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Stability of an AQCQ functional equation in non-Archimedean (n, β) -normed spaces

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Abstract: In this paper, we adopt direct method to prove the Hyers-Ulam-Rassias stability of an additive-quadratic-cubic-quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)$$

in non-Archimedean (n, β) -normed spaces.

Keywords: non-Archimedean (n, β) -normed space, Hyers-Ulam-Rassias stability, AQCQ functional equation

MSC: 39B82, 39B72

1 Introduction

In 1940, Ulam [1] first proposed the question of the stability problem of functional equations, concerning the stability of group homomorphisms. Given a group $(G_1, \cdot)$ and a metric group $(G_2, *)$ with the metric $d(\cdot, \cdot)$ and a number $\varepsilon > 0$, does there exist a $\delta > 0$ such that, if a mapping $h : G_1 \rightarrow G_2$ satisfies $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \rightarrow G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, the case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [3] proved a generalization of the Hyers theorem for additive mappings. Rassias's results made significant influence in the past 36 years on the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as the Hyers-Ulam-Rassias stability of functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found in [4–15].

In 2015, Yang et al. [16] introduced the notions of (n, β) -normed space and non-Archimedean (n, β) -normed space, and proved the Hyers-Ulam-Rassias stability of Cauchy functional equation, Jensen functional equation and Pexiderized Cauchy functional equation in (n, β) -normed spaces.

Park et al. [17] considered an additive-quadratic-cubic-quartic functional equation:

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \quad (1.1)$$

and proved the Hyers-Ulam stability in random normed spaces. In this paper, we study the Hyers-Ulam-Rassias stability of this functional equation in non-Archimedean (n, β) -normed spaces.

Now, we list some concepts and properties concerning (n, β) -normed space.

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Definition 1.1 [16] Let X be a linear space over $\mathbb{R}$ with $\dim X \geq n$, $n \in \mathbb{N}$ and $0 < \beta \leq 1$, let $\|\cdot, \dots, \cdot\|_\beta : X^n \rightarrow \mathbb{R}$ be a function satisfying the following properties:

- (a) $\|x_1, \dots, x_n\|_\beta = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;
 - (b) $\|x_1, \dots, x_n\|_\beta$ is invariant under permutations of $x_1, \dots, x_n$;
 - (c) $\|\alpha x_1, \dots, x_n\|_\beta = |\alpha|^\beta \|x_1, \dots, x_n\|_\beta$;
 - (d) $\|x_1, \dots, x_{n-1}, y + z\|_\beta \leq \|x_1, \dots, x_{n-1}, y\|_\beta + \|x_1, \dots, x_{n-1}, z\|_\beta$;
- for all $x_1, \dots, x_n \in X$ and $\alpha \in \mathbb{R}$. Then the function $\|\cdot, \dots, \cdot\|_\beta$ is called a (n, β) -norm on X and the pair $(X, \|\cdot, \dots, \cdot\|_\beta)$ is called a linear (n, β) -normed space or a (n, β) -normed space.

We remark that the concept of a linear (n, β) -normed space is a generalization of a linear n -normed space ($\beta = 1$) and of a β -normed space ($n = 1$).

Lemma 1.2 [16] Let $(X, \|\cdot, \dots, \cdot\|_\beta)$ be a linear (n, β) -normed space, $n \geq 2$, $0 < \beta \leq 1$. If $x_1 \in X$ and $\|x_1, y_1, \dots, y_{n-1}\|_\beta = 0$ for all $y_1, \dots, y_{n-1} \in X$, then $x_1 = 0$.

Remark 1.3 [16] Let $(X, \|\cdot, \dots, \cdot\|_\beta)$ be a linear (n, β) -normed space, $0 < \beta \leq 1$. One can show that the conditions (b) and (d) in Definition 1.1 imply that

$$|\|x, z_1, \dots, z_{n-1}\|_\beta - \|y, z_1, \dots, z_{n-1}\|_\beta| \leq \|x - y, z_1, \dots, z_{n-1}\|_\beta$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in X$.

Definition 1.4 [16] A sequence $\{x_m\}$ in a linear (n, β) -normed space X is called a convergent sequence if there is an $x \in X$ such that

$$\lim_{m \rightarrow \infty} \|x_m - x, y_1, \dots, y_{n-1}\|_\beta = 0$$

for all $y_1, \dots, y_{n-1} \in X$. In this case, we call that $\{x_m\}$ converges to x or that x is the limit of $\{x_m\}$, write $x_m \rightarrow x$ as $m \rightarrow \infty$ or $\lim_{m \rightarrow \infty} x_m = x$.

Definition 1.5 [16] A sequence $\{x_m\}$ in a linear (n, β) -normed space X is called a Cauchy sequence if

$$\lim_{m, k \rightarrow \infty} \|x_k - x_m, z_1, \dots, z_{n-1}\|_\beta = 0$$

for all $z_1, \dots, z_{n-1} \in X$.

We can easily get the following lemma by Remark 1.3.

Lemma 1.6 [16] For a convergent sequence $\{x_m\}$ in a linear (n, β) -normed space X , then

$$\lim_{m \rightarrow \infty} \|x_m, z_1, \dots, z_{n-1}\|_\beta = \lim_{m \rightarrow \infty} \|x_m, z_1, \dots, z_{n-1}\|_\beta$$

for all $z_1, \dots, z_{n-1} \in X$.

Definition 1.7 [16] A linear (n, β) -normed space in which every Cauchy sequence is convergent is called a complete (n, β) -normed space.

In 1897, Hensel [18] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [19–21]).

Definition 1.8 A field K equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty)$ is called a non-Archimedean field if the function $|\cdot| : K \rightarrow [0, \infty)$ satisfies the following conditions:

- (1) $|r| = 0$ if and only if $r = 0$;
- (2) $|rs| = |r||s|$;
- (3) $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$;
- (4) there exists a member $a_0 \in K$ such that $|a_0| \neq 0, 1$.

Definition 1.9 [22] Let X be a vector space over scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (1') $\|x\| = 0$ if and only if $x = 0$;
- (2') $\|rx\| = |r|\|x\|$;
- (3') $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ and $r \in K$.

The pair $(X, \|\cdot\|)$ is called a non-Archimedean space if $\|\cdot\|$ is a non-Archimedean norm on X .

Definition 1.10 [16] Let X be a real vector space with $\dim X \geq n$ over scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$, where n is a positive integer. β is a constant and $0 < \beta \leq 1$. A real-valued function $\|\cdot, \dots, \cdot\|_\beta : X^n \rightarrow \mathbb{R}$ is called a (n, β) -norm on X if the following conditions hold:

(N1') $\|x_1, \dots, x_n\|_\beta = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2') $\|x_1, \dots, x_n\|_\beta$ is invariant under permutations of $x_1, \dots, x_n$;

(N3') $\|\alpha x_1, x_2, \dots, x_n\|_\beta = |\alpha|^\beta \|x_1, x_2, \dots, x_n\|_\beta$;

(N4') $\|x_0 + x_1, x_2, \dots, x_n\|_\beta \leq \max\{\|x_0, x_2, \dots, x_n\|_\beta, \|x_1, x_2, \dots, x_n\|_\beta\}$;

for all $\alpha \in K$ and $x_0, x_1, \dots, x_n \in X$. Then $(X, \|\cdot, \dots, \cdot\|_\beta)$ is called a non-Archimedean (n, β) -normed space.

It follows from the preceding definition that the non-Archimedean (n, β) -normed space is a non-Archimedean n -normed space if $\beta = 1$, and a non-Archimedean β -normed space if $n = 1$, respectively.

Remark 1.11 [16] A sequence $\{x_m\}$ in a non-Archimedean (n, β) -normed space X is a Cauchy sequence if and only if $\{x_{m+1} - x_m\}$ converges to zero.

One can easily show that an odd mapping $f : X \rightarrow Y$ satisfies (1.1) if and only if the odd mapping $f : X \rightarrow Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x).$$

It was shown in [23, Lemma 2.2] that $g(x) = f(2x) - 8f(x)$ and $h(x) = f(2x) - 2f(x)$ are additive and cubic, respectively, and that $f(x) = \frac{1}{6}h(x) - \frac{1}{6}g(x)$.

One can easily show that an even mapping $f : X \rightarrow Y$ satisfies (1.1) if and only if the even mapping $f : X \rightarrow Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + 2f(2y) - 8f(y).$$

It was shown in [24, Lemma 2.1] that $g(x) = f(2x) - 16f(x)$ and $h(x) = f(2x) - 4f(x)$ are quadratic and quartic, respectively, and that $f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x)$.

Lemma 1.12 [17] Let X and Y be real vector spaces. If an odd mapping $f : X \rightarrow Y$ satisfies (1.1), then f is an additive-cubic. If an even mapping $f : X \rightarrow Y$ satisfies (1.1), then f is a quadratic-quartic.

Throughout this paper, Let $\mathbb{N}$ denote the nonnegative integers and $i, k, m, n \in \mathbb{N}$, and let $n \geq 2$ be fixed and $|2| < 1$. For notational convenience, given a function $f : X \rightarrow Y$, we define the difference operator $Df(x, y) : X \times X \rightarrow [0, \infty)$ by

$$Df(x, y) = f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 6f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) \quad (1.2)$$

for all $x, y \in X$.

2 Stability of (1.1): an odd mapping case

In this section, we investigate the Hyers-Ulam-Rassias stability of (1.1) in non-Archimedean (n, β) -normed space: an odd mapping case.

Theorem 2.1 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |2|^{m\beta} \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right) = 0 \quad (2.1)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max\{|2|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |2|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) : 0 \leq i \leq m-1\} \quad (2.2)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an odd mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (2.3)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists an additive mapping $A : X \rightarrow Y$ such that

$$\|f(2x) - 8f(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (2.4)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{ |2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} = 0 \quad (2.5)$$

for all $x \in X$, then A is a unique additive mapping satisfying (2.4).

Proof. Putting $x = y$ in (2.3) and multiplying both sides by $|4|^\beta$, we get

$$\|4f(3y) - 16f(2y) + 20f(y), z_1, \dots, z_{n-1}\|_\beta \leq |4|^\beta \varphi(y, y)\psi(z_1, \dots, z_{n-1}) \quad (2.6)$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $2y$ in (2.3), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(2y, y)\psi(z_1, \dots, z_{n-1}) \quad (2.7)$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Combining (2.6) and (2.7) yields

$$\|f(4y) - 10f(2y) + 16f(y), z_1, \dots, z_{n-1}\|_\beta \leq \max\{|4|^\beta \varphi(y, y), \varphi(2y, y)\}\psi(z_1, \dots, z_{n-1}) \quad (2.8)$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing y by $\frac{x}{2}$ in (2.8), we get

$$\left\| f(2x) - 10f(x) + 16f\left(\frac{x}{2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi(x, \frac{x}{2})\}\psi(z_1, \dots, z_{n-1}) \quad (2.9)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Let $g(x) = f(2x) - 8f(x)$, by (2.9) we obtain

$$\left\| g(x) - 2g\left(\frac{x}{2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi(x, \frac{x}{2})\}\psi(z_1, \dots, z_{n-1}) \quad (2.10)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2^i}$ in (2.10) and multiplying both sides by $|2|^{i\beta}$, we get

$$\|2^i g\left(\frac{x}{2^i}\right) - 2^{i+1} g\left(\frac{x}{2^{i+1}}\right), z_1, \dots, z_{n-1}\|_\beta \leq \max\{|2|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |2|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\}\psi(z_1, \dots, z_{n-1}) \quad (2.11)$$

for all $i \in \mathbb{N}$, $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $i \rightarrow \infty$ in (2.11) and considering (2.1), we can obtain

$$\lim_{i \rightarrow \infty} \left\| 2^i g\left(\frac{x}{2^i}\right) - 2^{i+1} g\left(\frac{x}{2^{i+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Lemma 1.2 and Remark 1.11, we know that $\{2^m g(\frac{x}{2^m})\}$ is a Cauchy sequence. Since Y is a complete space, we can define the mapping $A : X \rightarrow Y$ by

$$A(x) = \lim_{m \rightarrow \infty} 2^m g\left(\frac{x}{2^m}\right)$$

for all $x \in X$.

Next, we prove that A is additive. Then we have

$$\begin{aligned} \|A(2x) - 2A(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} \|2^m g\left(\frac{x}{2^{m-1}}\right) - 2^{m+1} g\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |2|^\beta \|2^{m-1} g\left(\frac{x}{2^{m-1}}\right) - 2^m g\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= 0 \end{aligned} \quad (2.12)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. On the other hand, it is easy to show that

$$Dg(x, y) = Df(2x, 2y) - 8Df(x, y) \quad (2.13)$$

for all $x, y \in X$. Hence it follows from (2.3)

$$\begin{aligned} \|DA(x, y), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} |2|^{m\beta} \|Dg\left(\frac{x}{2^m}, \frac{y}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |2|^{m\beta} \|Df\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 2^3 Df\left(\frac{x}{2^m}, \frac{y}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max\{|2|^{m\beta} \varphi\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right), |2|^{(m+3)\beta} \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\}\psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned} \quad (2.14)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$. This means that A satisfies (1.1). Then by Lemma 1.12, $x \rightarrow A(2x) - 8A(x)$ is additive. Thus (2.12) implies that A is additive.

Replacing x by $\frac{x}{2}$ in (2.10) and multiplying both sides by $|2|^\beta$, we get

$$\|2g(\frac{x}{2}) - 2^2g(\frac{x}{2^2}), z_1, \dots, z_{n-1}\|_\beta \leq \max\{|2|^\beta |4|^\beta \varphi(\frac{x}{2^2}, \frac{x}{2^2}), |2|^\beta \varphi(\frac{x}{2}, \frac{x}{2^2})\} \psi(z_1, \dots, z_{n-1}) \quad (2.15)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (2.10) and (2.15), we obtain

$$\|g(x) - 2^2g(\frac{x}{2^2}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq 1} \{|2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (2.16)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By induction on m , we get

$$\|g(x) - 2^m g(\frac{x}{2^m}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq m-1} \{|2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (2.17)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2}$ in (2.17) and multiplying both sides by $|2|^\beta$, we get

$$\|2g(\frac{x}{2}) - 2^{m+1}g(\frac{x}{2^{m+1}}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq m-1} \{|2|^{(i+1)\beta} |4|^\beta \varphi(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}}), |2|^{(i+1)\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}})\} \psi(z_1, \dots, z_{n-1}) \quad (2.18)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (2.10) and (2.18), we get

$$\|g(x) - 2^{m+1}g(\frac{x}{2^{m+1}}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq m} \{|2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (2.19)$$

for all $x \in X$, $m \in \mathbb{N}$ and $z_1, \dots, z_{n-1} \in Y$. This completes the proof of (2.17).

Taking the limit as $m \rightarrow \infty$ in (2.17), we can obtain

$$\begin{aligned} \|g(x) - A(x), z_1, \dots, z_{n-1}\|_\beta &= \|f(2x) - 8f(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \\ &= \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \end{aligned}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$, which proves (2.4).

Now we need to prove the uniqueness of A . Let A' be another additive mapping satisfying (2.4). Since

$$\begin{aligned} \lim_{k \rightarrow \infty} |2|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) &= \lim_{k \rightarrow \infty} |2|^{k\beta} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^\beta |2|^{i\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}})\} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^\beta |2|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |2|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}})\} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{|4|^\beta |2|^{i\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. It follows from (2.5) that

$$\begin{aligned} \|A(x) - A'(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{k \rightarrow \infty} |2|^{k\beta} \|A(\frac{x}{2^k}) - A'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{k \rightarrow \infty} |2|^{k\beta} \max\{\|A(\frac{x}{2^k}) - g(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta, \|g(\frac{x}{2^k}) - A'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta\} \\ &\leq \lim_{k \rightarrow \infty} |2|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Lemma 1.2, we prove that A is unique. $\square$

Theorem 2.2 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |8|^{m\beta} \varphi(\frac{x}{2^m}, \frac{y}{2^m}) = 0 \quad (2.20)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max\{|8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) : 0 \leq i \leq m-1\} \quad (2.21)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an odd mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (2.22)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists a cubic mapping $C : X \rightarrow Y$ such that

$$\|f(2x) - 2f(x) - C(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (2.23)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{|8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} = 0 \quad (2.24)$$

for all $x \in X$, then C is a unique cubic mapping satisfying (2.23).

Proof. It is easy to show that f satisfies (2.9). Setting $h(x) = f(2x) - 2f(x)$, by (2.9) we obtain

$$\left\| h(x) - 8h\left(\frac{x}{2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (2.25)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2^i}$, and multiplying both sides by $|8|^{i\beta}$ we get

$$\left\| 8^i h\left(\frac{x}{2^i}\right) - 8^{i+1} h\left(\frac{x}{2^{i+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|8|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |8|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (2.26)$$

for all $i \in \mathbb{N}$, $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $i \rightarrow \infty$ in (2.26) and considering (2.20), we can obtain

$$\lim_{i \rightarrow \infty} \left\| 8^i h\left(\frac{x}{2^i}\right) - 8^{i+1} h\left(\frac{x}{2^{i+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Lemma 1.2 and Remark 1.11, we know that $\{8^m h(\frac{x}{2^m})\}$ is a Cauchy sequence. Since Y is a complete space, we can define the mapping $C : X \rightarrow Y$ by

$$C(x) = \lim_{m \rightarrow \infty} 8^m h\left(\frac{x}{2^m}\right)$$

for all $x \in X$.

Next, we prove that C is cubic. Then we have

$$\begin{aligned} \|C(2x) - 8C(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} \|8^m h\left(\frac{x}{2^{m-1}}\right) - 8^{m+1} h\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} \|8|^\beta \|8^{m-1} h\left(\frac{x}{2^{m-1}}\right) - 8^m h\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= 0 \end{aligned} \quad (2.27)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. On the other hand, it is easy to show that

$$Dh(x, y) = Df(2x, 2y) - 2Df(x, y) \quad (2.28)$$

for all $x, y \in X$. Hence it follows from (2.22)

$$\begin{aligned} \|DC(x, y), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} |8|^{m\beta} \|Dh\left(\frac{x}{2^m}, \frac{y}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |8|^{m\beta} \|Df\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 2Df\left(\frac{x}{2^m}, \frac{y}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max\{|8|^{m\beta} \varphi\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right), |8|^{m\beta} |2|^\beta \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\} \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned} \quad (2.29)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Since C is an odd mapping, C satisfies (1.1). Then by Lemma 1.12 and (2.27), we conclude that $C(3x) = 27C(x)$ for all $x \in X$. Then C is cubic.

Replacing x by $\frac{x}{2}$ in (2.25) and multiplying both sides by $|8|^\beta$, we get

$$\left\| 8h\left(\frac{x}{2}\right) - 8^2 h\left(\frac{x}{2^2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|8|^\beta |4|^\beta \varphi\left(\frac{x}{2^2}, \frac{x}{2^2}\right), |8|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2^2}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (2.30)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (2.25) and (2.30), we obtain

$$\left\| h(x) - 8^2 h\left(\frac{x}{2^2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max_{0 \leq i \leq 1} \{|8|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |8|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (2.31)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By induction on m , we get

$$\|h(x) - 8^m h(\frac{x}{2^m}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq m-1} \{ |8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \quad (2.32)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2}$ in (2.32) and multiplying both sides by $|8|^\beta$, we get

$$\begin{aligned} \|8h(\frac{x}{2}) - 8^{m+1} h(\frac{x}{2^{m+1}}), z_1, \dots, z_{n-1}\|_\beta &\leq \max_{0 \leq i \leq m-1} \{ |8|^{(i+1)\beta} |4|^\beta \varphi(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}}), |8|^{(i+1)\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}}) \} \psi(z_1, \dots, z_{n-1}) \\ &= \max_{1 \leq i \leq m} \{ |8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \end{aligned} \quad (2.33)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (2.25) and (2.33), we get

$$\|h(x) - 8^{m+1} h(\frac{x}{2^{m+1}}), z_1, \dots, z_{n-1}\|_\beta = \max_{0 \leq i \leq m} \{ |8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \quad (2.34)$$

for all $x \in X$, $m \in \mathbb{N}$ and $z_1, \dots, z_{n-1} \in Y$. This completes the proof of (2.32).

Taking the limit as $m \rightarrow \infty$ in (2.32), we can obtain

$$\begin{aligned} \|h(x) - C(x), z_1, \dots, z_{n-1}\|_\beta &= \|f(2x) - 2f(x) - C(x), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{ |8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \\ &= \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \end{aligned}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$, which proves (2.23).

Now we need to prove the uniqueness of C . Let C' be another cubic mapping satisfying (1.1) and (2.23). Since

$$\begin{aligned} \lim_{k \rightarrow \infty} |8|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) &= \lim_{k \rightarrow \infty} |8|^{k\beta} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{ |4|^\beta |8|^{i\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}}) \} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{ |4|^\beta |8|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |8|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}}) \} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{ |4|^\beta |8|^{i\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$, it follows from (2.24) that

$$\begin{aligned} \|C(x) - C'(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{k \rightarrow \infty} |8|^{k\beta} \|C(\frac{x}{2^k}) - C'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{k \rightarrow \infty} |8|^{k\beta} \max \{ \|C(\frac{x}{2^k}) - h(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta, \|h(\frac{x}{2^k}) - C'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta \} \\ &\leq \lim_{k \rightarrow \infty} |8|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Lemma 1.2, we prove that C is unique. $\square$

Theorem 2.3 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |2|^{m\beta} \varphi(\frac{x}{2^m}, \frac{y}{2^m}) = 0 \quad (2.35)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max \{ |2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) : 0 \leq i \leq m-1 \} \quad (2.36)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an odd mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y) \psi(z_1, \dots, z_{n-1}) \quad (2.37)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists a cubic mapping $C : X \rightarrow Y$ and an additive $A : X \rightarrow Y$ such that

$$\|f(x) - C(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \leq \frac{1}{6} |\beta| \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \quad (2.38)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{ |2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} = 0 \quad (2.39)$$

for all $x \in X$, then C and A are unique cubic and additive mappings respectively satisfying (2.38).

Proof. By Theorem 2.1 and Theorem 2.2, there exist an additive $A_0 : X \rightarrow Y$ and a cubic mapping $C_0 : X \rightarrow Y$ such that

$$\|f(2x) - 8f(x) - A_0(x), z_1, \dots, z_{n-1}\|_\beta \leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{ |2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \quad (2.40)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$, and

$$\|f(2x) - 2f(x) - C_0(x), z_1, \dots, z_{n-1}\|_\beta \leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{ |8|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |8|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \quad (2.41)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Property (3) of Definition 1.8 implies $|8| \leq |2|$, then by (2.40) and (2.41), we get

$$\begin{aligned} & \|f(x) - \frac{1}{6} C_0(x) + \frac{1}{6} A_0(x), z_1, \dots, z_{n-1}\|_\beta \\ & \leq \lim_{m \rightarrow \infty} \left\{ \frac{1}{6} |\beta| \max_{0 \leq i \leq m-1} \{ |2|^{i\beta}, |8|^{i\beta} \} \max_{0 \leq i \leq m-1} \{ |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \right\} \\ & = \lim_{m \rightarrow \infty} \left\{ \frac{1}{6} |\beta| \max_{0 \leq i \leq m-1} \{ |2|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |2|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) \} \psi(z_1, \dots, z_{n-1}) \right\} \end{aligned} \quad (2.42)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. So we get (2.38) by letting $A(x) = -\frac{1}{6} A_0(x)$ and $C(x) = \frac{1}{6} C_0(x)$ for all $x \in X$.

Now we need to prove the uniqueness of C and A , let $A_1, C_1 : X \rightarrow Y$ be other additive and cubic mappings satisfying (2.38). Let $A' = A - A_1$, $C' = C_1 - C$. Then

$$\begin{aligned} \|A'(x) - C'(x), z_1, \dots, z_{n-1}\|_\beta &= \|A(x) - A_1(x) + C(x) - C_1(x) + f(x) - f(x), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \max \{ \|f(x) - A(x) - C(x), z_1, \dots, z_{n-1}\|_\beta, \|f(x) - A_1(x) - C_1(x), z_1, \dots, z_{n-1}\|_\beta \} \\ &\leq \frac{1}{6} |\beta| \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \end{aligned} \quad (2.43)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Together with (2.39), we have

$$\lim_{k \rightarrow \infty} |2|^{k\beta} \|A'(\frac{x}{2^k}) - C'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta = 0 \quad (2.44)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$.

Since $A'(\frac{x}{2^k}) = \frac{1}{2^k} A'(x)$ and $|\frac{1}{4}| > 1$, we obtain that $C'(x) = 0$ for all $x \in X$. Again by (2.44), we have $A'(x) = 0$ for all $x \in X$. We prove that A and C are unique. $\square$

Theorem 2.4 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} \left| \frac{1}{2} \right|^{(m+1)\beta} \varphi(2^m x, 2^m y) = 0 \quad (2.45)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \left\{ \left| \frac{1}{2} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{2} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} \quad (2.46)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an odd mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y) \psi(z_1, \dots, z_{n-1}) \quad (2.47)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then the limit

$$A(x) = \lim_{m \rightarrow \infty} 2^{-m}(f(2^{m+1}x) - 8f(2^m x)) \quad (2.48)$$

exists for all $x \in X$, and $A(x)$ is an additive mapping satisfying (1.1) and the inequality

$$\|f(2x) - 8f(x) - A(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (2.49)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ \left| \frac{1}{2} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{2} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} = 0 \quad (2.50)$$

for all $x \in X$, then A is a unique additive mapping satisfying (2.49).

Proof. The proof is similar to the proof of Theorem 2.1. $\square$

Theorem 2.5 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} \left| \frac{1}{8} \right|^{(m+1)\beta} \varphi(2^m x, 2^m y) = 0 \quad (2.51)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \left\{ \left| \frac{1}{8} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{8} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} \quad (2.52)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an odd mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (2.53)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then the limit

$$C(x) = \lim_{m \rightarrow \infty} 8^{-m}(f(2^{m+1}x) - 2f(2^m x)) \quad (2.54)$$

exists for all $x \in X$, and $C(x)$ is a cubic mapping satisfying (1.1) and the inequality

$$\|f(2x) - 2f(x) - C(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (2.55)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ \left| \frac{1}{8} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{8} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} = 0 \quad (2.56)$$

for all $x \in X$, then C is a unique cubic mapping satisfying (2.55).

Proof. The proof is similar to the proof of Theorem 2.2. $\square$

Theorem 2.6 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} \left| \frac{1}{8} \right|^{(m+1)\beta} \varphi(2^m x, 2^m y) = 0 \quad (2.57)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \left\{ \left| \frac{1}{8} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{8} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} \quad (2.58)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an odd mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (2.59)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exist a cubic mapping $C : X \rightarrow Y$ and an additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x) - C(x), z_1, \dots, z_{n-1}\|_\beta \leq \left|\frac{1}{6}\right|^\beta \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (2.60)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ \left|\frac{1}{8}\right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left|\frac{1}{8}\right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} = 0 \quad (2.61)$$

for all $x \in X$, then C and A are unique cubic and additive mappings respectively satisfying (2.60).

Proof. The proof is similar to the proof of Theorem 2.3. $\square$

3 Stability of (1.1): an even mapping case

In this section, we will investigate the Hyers-Ulam-Rassias stability of (1.1) in non-Archimedean (n, β) -normed space: an even mapping case.

Theorem 3.1 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |4|^{m\beta} \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right) = 0 \quad (3.1)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max\{|4|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |4|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) : 0 \leq i \leq m-1\} \quad (3.2)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (3.3)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists a quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(2x) - 16f(x) - Q(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (3.4)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ |4|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |4|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) \right\} = 0 \quad (3.5)$$

for all $x \in X$, then Q is a unique quadratic mapping satisfying (3.4).

Proof. Putting $x = y$ in (3.3) and multiplying both sides by $|4|^\beta$, we get

$$\|4f(3y) - 24f(2y) + 60f(y), z_1, \dots, z_{n-1}\|_\beta \leq |4|^\beta \varphi(y, y)\psi(z_1, \dots, z_{n-1}) \quad (3.6)$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $2y$ in (3.3), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(2y, y)\psi(z_1, \dots, z_{n-1}) \quad (3.7)$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Together with (3.6), we get

$$\|f(4y) - 20f(2y) + 64f(y), z_1, \dots, z_{n-1}\|_\beta \leq \max\{|4|^\beta \varphi(y, y), \varphi(2y, y)\}\psi(z_1, \dots, z_{n-1}) \quad (3.8)$$

for all $y \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing y by $\frac{x}{2}$ in (3.8), we get

$$\left\|f(2x) - 20f(x) + 64f\left(\frac{x}{2}\right), z_1, \dots, z_{n-1}\right\|_\beta \leq \max\{|4|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right)\}\psi(z_1, \dots, z_{n-1}) \quad (3.9)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Let $g(x) = f(2x) - 16f(x)$, by (3.9) we obtain

$$\left\| g(x) - 4g\left(\frac{x}{2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi(x, \frac{x}{2})\} \psi(z_1, \dots, z_{n-1}) \quad (3.10)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2^i}$, and multiplying both sides by $|4|^{i\beta}$, we get

$$\left\| 4^i g\left(\frac{x}{2^i}\right) - 4^{i+1} g\left(\frac{x}{2^{i+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |4|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (3.11)$$

for all $i \in \mathbb{N}$, $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $i \rightarrow \infty$ in (3.11) and considering (3.1), we can obtain

$$\lim_{i \rightarrow \infty} \left\| 4^i g\left(\frac{x}{2^i}\right) - 4^{i+1} g\left(\frac{x}{2^{i+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Remark 1.11, we know that $\{4^m g(\frac{x}{2^m})\}$ is a Cauchy sequence. Since Y is a complete space, we can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) = \lim_{m \rightarrow \infty} 4^m g\left(\frac{x}{2^m}\right)$$

for all $x \in X$.

Next, we prove that Q is quadratic. Then we have

$$\begin{aligned} \|Q(2x) - 4Q(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} \|4^m g\left(\frac{x}{2^{m-1}}\right) - 4^{m+1} g\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |4|^\beta \|4^{m-1} g\left(\frac{x}{2^{m-1}}\right) - 4^m g\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= 0 \end{aligned} \quad (3.12)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. On the other hand, it is easy to show that

$$Dg(x, y) = Df(2x, 2y) - 16Df(x, y) \quad (3.13)$$

for all $x, y \in X$. Hence it follows from (3.3)

$$\begin{aligned} \|DQ(x, y), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} |4|^{m\beta} \|Dg\left(\frac{x}{2^m}, \frac{y}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |4|^{m\beta} \|Df\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right) - 4^2 Df\left(\frac{x}{2^m}, \frac{y}{2^m}\right), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max\{|4|^{m\beta} \varphi\left(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}\right), |4|^{(m+2)\beta} \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\} \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned} \quad (3.14)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$. This means that Q satisfies (1.1). Since $f : X \rightarrow Y$ is even, $Q : X \rightarrow Y$ is even. Then by Lemma 1.12, the mapping Q is quadratic.

Replacing x by $\frac{x}{2}$ in (3.10) and multiplying both sides by $|4|^\beta$, we get

$$\left\| 4g\left(\frac{x}{2}\right) - 4^2 g\left(\frac{x}{2^2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^\beta |4|^\beta \varphi\left(\frac{x}{2^2}, \frac{x}{2^2}\right), |4|^\beta \varphi\left(\frac{x}{2}, \frac{x}{2^2}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (3.15)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (3.10) and (3.15), we obtain

$$\left\| g(x) - 4^2 g\left(\frac{x}{2^2}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max_{0 \leq i \leq 1} \{|4|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |4|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (3.16)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By induction on m , we get

$$\left\| g(x) - 4^m g\left(\frac{x}{2^m}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max_{0 \leq i \leq m-1} \{|4|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |4|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (3.17)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2}$ in (3.17) and multiplying both sides by $|4|^\beta$, we get

$$\left\| 4g\left(\frac{x}{2}\right) - 4^{m+1} g\left(\frac{x}{2^{m+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max_{0 \leq i \leq m-1} \{|4|^{(i+1)\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}}\right), |4|^{(i+1)\beta} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (3.18)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (3.10) and (3.18), we get

$$\left\| g(x) - 4^{m+1} g\left(\frac{x}{2^{m+1}}\right), z_1, \dots, z_{n-1} \right\|_\beta \leq \max_{0 \leq i \leq m} \{|4|^{i\beta} |4|^\beta \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), |4|^{i\beta} \varphi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right)\} \psi(z_1, \dots, z_{n-1}) \quad (3.19)$$

for all $x \in X$, $m \in \mathbb{N}$ and $z_1, \dots, z_{n-1} \in Y$. This completes the proof of (3.17).

Taking the limit as $m \rightarrow \infty$ in (3.17), we can obtain

$$\begin{aligned} \|g(x) - Q(x), z_1, \dots, z_{n-1}\|_\beta &= \|f(2x) - 16f(x) - Q(x), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \\ &= \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \end{aligned}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$, which proves (3.4).

Now we need to prove the uniqueness of Q . Let Q' be another quadratic mapping satisfying (3.4). Since

$$\begin{aligned} \lim_{k \rightarrow \infty} |4|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) &= \lim_{k \rightarrow \infty} |4|^{k\beta} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^\beta |4|^{i\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}})\} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^\beta |4|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |4|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}})\} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{|4|^\beta |4|^{i\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \\ &= 0 \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. It follows from (3.5) that

$$\begin{aligned} \|Q(x) - Q'(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{k \rightarrow \infty} |4|^{k\beta} \|Q(\frac{x}{2^k}) - Q'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{k \rightarrow \infty} |4|^{k\beta} \max\{\|Q(\frac{x}{2^k}) - g(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta, \|g(\frac{x}{2^k}) - Q'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta\} \\ &\leq \lim_{k \rightarrow \infty} |4|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Lemma 1.2, we prove that Q is unique. $\square$

Theorem 3.2 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |16|^{m\beta} \varphi(\frac{x}{2^m}, \frac{y}{2^m}) = 0 \quad (3.20)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max\{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) : 0 \leq i \leq m-1\} \quad (3.21)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y) \psi(z_1, \dots, z_{n-1}) \quad (3.22)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists a quartic mapping $T : X \rightarrow Y$ such that

$$\|f(2x) - 4f(x) - T(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \quad (3.23)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} = 0 \quad (3.24)$$

for all $x \in X$, then T is a unique quartic mapping satisfying (3.23).

Proof. It is easy to show that f satisfies (3.9). Setting $h(x) = f(2x) - 4f(x)$, by (3.9) we obtain

$$\left\| h(x) - 16h(\frac{x}{2}), z_1, \dots, z_{n-1} \right\|_\beta \leq \max\{|4|^\beta \varphi(\frac{x}{2}, \frac{x}{2}), \varphi(x, \frac{x}{2})\} \psi(z_1, \dots, z_{n-1}) \quad (3.25)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2^i}$, and multiplying both sides by $|16|^{i\beta}$ we get

$$\|16^i h(\frac{x}{2^i}) - 16^{i+1} h(\frac{x}{2^{i+1}}), z_1, \dots, z_{n-1}\|_\beta \leq \max\{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (3.26)$$

for all $i \in \mathbb{N}$, $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Taking the limit as $i \rightarrow \infty$ in (3.26) and considering (3.20), we can obtain

$$\lim_{i \rightarrow \infty} \|16^i h(\frac{x}{2^i}) - 16^{i+1} h(\frac{x}{2^{i+1}}), z_1, \dots, z_{n-1}\|_\beta = 0$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Remark 1.11, we know that $\{16^m h(\frac{x}{2^m})\}$ is a Cauchy sequence. Since Y is a complete space, we can define the mapping $T : X \rightarrow Y$ by

$$T(x) = \lim_{m \rightarrow \infty} 16^m h(\frac{x}{2^m})$$

for all $x \in X$.

Next, we prove that T is quartic. Then we have

$$\begin{aligned} \|T(2x) - 16T(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} \|16^m h(\frac{x}{2^{m-1}}) - 16^{m+1} h(\frac{x}{2^m}), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |16|^\beta \|16^{m-1} h(\frac{x}{2^{m-1}}) - 16^m h(\frac{x}{2^m}), z_1, \dots, z_{n-1}\|_\beta \\ &= 0 \end{aligned} \quad (3.27)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. On the other hand, it is easy to show that

$$Dh(x, y) = Df(2x, 2y) - 4Df(x, y) \quad (3.28)$$

for all $x, y \in X$. Hence it follows from (3.22)

$$\begin{aligned} \|DT(x, y), z_1, \dots, z_{n-1}\|_\beta &= \lim_{m \rightarrow \infty} |16|^{m\beta} \|Dh(\frac{x}{2^m}, \frac{y}{2^m}), z_1, \dots, z_{n-1}\|_\beta \\ &= \lim_{m \rightarrow \infty} |16|^{m\beta} \|Df(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}) - 4Df(\frac{x}{2^m}, \frac{y}{2^m}), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max\{|16|^{m\beta} \varphi(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}), |16|^{m\beta} |4|^\beta \varphi(\frac{x}{2^m}, \frac{y}{2^m})\} \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned} \quad (3.29)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$. This means that T satisfies (1.1). Since $f : X \rightarrow Y$ is even, $T : X \rightarrow Y$ is even. Then by Lemma 1.12, the mapping T is quartic.

Replacing x by $\frac{x}{2}$ in (3.25) and multiplying both sides by $|16|^\beta$, we get

$$\|16h(\frac{x}{2}) - 16^2 h(\frac{x}{2^2}), z_1, \dots, z_{n-1}\|_\beta \leq \max\{|16|^\beta |4|^\beta \varphi(\frac{x}{2^2}, \frac{x}{2^2}), |16|^\beta \varphi(\frac{x}{2}, \frac{x}{2^2})\} \psi(z_1, \dots, z_{n-1}) \quad (3.30)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (3.25) and (3.30), we obtain

$$\|h(x) - 16^2 h(\frac{x}{2^2}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq 1} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (3.31)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By induction on m , we get

$$\|h(x) - 16^m h(\frac{x}{2^m}), z_1, \dots, z_{n-1}\|_\beta \leq \max_{0 \leq i \leq m-1} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (3.32)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Replacing x by $\frac{x}{2}$ in (3.32) and multiplying both sides by $|16|^\beta$, we get

$$\begin{aligned} \|16h(\frac{x}{2}) - 16^{m+1} h(\frac{x}{2^{m+1}}), z_1, \dots, z_{n-1}\|_\beta &\leq \max_{0 \leq i \leq m-1} \{|16|^{(i+1)\beta} |4|^\beta \varphi(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}}), |16|^{(i+1)\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}})\} \psi(z_1, \dots, z_{n-1}) \\ &= \max_{1 \leq i \leq m} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \end{aligned} \quad (3.33)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. By (3.25) and (3.33), we get

$$\|h(x) - 16^{m+1} h(\frac{x}{2^{m+1}}), z_1, \dots, z_{n-1}\|_\beta = \max_{0 \leq i \leq m} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (3.34)$$

for all $x \in X$, $m \in \mathbb{N}$ and $z_1, \dots, z_{n-1} \in Y$. This completes the proof of (3.32).

Taking the limit as $m \rightarrow \infty$ in (3.32), we can obtain

$$\begin{aligned} \|h(x) - T(x), z_1, \dots, z_{n-1}\|_\beta &= \|f(2x) - 4f(x) - T(x), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \\ &= \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \end{aligned}$$

for all $x \in X$, $z_1, \dots, z_{n-1} \in Y$ and $m \in \mathbb{N}$, which proves (3.23).

Now we need to prove the uniqueness of T . Let T' be another quartic mapping satisfying (1.1) and (3.23).

Since

$$\begin{aligned} \lim_{k \rightarrow \infty} |16|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) &= \lim_{k \rightarrow \infty} |16|^{k\beta} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^\beta |16|^{i\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}})\} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^\beta |16|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i+1}}, \frac{x}{2^{k+i+1}}), |16|^{(i+k)\beta} \varphi(\frac{x}{2^{k+i}}, \frac{x}{2^{k+i+1}})\} \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{|4|^\beta |16|^{i\beta} \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \\ &= 0 \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. It follows from (3.24) that

$$\begin{aligned} \|T(x) - T'(x), z_1, \dots, z_{n-1}\|_\beta &= \lim_{k \rightarrow \infty} |16|^{k\beta} \|T(\frac{x}{2^k}) - T'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta \\ &\leq \lim_{k \rightarrow \infty} |16|^{k\beta} \max\{\|T(\frac{x}{2^k}) - h(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta, \|h(\frac{x}{2^k}) - T'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta\} \\ &\leq \lim_{k \rightarrow \infty} |16|^{k\beta} \tilde{\varphi}(\frac{x}{2^k}) \psi(z_1, \dots, z_{n-1}) \\ &= 0 \end{aligned}$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Considering Lemma 1.2, we prove that T is unique. $\square$

Theorem 3.3 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |4|^{m\beta} \varphi(\frac{x}{2^m}, \frac{y}{2^m}) = 0 \quad (3.35)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max\{|4|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}}) : 0 \leq i \leq m-1\} \quad (3.36)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y) \psi(z_1, \dots, z_{n-1}) \quad (3.37)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exists a quadratic mapping $Q : X \rightarrow Y$ and a quartic mapping $T : X \rightarrow Y$ such that

$$\|f(x) - Q(x) - T(x), z_1, \dots, z_{n-1}\|_\beta \leq \frac{1}{12} |\beta| \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \quad (3.38)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \{|4|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} = 0 \quad (3.39)$$

for all $x \in X$, then Q and T are unique quadratic and quartic mappings respectively satisfying (3.38).

Proof. By Theorem 3.1 and Theorem 3.2, there exist a quadratic mapping $Q_0 : X \rightarrow Y$ and a quartic mapping $T_0 : X \rightarrow Y$ such that

$$\|f(2x) - 16f(x) - Q_0(x), z_1, \dots, z_{n-1}\|_\beta \leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|4|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (3.40)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$, and

$$\|f(2x) - 4f(x) - T_0(x), z_1, \dots, z_{n-1}\|_\beta \leq \lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|16|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |16|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \quad (3.41)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Property (3) of definition 1.8 implies $|16| \leq |4|$, then by (3.40) and (3.41), we get

$$\begin{aligned} & \|f(x) - \frac{1}{12}T_0(x) + \frac{1}{12}Q_0(x), z_1, \dots, z_{n-1}\|_\beta \\ & \leq \lim_{m \rightarrow \infty} \frac{1}{12} |4|^\beta \max_{0 \leq i \leq m-1} \{|4|^{i\beta}, |16|^{i\beta}\} \max_{0 \leq i \leq m-1} \{|4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \\ & = \lim_{m \rightarrow \infty} \frac{1}{12} |4|^\beta \max_{0 \leq i \leq m-1} \{|4|^{i\beta} |4|^\beta \varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), |4|^{i\beta} \varphi(\frac{x}{2^i}, \frac{x}{2^{i+1}})\} \psi(z_1, \dots, z_{n-1}) \end{aligned} \quad (3.42)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. So we get (3.38) by letting $Q(x) = -\frac{1}{12}Q_0(x)$ and $T(x) = \frac{1}{12}T_0(x)$ for all $x \in X$.

Now we need to prove the uniqueness of Q and T , let $Q_1, T_1 : X \rightarrow Y$ be other quadratic and quartic mappings satisfying (3.38). Let $Q' = Q - Q_1$, $T' = T_1 - T$. Then

$$\begin{aligned} & \|Q'(x) - T'(x), z_1, \dots, z_{n-1}\|_\beta \\ & = \|Q(x) - Q_1(x) + T(x) - T_1(x) + f(x) - f(x), z_1, \dots, z_{n-1}\|_\beta \\ & \leq \max\{\|f(x) - Q(x) - T(x), z_1, \dots, z_{n-1}\|_\beta, \|f(x) - Q_1(x) - T_1(x), z_1, \dots, z_{n-1}\|_\beta\} \\ & \leq \frac{1}{12} |\tilde{\varphi}(x)| \psi(z_1, \dots, z_{n-1}) \end{aligned} \quad (3.43)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$.

Together with (3.39), we have

$$\lim_{k \rightarrow \infty} |4|^{k\beta} \|Q'(\frac{x}{2^k}) - T'(\frac{x}{2^k}), z_1, \dots, z_{n-1}\|_\beta = 0 \quad (3.44)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$.

Since $Q'(\frac{x}{2^k}) = \frac{1}{4^k} Q'(x)$ and $|\frac{1}{4}| > 1$, we obtain that $T'(x) = 0$ for all $x \in X$. Again by (3.44), we have $Q'(x) = 0$ for all $x \in X$. We prove that Q and T are unique. $\square$

Theorem 3.4 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} |\frac{1}{4}|^{(m+1)\beta} \varphi(2^m x, 2^m y) = 0 \quad (3.45)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \{|\frac{1}{4}|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), |\frac{1}{4}|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x)\} \quad (3.46)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y) \psi(z_1, \dots, z_{n-1}) \quad (3.47)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then the limit

$$Q(x) = \lim_{m \rightarrow \infty} 4^{-m} (f(2^{m+1}x) - 16f(2^m x)) \quad (3.48)$$

exists for all $x \in X$, and $Q(x)$ is a quadratic mapping satisfying (1.1) and the inequality

$$\|f(2x) - 16f(x) - Q(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x) \psi(z_1, \dots, z_{n-1}) \quad (3.49)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ \left| \frac{1}{4} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{4} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} = 0 \quad (3.50)$$

for all $x \in X$, then Q is a unique quadratic mapping satisfying (3.49).

Proof. The proof is similar to the proof of Theorem 3.1. $\square$

Theorem 3.5 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} \left| \frac{1}{16} \right|^{(m+1)\beta} \varphi(2^m x, 2^m y) = 0 \quad (3.51)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \left\{ \left| \frac{1}{16} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{16} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} \quad (3.52)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (3.53)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then the limit

$$T(x) = \lim_{m \rightarrow \infty} 16^{-m} (f(2^{m+1} x) - 4f(2^m x)) \quad (3.54)$$

exists for all $x \in X$, and $T(x)$ is a quartic mapping satisfying (1.1) and the inequality

$$\|f(2x) - 4f(x) - T(x), z_1, \dots, z_{n-1}\|_\beta \leq \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (3.55)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ \left| \frac{1}{16} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{16} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} = 0 \quad (3.56)$$

for all $x \in X$, then T is a unique quartic mapping satisfying (3.55).

Proof. The proof is similar to the proof of Theorem 3.2. $\square$

Theorem 3.6 Let X be a vector space and Y be a complete non-Archimedean (n, β) -normed space, where $n \geq 2$ and $0 < \beta \leq 1$. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\lim_{m \rightarrow \infty} \left| \frac{1}{16} \right|^{(m+1)\beta} \varphi(2^m x, 2^m y) = 0 \quad (3.57)$$

for all $x, y \in X$, and $\psi : \underbrace{Y \times Y \times \dots \times Y}_{n-1} \rightarrow [0, \infty)$ be a function. The limit

$$\lim_{m \rightarrow \infty} \max_{0 \leq i \leq m-1} \left\{ \left| \frac{1}{16} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{16} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} \quad (3.58)$$

exists for all $x \in X$, and is denoted by $\tilde{\varphi}(x)$. If an even mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y), z_1, \dots, z_{n-1}\|_\beta \leq \varphi(x, y)\psi(z_1, \dots, z_{n-1}) \quad (3.59)$$

for all $x, y \in X$ and $z_1, \dots, z_{n-1} \in Y$, then there exist a quadratic mapping $Q : X \rightarrow Y$ and a quartic mapping $T : X \rightarrow Y$ such that

$$\|f(x) - Q(x) - T(x), z_1, \dots, z_{n-1}\|_\beta \leq \left| \frac{1}{12} \right|^\beta \tilde{\varphi}(x)\psi(z_1, \dots, z_{n-1}) \quad (3.60)$$

for all $x \in X$ and $z_1, \dots, z_{n-1} \in Y$. Moreover, if

$$\lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \max_{k \leq i \leq m+k-1} \left\{ \left| \frac{1}{16} \right|^{(i+1)\beta} |4|^\beta \varphi(2^i x, 2^i x), \left| \frac{1}{16} \right|^{(i+1)\beta} \varphi(2^{i+1} x, 2^i x) \right\} = 0 \quad (3.61)$$

for all $x \in X$, then Q and T are unique quadratic and quartic mappings respectively satisfying (3.60).

Proof. The proof is similar to the proof of Theorem 3.3. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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