Research Article

Open Access

Faruk Özger*

Characterizations of compact operators on ℓ_p -type fractional sets of sequences

<https://doi.org/10.1515/dema-2019-0015>

Received August 6, 2018; accepted January 30, 2019

Abstract: Among the sets of sequences studied, difference sets of sequences are probably the most common type of sets. This paper considers some ℓ_p -type fractional difference sets via the gamma function. Although, we characterize compactness conditions on those sets using the main key of Hausdorff measure of noncompactness, we can only obtain sufficient conditions when the final space is ℓ_∞ . However, we use some recent results to exactly characterize the classes of compact matrix operators when the final space is the set of bounded sequences.

Keywords: gamma function, fractional operator, operator norm, compact operator, Hausdorff measure of noncompactness

MSC: 46B45, 47B37

1 Introduction

The gamma function of a real number x (except zero and the negative integers) is defined by an improper integral:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

It is known that for any natural number n , $\Gamma(n+1) = n!$, and $\Gamma(n+1) = n\Gamma(n)$ hold for any real number $n \notin \{0, -1, -2, \dots\}$. The fractional difference operator for a fraction $\tilde{\alpha}$ was defined in [1] as

$$\Delta^{(\tilde{\alpha})}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i! \Gamma(\tilde{\alpha}-i+1)} x_{k-i}. \quad (1.1)$$

It is assumed that the series defined in Eq. (1.1) is convergent for $x \in \omega$. The infinite sum in Eq. (1.1) becomes a finite sum if $\tilde{\alpha}$ is a nonnegative integer. We use the usual convention that any term with a negative subscript is equal to naught, throughout the paper.

Let m be a positive integer, then recall the difference operators $\Delta^{(1)}$ and $\Delta^{(m)}$ are defined by:

$$(\Delta^{(1)}x)_k = \Delta^{(1)}x_k = x_k - x_{k-1}$$

and

$$(\Delta^{(m)}x)_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k-i}.$$

*Corresponding Author: Faruk Özger: Department of Engineering Sciences, İzmir Katip Çelebi University, 35620, İzmir, Turkey; E-mail: farukozger@gmail.com

We write Δ and $\Delta^{(m)}$ for the matrices with $\Delta_{nk} = (\Delta^{(1)} e^{(k)})_n$ and $\Delta_{nk}^{(m)} = (\Delta^{(m)} e^{(k)})_n$ for all n and k . The topological properties of some spaces that are constructed by the matrix operator $\Delta^{(m)}$ were studied in the paper [2]. Some identities and estimates for the Hausdorff measure of noncompactness of matrix operators from $\Delta^{(m)}$ -type spaces into the sets of bounded ℓ_∞ , convergent c , null sequences c_0 and also absolutely convergent series were established in [3].

We can write the fractional difference operator defined in Eq. (1.1) as an infinite matrix:

$$\Delta_{nk}^{(\tilde{\alpha})} = \begin{cases} (-1)^{n-k} \frac{\Gamma(\tilde{\alpha}+1)}{(n-k)! \Gamma(\tilde{\alpha}-n+k+1)} & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases}$$

Remark 1.1. [1] The inverse of fractional difference matrix is given by

$$\Delta_{nk}^{(-\tilde{\alpha})} = \begin{cases} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-k)! \Gamma(-\tilde{\alpha}-n+k+1)} & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases}$$

For some values of $\tilde{\alpha}$, we have

$$\begin{aligned} \Delta^{1/2} x_k &= x_k - \frac{1}{2} x_{k-1} - \frac{1}{8} x_{k-2} - \frac{1}{16} x_{k-3} - \frac{5}{128} x_{k-4} - \cdots, \\ \Delta^{-1/2} x_k &= x_k + \frac{1}{2} x_{k-1} + \frac{3}{8} x_{k-2} + \frac{5}{16} x_{k-3} + \frac{35}{128} x_{k-4} + \cdots, \\ \Delta^{2/3} x_k &= x_k - \frac{2}{3} x_{k-1} - \frac{1}{9} x_{k-2} - \frac{4}{81} x_{k-3} - \frac{7}{243} x_{k-4} - \cdots. \end{aligned}$$

The idea of constructing new sets of sequences via infinite matrices started with [4] and then it has been developed by numerous researchers using different triangles [5–10]. In the papers [11–19] different difference sets of sequences have been studied based on some newly defined infinite matrices. Some new results on the visualization and animations of the topologies of certain sets were illustrated in [20–22]. The authors applied their software package for this purpose. Those results have an interesting and important applications in crystallography. The fractional Banach sets of difference sequences $\ell(\Delta^{(\tilde{\alpha})}, p)$ were geometrically characterized and the modular structure of those sets were investigated in [23].

Many authors have made efforts to apply Hausdorff measure of noncompactness to find compactness conditions of certain sets of sequences during the past decade [24–27]. Note that, necessary and sufficient compactness conditions for a matrix operator from fractional sets of sequences $c_0(\Delta^{(\tilde{\alpha})})$, $c(\Delta^{(\tilde{\alpha})})$ and $\ell_\infty(\Delta^{(\tilde{\alpha})})$ to the classical sets of sequences have been very recently determined in [28].

Fractional difference sequence spaces have been studied in the literature recently [1, 29, 30]. The authors of those papers especially studied the properties of fractional operators in their research in addition to focusing on certain fractional sequence spaces.

In this work, we consider the fractional sets of sequences $\ell_p(\Delta^{(\tilde{\alpha})})$ for $1 \leq p \leq \infty$ and determine operator norms of our spaces. We establish some identities and estimates for the Hausdorff measures of noncompactness of certain operators on the sets $\ell_p(\Delta^{(\tilde{\alpha})})$ of fractional orders. We characterize some classes of compact operators on these sets. Note that, we can only obtain sufficient conditions when the final space is ℓ_∞ . However, we use the results of [31, 32] to obtain necessary and sufficient conditions for the classes of compact matrix operators from $\ell_p(\Delta^{(\tilde{\alpha})})$ spaces into the sets of bounded sequences and for the classes $(\ell_1(\Delta^{(\tilde{\alpha})}), \ell_\infty)$ and $(\ell_\infty(\Delta^{(\tilde{\alpha})}), \ell_\infty)$.

2 Preliminaries

The β dual of a set X is defined by

$$X^\beta = \{a \in \omega : a \cdot x \in cs \text{ for all } x \in X\}.$$

Note that $c_0^\beta = c^\beta = \ell_\infty^\beta = \ell_1$ and $\ell_p^\beta = \ell_q$. Here q is the conjugate of p , it means $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$ and $q = 1$ for $p = \infty$.

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers and any sequence x , we write $A_n = (a_{nk})_{k=0}^\infty$ for the sequence in the n^{th} row of A , $A_n x = \sum_{k=0}^\infty a_{nk} x_k$ ($n = 0, 1, \dots$) and $Ax = (A_n x)_{n=0}^\infty$, provided $A_n \in X^\beta$ for all n .

If X and Y are subsets of ω , then

$$X_A = \{x \in \omega : Ax \in X\}$$

denotes the matrix domain of A in X and (X, Y) is the class of all infinite matrices that map X into Y ; so $A \in (X, Y)$ if and only if $X \subset Y_A$.

An infinite matrix $T = (t_{nk})$ is called a triangle if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$. We denote its inverse by S . A BK space is a Banach space with continuous coordinates. A BK space X is said to have AK if every sequence $x \in X$ has the unique representation $x = \sum_k x_k e^{(k)}$.

Consider the following fractional difference sets of sequences for $1 \leq p < \infty$:

$$\ell_p(\Delta^{(\tilde{\alpha})}) := \left\{ x = (x_k) \in \omega : \sum_{n=0}^\infty \left| \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i! \Gamma(\tilde{\alpha}-i+1)} x_{n-i} \right|^p < \infty \right\}.$$

Let us define the sequence $y = (y_k)$ which will be used, by the $\Delta^{(\tilde{\alpha})}$ -transform of a sequence $x = (x_k)$, that is,

$$\begin{aligned} y_k &= x_k - \tilde{\alpha} x_{k-1} + \frac{\tilde{\alpha}(\tilde{\alpha}-1)}{2!} x_{k-2} - \frac{\tilde{\alpha}(\tilde{\alpha}-1)(\tilde{\alpha}-2)}{3!} x_{k-3} + \dots \\ &= \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i! \Gamma(\tilde{\alpha}-i+1)} x_{k-i}. \end{aligned}$$

It seems those spaces can be considered as the matrix domains of the triangle $\Delta^{(\tilde{\alpha})}$ in the classical sequence spaces ℓ_p , where $1 \leq p < \infty$. We also have the following relation between the sequences $x = (x_k)$ and $y = (y_k)$:

$$x_k = \sum_{i=0}^\infty (-1)^i \frac{\Gamma(-\tilde{\alpha}+1)}{i! \Gamma(-\tilde{\alpha}-i+1)} y_{k-i}. \quad (2.1)$$

Lemma 2.1. [33, Theorem 4.3.12, p. 63] Let $(X, \|\cdot\|)$ be a BK space. Then X_T is a BK space with $\|\cdot\|_T = \|T(\cdot)\|$.

By Lemma 2.1, the defined fractional difference sequence spaces are complete, linear, BK spaces with the following norm:

$$\|x\| = \left(\sum_n \left| \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i! \Gamma(\tilde{\alpha}-i+1)} x_{n-i} \right|^p \right)^{\frac{1}{p}}.$$

Let X be a normed space. Then $S_X = \{x \in X : \|x\| = 1\}$ and $\bar{B}_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and closed unit ball in X , where X is a normed space.

By $\mathcal{F}_r$ ($r = 0, 1, \dots$), we denote the subcollection of $\mathcal{F}$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with terms that are greater than r , that is

$$\mathcal{F}_r = \{N \in \mathcal{F} : n > r \text{ for all } n \in N\}, \quad (r = 0, 1, \dots).$$

Given $a \in \omega$, we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=1}^\infty a_k x_k \right|$$

provided the expression on the right hand side is defined and finite which is the case whenever X is a BK space and $a \in X^\beta$.

Lemma 2.2. Let X and Y be BK spaces.

- (i) Then, we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ for all $x \in X$ (see [34, Theorem 1.23]).

- (ii) We have $\|x\|_X^* = \|x\|_{X^\beta}$ for all $x \in X^\beta$, where $\|\cdot\|_{X^\beta}$ is the natural norm on the dual set X^β (see [35, Theorem 3.2]).

Lemma 2.3. Let Y be an arbitrary subset of ω and X be a BK space with AK or $X = \ell_\infty$, and $R = S^t$. Then $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all $n = 0, 1, \dots$. Here $\hat{A}$ is the matrix with rows $\hat{A}_n = RA_n$ for $n = 0, 1, \dots$, and the triangles $W^{(n)}$ ($n = 0, 1, \dots$) are defined by $w_{mk}^{(n)} = \sum_{j=m}^{\infty} a_{nj} s_{jk}$. Moreover, if $A \in (X, Y)$ then we have $Az = \hat{A}(Tz)$ for all $z \in Z = X_T$ (see [35, Theorem 3.4, Remark 3.5(a)]).

We have the following results for operator norms of bounded operators following [36, Theorem 2.8].

Lemma 2.4. Let X be a BK space.

- (i) If $A \in (X, \ell_1)$, then

$$\|A\|_{(X, \ell_1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(X, \ell_1)},$$

where

$$\|A\|_{(X, \ell_1)} = \sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} A_n \right\|_X^* < \infty.$$

- (ii) If $A \in (X, Y)$, then

$$\|L_A\| = \|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* < \infty,$$

where Y is any of the spaces c_0, c, ℓ_∞ .

We obtain the following result as an immediate consequence of Lemma 2.2(ii).

Lemma 2.5. Let $1 \leq p < \infty$, then we have $\|x\|_{\ell_p}^* = \|x\|_{\ell_q}$ for all $x = (x_k) \in \ell_q$.

Remark 2.6. Let $1 \leq p < \infty$, A be an infinite matrix and X be a BK space. If $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), Y)$, then $\hat{A} \in (\ell_p, Y)$ such that $Ax = \hat{A}y$ for all $x \in \ell_p(\Delta^{(\tilde{\alpha})})$ and $y \in \ell_p$. Here x and y are connected by Eq. (2.1) and $\hat{A} = (\hat{a}_{nk})$ is defined by

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha} + 1)}{(j-k)! \Gamma(-\tilde{\alpha} + j - k + 1)} a_{nj} \text{ for all } n, k \in \mathbb{N}_0; \quad (2.2)$$

and for $k = 0, 1, \dots$ we also define $\hat{\alpha} = (\hat{\alpha}_k)_{k=0}^\infty$ by

$$\hat{\alpha}_k = \lim_{n \rightarrow \infty} \hat{a}_{nk}. \quad (2.3)$$

Proof. Let $1 \leq p < \infty$, $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), Y)$ and $x \in \ell_p(\Delta^{(\tilde{\alpha})})$. So, $A_n \in (\ell_p(\Delta^{(\tilde{\alpha})}))^\beta$ for all $n = 0, 1, \dots$. Then, we have $\hat{A}_n \in \ell_p^\beta = \ell_q$ for all $n = 0, 1, \dots$ and $Ax = \hat{A}y$ is satisfied. Therefore, $\hat{A}y \in Y$ and this means $\hat{A} \in (\ell_p, Y)$. $\square$

Lemma 2.7. If $a = (a_k) \in (\ell_p(\Delta^{(\tilde{\alpha})}))^\beta$, then $\bar{a} = (\bar{a}_k) \in \ell_q$ and the equality

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \bar{a}_k y_k \quad (2.4)$$

is satisfied for every $x = (x_k) \in \ell_p(\Delta^{(\tilde{\alpha})})$, where $1 \leq p \leq \infty$ and

$$\bar{a}_k = \sum_{i=k}^{\infty} (-1)^{i-k} \frac{\Gamma(-\tilde{\alpha} + 1)}{(i-k)! \Gamma(-\tilde{\alpha} - i + k + 1)} a_i. \quad (2.5)$$

Let us now denote $S = S_{\ell_p}$ and $\hat{S} = S_{\ell_p(\Delta^{(\tilde{\alpha})})}$ for the sake of brevity.

Lemma 2.8. Let $1 \leq p < \infty$ and $\bar{a} = (\bar{a}_k)$ be defined as in Eq. (2.5), then we have

$$\|a\|_{\ell_p(\Delta^{(\tilde{a})})}^* = \|\bar{a}\|_{\ell_q} = \begin{cases} \left(\sum_{k=0}^{\infty} |\bar{a}_k|^q \right)^{\frac{1}{q}} & (1 < p < \infty), \\ \sup_k |\bar{a}_k| & (p = 1), \end{cases}$$

for all $a = (a_k) \in (\ell_p(\Delta^{(\tilde{a})}))^\beta$.

Proof. Assume that $\bar{a} \in (\ell_p(\Delta^{(\tilde{a})}))^\beta$. So, we have by Lemma 2.7 that $\bar{a} \in \ell_q$ and the Eq. (2.4) is satisfied for all sequences $x = (x_k) \in \ell_p((\Delta^{(\tilde{a})}))$ and $y = (y_k) \in \ell_p$ which are connected by Eq. (2.1). Additionally, we have by Lemma 2.5 that $x \in \hat{S}$ if and only if $y \in S$. Hence, we have by Eq. (2.4) that

$$\|a\|_{\ell_p(\Delta^{(\tilde{a})})}^* = \sup_{x \in \hat{S}} \left| \sum_{k=0}^{\infty} a_k x_k \right| = \sup_{y \in S} \left| \sum_{k=0}^{\infty} \bar{a}_k y_k \right| = \|\bar{a}\|_{\ell_p}^* < \infty. \quad (2.6)$$

Moreover, we obtain by Eq. (2.6) and Lemma 2.5 that

$$\|a\|_{\ell_p(\Delta^{(\tilde{a})})}^* = \|\bar{a}\|_{\ell_p}^* = \|\bar{a}\|_{\ell_q}$$

because $\bar{a} \in \ell_q$. This completes the proof. $\square$

We obtain estimates and identities for the norms of the matrix operators for the classes $(\ell_p(\Delta^{(\tilde{a})}), c_0)$, $(\ell_p(\Delta^{(\tilde{a})}), c)$, $(\ell_p(\Delta^{(\tilde{a})}), \ell_\infty)$ and $(\ell_p(\Delta^{(\tilde{a})}), \ell_1)$.

Theorem 2.9. Let A be in any of the classes $(\ell_p(\Delta^{(\tilde{a})}), c_0)$, $(\ell_p(\Delta^{(\tilde{a})}), c)$ or $(\ell_p(\Delta^{(\tilde{a})}), \ell_\infty)$, then

$$\|L_A\| = \|A\|_{(\ell_p(\Delta^{(\tilde{a})}), \ell_\infty)},$$

where $1 \leq p < \infty$ and

$$\|A\|_{(\ell_p(\Delta^{(\tilde{a})}), \ell_\infty)} = \sup_n \left(\sum_{k=0}^{\infty} |\hat{a}_{nk}|^q \right)^{\frac{1}{q}} < \infty.$$

Proof. This is an immediate consequence of Lemma 2.2 and Lemma 2.4. $\square$

Theorem 2.10. Let $1 \leq p < \infty$ and $A \in (\ell_p(\Delta^{(\tilde{a})}), \ell_1)$, then

$$\|A\|_{(\ell_p(\Delta^{(\tilde{a})}), \ell_1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(\ell_p(\Delta^{(\tilde{a})}), \ell_1)},$$

where

$$\|A\|_{(\ell_p(\Delta^{(\tilde{a})}), \ell_1)} = \sup_{N \in \mathcal{F}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \hat{a}_{nk} \right| \right) < \infty.$$

Proof. This is an immediate consequence of Lemma 2.2 and Lemma 2.4. $\square$

3 Main results related to compact operators

The following notations are needed to establish estimates and identities for the Hausdorff measure of non-compactness of matrix operators and characterize the classes of compact operators. We also use the results in Katarina's paper [31] to prove our results.

We recall the definitions of Hausdorff measure of noncompactness of bounded subsets of a metric space, and Hausdorff measure of noncompactness of operators between Banach spaces.

If X and Y are infinite-dimensional complex Banach spaces then a linear operator $L : X \rightarrow Y$ is said to be compact if the domain of L is all of X , and, for every bounded sequence (x_n) in X , the sequence $(L(x_n))$ has a convergent subsequence. We denote the class of such operators by $\mathcal{C}(X, Y)$.

Definition 3.1. Let (X, d) be a metric space, $B(x_0, \delta) = \{x \in X : d(x, x_0) < \delta\}$ denote the open ball of radius $\delta > 0$ and center in $x_0 \in X$, and $\mathcal{M}_X$ be the collection of bounded sets in X . The Hausdorff measure of noncompactness of $Q \in \mathcal{M}_X$ is

$$\chi(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{k=1}^n B(x_k, \delta_k), x_k \in X, \delta_k < \epsilon, 1 \leq k \leq n, n \in \mathbb{N} \right\}.$$

Let X and Y be Banach spaces and χ_1 and χ_2 be measures of noncompactness on X and Y . Then the operator $L : X \rightarrow Y$ is called (χ_1, χ_2) -bounded if $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$ and there exists a positive constant C such that

$$\chi_2(L(Q)) \leq C\chi_1(Q) \quad (3.1)$$

for every $Q \in \mathcal{M}_X$. If an operator L is (χ_1, χ_2) -bounded then the number

$$\|L\|_{(\chi_1, \chi_2)} = \inf \{C \geq 0 : \text{Eq. (3.1) holds for all } Q \in \mathcal{M}_X\}$$

is called the (χ_1, χ_2) -measure of noncompactness of L . In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_\chi$ instead of $\|L\|_{(\chi, \chi)}$.

Lemma 3.2. [34, Corollary 2.26] Let X and Y be Banach spaces and $L \in \mathcal{B}(X, Y)$. Then, we have

$$\|L\|_\chi = \chi(L(\bar{B}_X)) = \chi(L(S_X)),$$

and

$$L \in \mathcal{C}(X, Y) \text{ if and only if } \|L\|_\chi = 0.$$

Lemma 3.3. [34, Theorem 2.23] Let X be a Banach space with Schauder basis $(b_n)_{n=0}^\infty$, $Q \in \mathcal{M}_X$, $P_n : X \rightarrow X$ be the projector onto the linear span of $\{b_0, b_1, \dots, b_n\}$ and let I be the identity map on X and $R_n = I - P_n$ ($n = 0, 1, \dots$). Then, we have

$$\frac{1}{a} \cdot \limsup_{n \rightarrow \infty} \left(\sup_{x \in Q} \|R_n(x)\| \right) \leq \chi(Q) \leq \limsup_{n \rightarrow \infty} \left(\sup_{x \in Q} \|R_n(x)\| \right),$$

where $a = \limsup_{n \rightarrow \infty} \|R_n\|$.

Lemma 3.4. [34, Theorem 2.8] Let Q be a bounded subset of the normed space X , where X is ℓ_p for $1 \leq p < \infty$ or c_0 . If $P_n : X \rightarrow X$ is the operator defined by $P_n(x) = x^{[n]}$ for $x = (x_k)_{k=0}^\infty \in X$, then we have

$$\chi(Q) = \lim_n \left(\sup_{x \in Q} \|R_n(x)\| \right).$$

When the first space is ℓ_1 , a problem takes place since β dual of ℓ_1 is ℓ_∞ , which has no AK. In this case, the following study of Sargent [32] is used in order to characterize compact operators.

Lemma 3.5. [32, Theorem 5] If $L \in \mathcal{B}(\ell_1, \ell_\infty)$, then L is compact if and only if

$$\lim_{m \rightarrow \infty} \sup_{1 \leq n \leq m} |a_{n, k_1} - a_{n, k_2}| = \sup_n |a_{n, k_1} - a_{n, k_2}|$$

uniformly in k_1 and k_2 , ($1 \leq k_1, k_2 < \infty$).

The following results (by [36, Corollary 3.6] and [27, Theorems 3.7 and 3.11]) are needed to determine estimates for the norms of continuous linear operators L_A on our spaces and establish the necessary and sufficient conditions for a matrix operator to be a compact operator.

Lemma 3.6. Let X be a BK space.

(i) If $A \in (X, c_0)$, then we have

$$\|L_A\|_X = \limsup_{n \rightarrow \infty} \|A_n\|_X^*.$$

(ii) If $A \in (X, c)$, then we have

$$\frac{1}{2} \cdot \limsup_{n \rightarrow \infty} \|A_n - \alpha\|_X^* \leq \|L_A\|_X \leq \limsup_{n \rightarrow \infty} \|A_n - \alpha\|_X^*.$$

(iii) If $A \in (X, \ell_1)$, then we have

$$\lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} A_n \right\|_X^* \right) \leq \|L_A\|_X \leq 4 \cdot \lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} A_n \right\|_X^* \right).$$

We are now ready to state the main results related to compact operators. We start with establishing some estimates for the norms of bounded linear operators L_A on the given fractional sequence spaces $\ell_p(\Delta^{(\tilde{\alpha})})$.

Theorem 3.7. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$.

(i) If $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), c_0)$, then we have

$$\|L_A\|_X = \lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sum_k |\hat{a}_{nk}|^q \right)^{\frac{1}{q}}. \quad (3.2)$$

(ii) If $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), c)$, then we have

$$\frac{1}{2} \cdot \lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sum_k |\hat{a}_{nk} - \hat{a}_k|^q \right)^{\frac{1}{q}} \leq \|L_A\|_X \leq \lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sum_k |\hat{a}_{nk} - \hat{a}_k|^q \right)^{\frac{1}{q}}.$$

(iii) If $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), \ell_1)$, then we have

$$\lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \hat{a}_{nk} \right|^q \right)^{\frac{1}{q}} \right) \leq \|L_A\|_X \leq 4 \cdot \lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \hat{a}_{nk} \right|^q \right)^{\frac{1}{q}} \right). \quad (3.3)$$

Proof. Let $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), c_0)$. By Lemma 2.8, we have

$$\|A\|_{\ell_p(\Delta^{(\tilde{\alpha})})}^* = \|\hat{A}_n\|_{\ell_q} = \left(\sum_{k=0}^{\infty} |\hat{a}_{nk}|^q \right)^{\frac{1}{q}}$$

for all $n = 0, 1, \dots$, because $A_n \in (\ell_p(\Delta^{(\tilde{\alpha})}))^\beta$ for all $n = 0, 1, \dots$. Taking into account this result and Lemma 3.6(i) we obtain Eq. (3.2).

In order to prove the second part, we start with $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), c)$. By Remark 2.2 $\hat{A} \in (\ell_p, c)$. So, we have by Lemma 3.6(ii) that

$$\frac{1}{2} \cdot \limsup_{n \rightarrow \infty} \|\hat{A}_n - \hat{\alpha}\|_{\ell_q} \leq \|L_A\|_X \leq \limsup_{n \rightarrow \infty} \|\hat{A}_n - \hat{\alpha}\|_{\ell_q},$$

where $\hat{\alpha}$ defined in Eq. (2.3). Hence, by Lemma 3.2, we get

$$\|L_A\|_X = \chi(L_A(\hat{S})) = \chi(\hat{A}\hat{S}) \quad (3.4)$$

and

$$\|L_{\hat{A}}\|_X = \chi(L_{\hat{A}}(S)) = \chi(\hat{A}S). \quad (3.5)$$

Moreover, $x \in \hat{S}$ if and only if $y \in S$. We have $A\hat{S} = \hat{A}S$ because $Ax = \hat{A}y$ by Remark 2.2. Taking into account the Eqs. (3.4) and (3.5), we conclude that $\|L_A\|_X = \|L_{\hat{A}}\|_X$, which completes the second part of the proof.

Finally, let $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), \ell_1)$. We derive from Lemma 2.6 that

$$\left\| \sum_{n \in N} A_n \right\|_{\ell_p(\Delta^{(\tilde{\alpha})})}^* = \left\| \sum_{n \in N} \hat{A}_n \right\|_{\ell_q}, \quad (3.6)$$

because $A_n \in (\ell_p(\Delta^{(\tilde{\alpha})}))^\beta$. Hence, we obtain the condition in Eq. (3.3) by the help of Eq. (3.6) and Lemma 3.6(iii). $\square$

Theorem 3.8. Let $X = \ell_1(\Delta^{(\tilde{\alpha})})$.

(i) If $A \in (X, c_0)$, then we have

$$\|L_A\|_X = \lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sup_k |\hat{a}_{nk}| \right).$$

(ii) If $A \in (X, c)$, then we have

$$\frac{1}{2} \cdot \lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sup_k |\hat{a}_{nk} - \hat{\alpha}_k| \right) \leq \|L_A\|_X \leq \lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sup_k |\hat{a}_{nk} - \hat{\alpha}_k| \right).$$

(iii) If $A \in (X, \ell_1)$, then we have

$$\|L_A\|_X = \lim_{r \rightarrow \infty} \sup_k \sum_{n=r}^{\infty} |\hat{a}_{nk}|.$$

Proof. This is an immediate consequence of Theorem 2.9, Lemma 3.6 and Theorem 3.7. $\square$

We now use the results of [31] and [32] to obtain the necessary and sufficient conditions for the classes of compact matrix operators in $(\ell_p(\Delta^{(\tilde{\alpha})}), \ell_\infty)$, $(\ell_1(\Delta^{(\tilde{\alpha})}), \ell_\infty)$ and $(\ell_\infty(\Delta^{(\tilde{\alpha})}), \ell_\infty)$.

Theorem 3.9. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$.

(i) If $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), \ell_\infty)$, then L_A is compact if and only if

$$\lim_{r \rightarrow \infty} \sup_n \left(\sum_{k=r+1}^{\infty} |\hat{a}_{nk}|^q \right)^{\frac{1}{q}} = 0.$$

(ii) If $A \in (\ell_1(\Delta^{(\tilde{\alpha})}), \ell_\infty)$, then L_A is compact if and only if

$$\lim_{m \rightarrow \infty} \sup_{1 \leq n \leq m} |\hat{a}_{n,k_1} - \hat{a}_{n,k_2}| = \sup_n |\hat{a}_{n,k_1} - \hat{a}_{n,k_2}|$$

uniformly in k_1 and k_2 ($1 \leq k_1, k_2 < \infty$).

Proof. Assume $A \in (\ell_p(\Delta^{(\tilde{\alpha})}), \ell_\infty)$. By Lemma 2.3, we obtain $L_A \in \mathcal{B}(\ell_p(\Delta^{(\tilde{\alpha})}), \ell_\infty)$ and $\hat{A} \in (\ell_p, \ell_\infty)$ such that $Ax = \hat{A}(\Delta^{(-\tilde{\alpha})}x)$ for all $x \in \ell_p(\Delta^{(\tilde{\alpha})})$, where $\Delta^{(-\tilde{\alpha})}$ is the inverse of the triangle $\Delta^{(\tilde{\alpha})}$ and $\hat{a}_{nk}$ defined in Eq. (2.2). We have $L_{\hat{A}} \in \mathcal{B}(\ell_p, \ell_\infty)$ and $\hat{A}x = L_{\hat{A}}(x)$ for all $x \in \ell_p$ because ℓ_p is a BK space. If we write $(\hat{A}_n)_r$ for the r th row of the matrix $\hat{A}$ with $\hat{a}_{nk}$ replaced by 0 for $k > r$,

$$(\hat{A}_n)_r = \sum_{k=0}^r \hat{a}_{nk} e^{(k)}$$

and applying [32, (b), p. 85], we obtain $L_{\hat{A}}$ is compact if and only if

$$\sup_n \|\hat{A}_n\|_q = \sup_n \left(\sum_k |\hat{a}_{nk}|^q \right)^{\frac{1}{q}} < \infty$$

and

$$\lim_{r \rightarrow \infty} \sup_n \|\hat{A}_n - (\hat{A}_n)_r\|_q = \lim_{r \rightarrow \infty} \sup_n \left(\sum_{k=r+1}^{\infty} |\hat{a}_{nk}|^q \right)^{\frac{1}{q}} = 0.$$

The first condition is satisfied from the characterizations of matrix transformations, [33, Example 8.4.5D and Example 8.4.6D]. Hence, $L_{\hat{A}}$ is compact if and only if the condition

$$\lim_{r \rightarrow \infty} \sup_n \left(\sum_{k=r+1}^{\infty} |\hat{a}_{nk}|^q \right)^{\frac{1}{q}} = 0$$

is satisfied. Taking into account the equalities (3.4) and (3.5), we have L_A is compact if and only if $L_{\hat{A}}$ is compact. This completes the first part of the proof.

We know that L_A is compact if and only if $L_{\hat{A}}$ is compact with the equalities (3.4) and (3.5). Also $A \in (\ell_1(\Delta^{(\tilde{\alpha})}), \ell_{\infty})$ implies $\hat{A} \in (\ell_1, \ell_{\infty})$. As an immediate consequence of Lemma 3.5 we prove the second part of the theorem. $\square$

Theorem 3.10. *Let $A \in (\ell_{\infty}(\Delta^{(\tilde{\alpha})}), \ell_{\infty})$. Then L_A is compact if and only if*

$$\lim_{r \rightarrow \infty} \sup_n \sum_{k=r+1}^{\infty} |\hat{a}_{nk}| = 0. \quad (3.7)$$

Proof. By the help of Lemma 2.3, we have $\hat{A} \in (\ell_{\infty}, \ell_{\infty})$ because ℓ_{∞} is a BK space and also by Lemma 2.2 the associated operator $L_{\hat{A}}$ is bounded and linear. Since $(\ell_{\infty})^{\beta} = \ell_1$, by the condition (b) in [32, p. 85] we have: the spaces ℓ_{∞} and c_0 have the same dual space ℓ_1 , and therefore the conditions [32, (b) and (f), p. 85] are the same. It follows that $L_{\hat{A}}$ is compact if and only if

$$\sup_n \|\hat{A}_n\|_1 = \sup_n \sum_{k=0}^{\infty} |\hat{a}_{nk}| < \infty$$

and

$$\lim_{r \rightarrow \infty} \sup_n \|\hat{A}_n - (\hat{A}_n)_r\|_1 = \lim_{r \rightarrow \infty} \sup_n \sum_{k=r+1}^{\infty} |\hat{a}_{nk}| = 0.$$

So, the first condition is satisfied [33, Example 8.4.5A], and it follows that $L_{\hat{A}}$ is compact if and only if the condition in Eq. (3.7) is satisfied. This completes the proof because L_A is compact if and only if $L_{\hat{A}}$ is compact by Lemma 2.2 and by the equalities of (3.4) and (3.5). $\square$

4 Conclusion

The main objective of this paper is to consider fractional sequence spaces $\ell_p(\Delta^{(\tilde{\alpha})})$ for $1 \leq p \leq \infty$ via the gamma function. We derive some estimates and identities for the norms of bounded linear operators on fractional sequence spaces $\ell_p(\Delta^{(\tilde{\alpha})})$. In general, Hausdorff measure of noncompactness is used to find necessary and sufficient conditions for a matrix operator on a given sequence space to be a compact operator. However, we can only obtain sufficient conditions when the final space is ℓ_{∞} , that is for a matrix class (X, ℓ_{∞}) . This is why we use the results of [31] and [32] to exactly obtain necessary and sufficient conditions for the classes of compact matrix operators in $(\ell_p(\Delta^{(\tilde{\alpha})}), \ell_{\infty})$ for $1 \leq p \leq \infty$. In addition to these characterizations, we apply Hausdorff measure of noncompactness to establish necessary and sufficient conditions for a matrix operator to be a compact operator from fractional sequence spaces (X, Y) , where $X = \ell_p(\Delta^{(\tilde{\alpha})})$ or $X = \ell_1(\Delta^{(\tilde{\alpha})})$ ($1 < p < \infty$) and Y is any of the spaces c_0 , c and ℓ_1 .

All these results are given with the following table:

1. $\lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} \right|^q \right)^{\frac{1}{q}} = 0.$
2. $\lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} - \lim_{n \rightarrow \infty} \hat{a}_{nk} \right|^q \right)^{\frac{1}{q}} = 0.$

Table 1: Necessary and sufficient conditions for an operator to be compact for $1 < p < \infty$.

	$\ell_p(\Delta^{(\tilde{\alpha})})$	$\ell_1(\Delta^{(\tilde{\alpha})})$
c_0	1.	5.
c	2.	6.
ℓ_∞	3.	7.
ℓ_1	4.	8.

3. $\lim_{r \rightarrow \infty} \sup_n \left(\sum_{k=r+1}^{\infty} \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} \right|^q \right)^{\frac{1}{q}} = 0.$
4. $\lim_{r \rightarrow \infty} \sup_{N \in \mathcal{F}_r} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} \right|^q \right)^{\frac{1}{q}} = 0.$
5. $\lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sup_k \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} \right| \right) = 0.$
6. $\lim_{r \rightarrow \infty} \sup_{n \geq r} \left(\sup_k \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} - \lim_{n \rightarrow \infty} \hat{a}_{nk} \right| \right) = 0.$
7. $\lim_{m \rightarrow \infty} \sup_{1 \leq n \leq m} |\hat{a}_{n,k_1} - \hat{a}_{n,k_2}| = \sup_n |\hat{a}_{n,k_1} - \hat{a}_{n,k_2}|, \text{ uniformly in } k_1 \text{ and } k_2, (1 \leq k_1, k_2 < \infty).$
8. $\lim_{r \rightarrow \infty} \sum_k \sum_{n=r}^{\infty} \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)! \Gamma(-\tilde{\alpha}+j-k+1)} a_{nj} \right| = 0.$

Acknowledgements: This work is part of the research project with project number 2015-GAP-MÜMF-0017 supported by İzmir Katip Çelebi University Scientific Research Project Coordination Unit.

I am thankful to the reviewers for their valuable suggestions leading to the overall improvement of the paper.

References

- [1] Baliarsingh P., Some new difference sequence spaces of fractional order and their dual spaces, *Appl. Math. Comput.*, 2015, 219(18), 9737–9742
- [2] Çolak R., Et M., On some generalized difference sequence spaces and related matrix transformations, *Hokkaido Math. J.*, 1997, 26(3), 483–492
- [3] Djolović I., Malkowsky E., Matrix transformations and compact operators on some new m th order difference sequences, *Appl. Math. Comput.*, 2008, 198(2), 700–714
- [4] Kızmaz H., On certain sequence spaces, *Canad. Math. Bull.*, 1981, 24(2), 169–176
- [5] Alotaibi A., Raj K., Alkhaldi A. H., Mohiuddine S. A., Lacunary sequence spaces defined by Euler transform and Orlicz functions, *J. Comput. Anal. Appl.*, 2019, 27(5), 770–780
- [6] Karaisa K., Özger F., Almost difference sequence space derived by using a generalized weighted mean, *J. Comput. Anal. Appl.*, 2015, 19(1), 27–38
- [7] Kama R., Altay B., Weakly unconditionally Cauchy series and Fibonacci sequence spaces, *J. Inequal. Appl.*, 2017, 2017(133), DOI: 10.1186/s13660-017-1407-y
- [8] Malkowsky E., Özger F., Veličković V., Some mixed paranorm spaces, *Filomat*, 2017, 31(4), 1079–1098
- [9] Malkowsky E., Özger F., A note on some sequence spaces of weighted means, *Filomat*, 2012, 26(3), 511–518
- [10] Nuray F., Ulusu U., DüNDAR E., Lacunary statistical convergence of double sequences of sets, *Soft Comput.*, 2016, 20, 2883–2888
- [11] Karaisa K., Özger F., On almost convergence and difference sequence spaces of order m with core theorems, *Gen. Math. Notes*, 2015, 26(1), 102–125
- [12] Aydın C., Başar F., Some new difference sequence spaces, *Appl. Math. Comput.*, 2004, 157(3), 677–693
- [13] Özger F., Başar F., Domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$ on some Maddox's spaces, *AIP Conf. Proc.*, 2012, 1470, 152–155

- [14] Özger F., Başar F., Domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$ on some Maddox's spaces, *Acta Math. Sci. Ser. B Engl. Ed.*, 2014, 34(2), 394–408
- [15] Polat H., Altay B., On some new Euler difference sequence spaces, *Southeast Asian Bull. Math.*, 2006, 30, 209–220
- [16] Kilinc G., Candan M., Some generalized Fibonacci difference sequence spaces defined by a sequence of modulus functions, *Facta Univ. Ser. Math. Inform.*, 2017, 32(1), 95–115
- [17] Kirişçi M., Başar F., Some new sequence spaces derived by the domain of generalized difference matrix, *Comput. Math. Appl.*, 2010, 60(5), 1299–1309
- [18] Mursaleen M., Karakaya V., Polat H., Şimşek N., Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means, *Comput. Math. Appl.*, 2011, 62(2), 814–820
- [19] Yavuz E., Talo Ö., Abel summability of sequences of fuzzy numbers, *Soft Comput.*, 2016, 20, 1041–1046
- [20] Malkowsky E., Özger F., Veličković V., Some spaces related to Cesaro sequence spaces and an application to crystallography, *MATCH Commun. Math. Comput. Chem.*, 2013, 70(3), 867–884
- [21] Malkowsky E., Özger F., Veličković V., Matrix transformations on mixed paranorm spaces, *Filomat*, 2017, 31(10), 2957–2966
- [22] Veličković V., Malkowsky E., Özger F., Visualization of the spaces $W(u, v; \ell_p)$ and their duals, *AIP Conf. Proc.*, 2016, 1759, DOI: 10.1063/1.4959634
- [23] Özger F., Some geometric characterizations of a fractional Banach set, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 2019, 68(1), 546–558
- [24] Malkowsky E., Özger F., Compact operators on spaces of sequences of weighted means, *AIP Conf. Proc.*, 2012, 1470, 179–182
- [25] Malkowsky E., Özger F., Alotatibi A., Some notes on matrix mappings and their Hausdorff measure of noncompactness, *Filomat*, 2014, 28(5), 1059–1072
- [26] Kara E. E., Başarır M., Mursaleen M., Compactness of matrix operators on some sequence spaces derived by Fibonacci Numbers, *Kragujevac J. Math.*, 2015, 39(2), 217–230
- [27] Mursaleen M., Noman A. K., Compactness by Hausdorff the measures of noncompactness, *Nonlinear Anal.*, 2010, 73(8), 2541–2557
- [28] Özger F., Compact operators on the sets of fractional difference sequences, *Sakarya University Journal of Science*, 2019, 23(3), DOI: 10.16984/aufenbilder.463368
- [29] Furkan H., On some λ difference sequence spaces of fractional order, *J. Egypt. Math. Soc.*, 2017, 25, 37–42
- [30] Kadak U., Baliarsingh P., On certain Euler difference sequence spaces of fractional order and related dual properties, *J. Nonlinear Sci. Appl.*, 2015, 8(6), 997–1004
- [31] Petković K., Some new results related to compact matrix operators in the classes $A \in ((\ell_p)_T, \ell_\infty)$, *Acta Math. Sci. Ser. B Engl. Ed.*, 2015, 31(8), 1339–1347
- [32] Sargent W. L. C., On compact matrix transformations between sectionally bounded BK spaces, *J. London Math. Soc.*, 1966, 41, 79–87
- [33] Wilansky A., *Summability through functional analysis*, North–Holland Mathematics Studies, 85, Amsterdam, New York, Oxford, 1984
- [34] Malkowsky E., Rakočević R., An introduction into the theory of sequence spaces and measures of noncompactness, *Zb. Rad. (Beogr.)*, 2000, 9(17), 143–234
- [35] Malkowsky E., Rakočević R., On matrix domains of triangles, *Appl. Math. Comput.*, 2007, 189, 1148–1163
- [36] Djolović I., Malkowsky E., A note on compact operators on matrix domains, *J. Math. Anal. Appl.*, 2008, 340, 291–303
- [37] Stieglitz M., Tietz H., Matrixtransformationen von Folgenräumen eine Ergebnisübersicht, *Math. Z.*, 1977, 154, 1–16