |d(\xi_{2n-1}, \xi_{2n}) - d(\xi_{2n}, \xi_{2n+1})|) [d(\xi_{2n-1}, \xi_{2n}) + |d(\xi_{2n-1}, \xi_{2n}) - d(\xi_{2n}, \xi_{2n+1})|] \\ &= \varphi(d_{2n-1} + |d_{2n-1} - d_{2n}|) (d_{2n-1} + |d_{2n-1} - d_{2n}|) \\ &< d_{2n-1} + |d_{2n-1} - d_{2n}|. \end{aligned}$$

If we suppose that $d_{2n-1} \leq d_{2n}$, then we obtain

$$0 < d_{2n} < d_{2n},$$

a contradiction. Hence, we conclude that

$$d_{2n} < d_{2n-1}, \text{ for all } n \in \mathbb{N}_0.$$

By combining these two cases, we conclude that the sequence $\{d_n\} = \{d(\xi_{n+1}, \xi_n)\}$ is non-increasing. Since the sequence $\{d_n\}$ is monotone and bounded, there exists $d \geq 0$, such that $\lim_{n \rightarrow \infty} d_n = d$. We claim that $d = 0$.

Observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} E_{T,S}(\xi_{2n+1}, \xi_{2n}) &= \lim_{n \rightarrow \infty} \left(d(\xi_{2n+1}, \xi_{2n}) + |d(\xi_{2n+1}, T\xi_{2n+1}) - d(\xi_{2n}, S\xi_{2n})| \right) \\ &= \lim_{n \rightarrow \infty} \left(d(\xi_{2n+1}, \xi_{2n}) + |d(\xi_{2n+1}, \xi_{2n+2}) - d(\xi_{2n}, \xi_{2n+1})| \right) \\ &= \lim_{n \rightarrow \infty} (d_{2n} + |d_{2n+1} - d_{2n}|) = d. \end{aligned}$$

Now, consider the following inequality:

$$\begin{aligned} d(\xi_{2n+2}, \xi_{2n+1}) &= d(T\xi_{2n+1}, S\xi_{2n}) \\ &\leq \varphi(E_{T,S}(\xi_{2n+1}, \xi_{2n})) E_{T,S}(\xi_{2n+1}, \xi_{2n}). \end{aligned}$$

By taking upper limits as $n \rightarrow \infty$ in the inequality above, we get that

$$d \leq \limsup_{n \rightarrow \infty} \varphi(E_{T,S}(\xi_{2n+1}, \xi_{2n})) d \leq d.$$

So, we find that $\lim_{n \rightarrow \infty} \varphi(E_{T,S}(\xi_{2n+1}, \xi_{2n})) = 1$. Using the properties of the function φ we get

$$d = \lim_{n \rightarrow \infty} E_{T,S}(\xi_{2n+1}, \xi_{2n}) = 0 \text{ and } \lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} d(\xi_n, \xi_{n+1}) = 0. \quad (2.7)$$

Next we shall prove that the sequence $\{\xi_n\}$ is Cauchy. By using the triangle inequality we have

$$d(\xi_{2n+1}, \xi_{2m}) \leq d(\xi_{2n+1}, \xi_{2n}) + d(\xi_{2n}, \xi_{2m}).$$

Since $d(\xi_n, \xi_{n+1}) \rightarrow 0$, the above inequality implies that if the sequence $\{\xi_{2n}\}$ is Cauchy, then the sequence $\{\xi_n\}$ is Cauchy.

By contradiction, we assume that there exists $\varepsilon > 0$ and two sequences $\{n(k)\}_{k=1}^{\infty}, \{m(k)\}_{k=1}^{\infty}$ of positive integers, such that

$$n(k) > m(k) > k, \text{ for all } k \in \mathbb{N}$$

and

$$d(\xi_{2n(k)}, \xi_{2m(k)}) \geq \varepsilon, \quad d(\xi_{2n(k)-2}, \xi_{2m(k)}) < \varepsilon.$$

Thus, we have

$$\begin{aligned} \varepsilon &\leq d(\xi_{2n(k)}, \xi_{2m(k)}) \leq d(\xi_{2n(k)}, \xi_{2n(k)-2}) + d(\xi_{2n(k)-2}, \xi_{2m(k)}) \\ &< d(\xi_{2n(k)}, \xi_{2n(k)-2}) + \varepsilon. \end{aligned}$$

From (2.7) it follows that

$$\lim_{k \rightarrow \infty} d(\xi_{2n(k)}, \xi_{2m(k)}) = \varepsilon.$$

Furthermore, by using the triangle inequality again we find that

$$\lim_{k \rightarrow \infty} d(\xi_{2n(k)-1}, \xi_{2m(k)}) = \varepsilon. \quad (2.8)$$

By taking (2.7) and (2.8) into account we also find that

$$\begin{aligned} \lim_{k \rightarrow \infty} E_{T,S}(\xi_{2n(k)-1}, \xi_{2m(k)}) &= \lim_{k \rightarrow \infty} \left(d(\xi_{2n(k)-1}, \xi_{2m(k)}) + |d(\xi_{2n(k)-1}, T\xi_{2n(k)-1}) - d(\xi_{2m(k)}, S\xi_{2m(k)})| \right) \\ &= \lim_{k \rightarrow \infty} \left(d(\xi_{2n(k)-1}, \xi_{2m(k)}) + |d(\xi_{2n(k)-1}, \xi_{2n(k)}) - d(\xi_{2m(k)}, \xi_{2m(k)+1})| \right) \\ &= \varepsilon. \end{aligned}$$

Finally, from the properties of the function φ and from (2.1) it follows that

$$\begin{aligned}\varepsilon &= \lim_{k \rightarrow \infty} d(\xi_{2n(k)}, \xi_{2m(k)+1}) = \lim_{k \rightarrow \infty} d(T\xi_{2n(k)-1}, S\xi_{2m(k)}) \\ &\leq \lim_{k \rightarrow \infty} \varphi(E_{T,S}(\xi_{2n(k)-1}, \xi_{2m(k)})) \lim_{k \rightarrow \infty} (E_{T,S}(\xi_{2n(k)-1}, \xi_{2m(k)})) \\ &\leq \lim_{k \rightarrow \infty} (E_{T,S}(\xi_{2n(k)-1}, \xi_{2m(k)})) = \varepsilon.\end{aligned}$$

So, we have $\lim_{k \rightarrow \infty} \varphi(E_{T,S}(\xi_{2n(k)-1}, \xi_{2m(k)})) = 1$, and hence,

$$\varepsilon = \lim_{k \rightarrow \infty} (E_{T,S}(\xi_{2n(k)-1}, \xi_{2m(k)})) = 0.$$

It is a contradiction. Consequently, $\{\xi_n\}_{n=1}^\infty$ is a Cauchy sequence. By completeness of (M, d) the sequence $\{\xi_n\}_{n=1}^\infty$ converges to some point $\xi^* \in M$. Moreover,

$$\xi_{2n} \rightarrow \xi^* \text{ and } \xi_{2n+1} \rightarrow \xi^*. \quad (2.9)$$

Next, we shall prove that $T\xi^* = \xi^* = S\xi^*$. In the beginning, we observe that if there exists $N_1 \in \mathbb{N}$ and a subsequence $\{\xi_{2n(k)}\}_{k=1}^\infty$ of the sequence $\{\xi_n\}$, such that $\xi_{2n(k)} = \xi^*$ for all $k > N_1$, from (2.7) we have

$$0 = \lim_{k \rightarrow \infty} d(\xi_{2n(k)}, \xi_{2n(k)+1}) = \lim_{k \rightarrow \infty} d(\xi_{2n(k)}, S\xi_{2n(k)}) = d(\xi^*, S\xi^*),$$

for every $k > N_1$. Then $S\xi^* = \xi^*$. In the same way, if there exists $N_2 \in \mathbb{N}$ and a subsequence $\{\xi_{2n(k)+1}\}_{k=1}^\infty$ of the $\{\xi_n\}$, such that $\xi_{2n(k)+1} = \xi^*$ for all $k > N_2$, we obtain $T\xi^* = \xi^*$.

Otherwise, taking $\xi = \xi^*$, $v = \xi_{2n}$ in (2.1) we have

$$\begin{aligned}d(T\xi^*, \xi_{2n+1}) &= d(T\xi^*, S\xi_{2n}) \leq \varphi(E_{T,S}(\xi^*, \xi_{2n}))(E_{T,S}(\xi^*, \xi_{2n})) \\ &= \varphi(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, S\xi_{2n})|)(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, S\xi_{2n})|) \\ &= \varphi(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|)(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|).\end{aligned}$$

Letting $n \rightarrow \infty$ we obtained

$$\begin{aligned}d(T\xi^*, \xi^*) &= \limsup_{n \rightarrow \infty} d(T\xi^*, \xi_{2n+1}) \\ &\leq \limsup_{n \rightarrow \infty} [\varphi(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|)] \\ &\quad \limsup_{n \rightarrow \infty} [(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|)] \\ &\leq \limsup_{n \rightarrow \infty} (d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|) \\ &= d(\xi^*, T\xi^*).\end{aligned}$$

Hence, by (2.7) we get

$$\lim_{n \rightarrow \infty} \varphi(d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|) = 1,$$

which implies

$$d(\xi^*, T\xi^*) = \lim_{n \rightarrow \infty} (d(\xi^*, \xi_{2n}) + |d(\xi^*, T\xi^*) - d(\xi_{2n}, \xi_{2n+1})|) = 0.$$

Hence, we get that $T\xi^* = \xi^*$.

Similarly, regarding (2.7) and (2.9), we observe that

$$\begin{aligned}d(\xi_{2n}, S\xi^*) &= d(T\xi_{2n-1}, S\xi^*) \\ &\leq \varphi(E_{T,S}(\xi_{2n-1}, \xi^*))(E_{T,S}(\xi_{2n-1}, \xi^*)),\end{aligned}$$

where

$$\begin{aligned} E_{T,S}(\xi_{2n-1}, \xi^*) &= d(\xi_{2n-1}, \xi^*) + |d(\xi_{2n-1}, T\xi_{2n-1}) - d(\xi^*, S\xi^*)| \\ &= d(\xi_{2n-1}, \xi^*) + |d(\xi_{2n-1}, \xi_{2n}) - d(\xi^*, S\xi^*)| \rightarrow d(\xi^*, S\xi^*). \end{aligned}$$

$$\begin{aligned} d(\xi^*, S\xi^*) &\leq \lim_{n \rightarrow \infty} \varphi(E_{T,S}(\xi_{2n-1}, \xi^*)) d(\xi^*, S\xi^*) \\ &\leq d(\xi^*, S\xi^*). \end{aligned}$$

It yields that

$$\lim_{n \rightarrow \infty} \varphi(E_{T,S}(\xi_{2n-1}, \xi^*)) = 1 \text{ and therefore } d(\xi^*, S\xi^*) = \lim_{n \rightarrow \infty} E_{T,S}(\xi_{2n-1}, \xi^*) = 0.$$

Hence, we find that $S\xi^* = \xi^*$. Accordingly, we conclude that T and S have a common fixed point ξ^* .

Finally, we prove that the common fixed point of T and S is unique. If there exists another point $v^* \in M$, $v^* \neq \xi^*$, such that $Tv^* = v^* = Sv^*$, then from (2.1) it follows that

$$\begin{aligned} 0 &< d(\xi^*, v^*) = d(T\xi^*, Sv^*) \\ &\leq \varphi(E_{T,S}(\xi^*, v^*)) E_{T,S}(\xi^*, v^*) \\ &= \varphi(d(\xi^*, v^*) + |d(\xi^*, T\xi^*) - d(v^*, Sv^*)|) (d(\xi^*, v^*) + |d(\xi^*, T\xi^*) - d(v^*, Sv^*)|) \\ &= \varphi(d(\xi^*, v^*)) d(\xi^*, v^*) < d(\xi^*, v^*). \end{aligned}$$

This is a contradiction. Then $d(\xi^*, v^*) = 0$, that is, $\xi^* = v^*$. This proves that the common fixed point is unique. $\square$

Example 6. Let $X = [0, 1]$ be endowed with the metric $d(\xi, v) = \begin{cases} \max\{\xi, v\}, & \xi \neq v \\ 0, & \xi = v \end{cases}$.

Let $T, S : M \rightarrow M$, defined by

$$T\xi = \frac{\xi}{2}, \quad S\xi = 0.$$

Then

$$\begin{aligned} d(T\xi, Sv) &= \max\left\{\frac{\xi}{2}, 0\right\} = \frac{\xi}{2}, \quad d(\xi, v) = \max\{\xi, v\}, \\ d(\xi, T\xi) &= \max\left\{\xi, \frac{\xi}{2}\right\} = \xi, \quad d(v, Sv) = \max\{v, 0\} = v \end{aligned}$$

and

$$\begin{aligned} E_{T,S}(\xi, v) &= d(\xi, v) + |d(\xi, T\xi) - d(v, Sv)| \\ &= \max\{\xi, v\} + |\xi - v| \\ &= \begin{cases} 2\xi - v, & \xi \geq v \\ 2v - \xi, & \xi < v \end{cases}. \end{aligned}$$

Considering $\varphi : [0, \infty) \rightarrow [0, 1)$, $\varphi(t) = \begin{cases} \frac{1}{1+t}, & t > 0 \\ \frac{1}{2}, & t = 0 \end{cases}$, we have

(i) for $\xi \geq v$

$$\frac{\xi}{2} = d(T\xi, Sv) \leq \varphi(E_{T,S}(\xi, v)) E_{T,S}(\xi, v) = \frac{2\xi - v}{1 + 2\xi - v},$$

so

$$\begin{aligned}\frac{\xi}{2} &\leq \frac{2\xi - \nu}{1 + 2\xi - \nu} \\ \xi + 2\xi^2 - \xi\nu &\leq 4\xi - 2\nu \\ \nu(2 - \xi) &\leq 3\xi - 2\xi^2 \\ \nu &\leq \frac{3\xi - 2\xi^2}{2 - \xi}.\end{aligned}$$

Because $\eta \leq \xi$, we get

$$\begin{aligned}\xi &\leq \frac{3\xi - 2\xi^2}{2 - \xi} \\ 2\xi - \xi^2 &\leq 3\xi - 2\xi^2 \\ \xi(\xi - 1) &\leq 0.\end{aligned}$$

(ii) for $\xi < \nu$

$$\begin{aligned}\frac{\xi}{2} = d(T\xi, S\nu) &\leq \frac{2\eta - \xi}{1 + 2\nu - \xi} \\ \xi + 2\xi\nu - \xi^2 &\leq 4\nu - 2\xi \\ \nu &\geq \frac{3\xi - \xi^2}{4 - 2\xi}.\end{aligned}$$

But $\nu > \xi$, so we have

$$\begin{aligned}\xi &\geq \frac{3\xi - \xi^2}{4 - 2\xi} \\ 4\xi - 2\xi^2 &\geq 3\xi - \xi^2 \\ \xi(\xi - 1) &\leq 0.\end{aligned}$$

Consequently T, S satisfies Theorem 5.

3 Consequences and conclusion

In this section we give some consequences of the main results. The immediate outcome of the main result is Theorem 2 which is obtained by taking $S = T$ in Theorem 5. Moreover, letting $\varphi(t) = kt$ in Theorem 5, we derive the following result:

Theorem 7. Suppose that T, S are two self-mappings on a metric space (M, d) . Suppose also that there is a $k \in [0, 1)$, such that the inequality

$$d(T\xi, S\nu) \leq kE_{T,S}(\xi, \nu), \text{ for all } \xi, \nu \in M,$$

is satisfied, where

$$E_{T,S}(\xi, \nu) = d(\xi, \nu) + |d(\xi, T\xi) - d(\nu, S\nu)|.$$

Then the pair of the mappings T, S has a unique common fixed point ξ^* , that is, $T\xi^* = \xi^* = S\xi^*$.

The following result is derived by taking $S = T$ and $\varphi(t) = kt$ in Theorem 5.

Theorem 8. Suppose that T is a self-mapping on a complete metric space (M, d) . If there exists $\lambda \in [0, 1)$, such that

$$d(T\xi, T\nu) \leq \lambda E(\xi, \nu), \text{ for all } \xi, \nu \in M, \quad (3.1)$$

where

$$E(\xi, v) = d(\xi, v) + |d(\xi, T\xi) - d(v, Tv)|,$$

then T has a unique fixed point $\xi^* \in M$. Additionally, for any initial point $\xi_0 \in M$, the recursive sequence $\{T^n \xi_0\}_{n=1}^\infty$ converges to ξ^* .

Theorem 9 (Banach Contraction Mapping Principle). Suppose that T is self-mapping on a complete metric space (M, d) . If there exists a $k \in [0, 1)$, such that

$$d(T\xi, Tv) \leq kd(\xi, v), \text{ for all } \xi, v \in M, \quad (3.2)$$

then T has a unique fixed point $\xi^* \in M$. Moreover, for any initial point $\xi_0 \in M$, the iterative sequence $\{T^n \xi_0\}_{n=1}^\infty$ converges to ξ^* .

3.1 Conclusion

We assert that our results are an improvement of the existing results in the literature. Indeed, if we consider a simple example $Tx = x/2$ on $\mathbb{R}$ with a standard metric, we realize that the ratio of $d(T\xi_n, T\xi_{n+1})/d(\xi_n, \xi_{n+1})$ is 0.5 but the ratio of $d(T\xi_n, T\xi_{n+1})/E(\xi_n, \xi_{n+1})$ is 0.3. We also notice that in the other famous fixed point theorems (such as, Kannan Type, Chatterjee Type, Reich-Rus Type, Hardy-Rogers Type, Ćirić Type etc.) the corresponding ratios are larger than 0.3. Hence, the convergence in our case should be faster than the others.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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