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Two types of separation axioms on supra soft topological spaces

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Abstract: In 2011, Shabir and Naz [1] employed the notion of soft sets to introduce the concept of soft topologies; and in 2014, El-Sheikh and Abd El-Latif [2] relaxed the conditions of soft topologies to construct a wider and more general class, namely supra soft topologies. In this disquisition, we continue studying supra soft topologies by presenting two kinds of supra soft separation axioms, namely supra soft T_i -spaces and supra p-soft T_i -spaces for $i = 0, 1, 2, 3, 4$. These two types are formulated with respect to the ordinary points; and the difference between them is attributed to the applied non belong relations in their definitions. We investigate the relationships between them and their parametric supra topologies; and we provide many examples to separately elucidate the relationships among spaces of each type. Then we elaborate that supra p-soft T_i -spaces are finer than supra soft T_i -spaces in the case of $i = 0, 1, 4$; and we demonstrate that supra soft T_3 -spaces are finer than supra p-soft T_3 -spaces. We point out that supra p-soft T_i -axioms imply supra p-soft T_{i-1} , however, this characterization does not hold for supra soft T_i -axioms (see, Remark (3.30)). Also, we give a complete description of the concepts of supra p-soft T_i -spaces ($i = 1, 2$) and supra p-soft regular spaces. Moreover, we define the finite product of supra soft spaces and manifest that the finite product of supra soft T_i (supra p-soft T_i) is supra soft T_i (supra p-soft T_i) for $i = 0, 1, 2, 3$. After investigating some properties of these axioms in relation with some of the supra soft topological notions such as supra soft subspaces and enriched supra soft topologies, we explore the images of these axioms under soft S^* -continuous mappings. Ultimately, we provide an illustrative diagram to show the interrelations between the initiated supra soft spaces.

Keywords: supra soft T_i -space, supra p-soft T_i -space, supra soft subspace, product supra soft spaces, supra soft topological space

MSC: 54D10, 54D15, 54D70

1 Introduction

The concept of soft sets [3] first appeared in 1999 as a virtual mathematical approach to overcome problems containing uncertainties or incomplete data. This approach is free from the difficulties that have troubled the existing mathematical approaches which utilized to solve problems associated with ambiguity such as fuzzy sets and rough sets. The theory of soft sets provides a general framework for modeling practical phenomena and brings about a rich potential for real life applications (see, for example, [4–6]).

Based on the soft set theory, the concept of soft topologies [1] was defined in 2011 as a new soft structure. Later on, many studies were carried out to study the properties of soft topologies and to compare them with those properties which exist via classical topologies (see, for example, [7–10]). In particular, the study of soft separation axioms was a great interest to researchers who studied this topic from many perspectives (see, for example, [11–14]). Recently, [15–18] corrected some errors related to soft separation axioms. The variety of the

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study of the soft separation axioms is attributed to twofold: One is the distinguished objects that we desire to separate: Are they soft points or ordinary points?, and the second one is the nature of the belong and non belong relations which we utilized in the definitions: Are they natural belong or natural non belong or partial belong or total non belong relations?

The concept of supra topologies [19] was introduced in 1983 as a wider class of general topologies. Since then several studies were done on the supra topologies and their applications (see, for example, [20–25]). In connection to generalizing the crisp mathematical structures to the soft mathematical structures, the concept of supra soft topologies [2] was established in 2014 as an extension of supra topologies. Some fundamental notions via supra soft topologies such as supra soft continuity [2], supra soft compactness [26], various types of generalized supra soft open sets and supra soft separation axioms [27–31] were introduced and investigated. In 2017, Hosny and Al-Kadi [32] generated a supra soft topology from a soft topology by utilizing a soft stack. It turns out that the study of the properties of supra soft topologies has not yet taken its due attention and many fundamental supra soft topological notions need to be formulated and discussed.

It should be noted that the existing supra soft separation axioms were studied using the notion of soft points, so the current work aims to introduce two types of supra soft separation axioms with respect to ordinary points, namely supra soft T_i -spaces and supra p-soft T_i -spaces for $i = 0, 1, 2, 3, 4$. One of the motivations to study supra soft T_i -spaces is to establish a wider family which can be easily applied to classify the objects of study; and one of the motivations to study supra p-soft T_i -spaces is to point out the significant role of a total non belong relation in obtaining results similar to the ones achieved via supra topologies, and relaxing the conditions on the concept of supra soft regular spaces.

The layout of this article is as follows: In Section 1, we recall some definitions and properties of soft sets and supra soft topologies. In Section 2, we introduce the notions of supra soft T_i -spaces ($i = 0, 1, 2, 3, 4$); and study some properties. In Section 3, we introduce the notions of supra p-soft T_i -spaces ($i = 0, 1, 2, 3, 4$); and investigate some properties. Also, we elucidate their relationships to supra soft separation axioms which are presented in Section 2. Finally, we give a conclusion and make a plan for future works.

2 Preliminaries

In this section, we recall some definitions and results as they are important for the material of our study.

Definition 2.1. [3] A pair (G, E) is said to be a soft set over a non-empty set X provided that G is a mapping of a set of parameters E into 2^X .

In this work, a soft set is denoted by G_E instead of (G, E) and it is identified as $G_E = \{(e, G(e)) : e \in E \text{ and } G(e) \in 2^X\}$. And the set of all soft sets over X under a parameters set E , is denoted by $S(X_E)$.

Definition 2.2. [33] The relative complement of a soft set G_E , denoted by G_E^c , is given by $G_E^c = (G^c)_E$, where $G^c : E \rightarrow 2^X$ is a mapping defined by $G^c(e) = X \setminus G(e)$ for each $e \in E$.

Definition 2.3. [34] A soft set G_E over X is said to be a null soft set, denoted by $\tilde{\Phi}$, if $G(e) = \emptyset$ for each $e \in E$. And its relative complement is said to be an absolute soft set, denoted by $\tilde{X}$.

Definition 2.4. [33] The intersection of two soft sets G_A and F_B over X , denoted by $G_A \tilde{\cap} F_B$, is a soft set H_D , where $D = A \cap B \neq \emptyset$, and a mapping $H : D \rightarrow 2^X$ is given by $H(d) = G(d) \cap F(d)$.

Definition 2.5. [34] The union of two soft sets G_A and F_B over X , denoted by $G_A \tilde{\cup} F_B$, is a soft set H_D , where $D = A \cup B$ and a mapping $H : D \rightarrow 2^X$ is given as follows:

$$H(d) = \begin{cases} G(d) & : d \in A - B \\ F(d) & : d \in B - A \\ G(d) \cup F(d) & : d \in A \cap B \end{cases}$$

Definition 2.6. [35] A soft set G_A is a soft subset of a soft set F_B , denoted by $G_A \widetilde{\subseteq} F_B$, if $A \subseteq B$ and for all $a \in A$, we have $G(a) \subseteq F(a)$.

The soft sets G_A and G_B are soft equal if each one of them is a soft subset of the other.

In the literature, there are many types of soft subsets, soft unions and soft intersections between two soft sets introduced and studied. For more details on this subject, we refer the reader to [36, 37] and the references mentioned therein.

Definition 2.7. [9, 10] A soft set P_E over X is called a soft point if there exists $e \in E$ and there exists $x \in X$ such that $P(e) = \{x\}$ and $P(\alpha) = \emptyset$ for each $\alpha \in E \setminus \{e\}$. A soft point will be shortly denoted by P_e^x .

Definition 2.8. [7] Let G_A and H_B be soft sets over X and Y , respectively. Then the cartesian product of G_A and H_B , denoted by $(G \times H)_{A \times B}$, is defined as $(G \times H)(a, b) = G(a) \times H(b)$ for each $(a, b) \in A \times B$.

Definition 2.9. [8] A soft mapping between $S(X_A)$ and $S(Y_B)$ is a pair (f, ϕ) , denoted also by f_ϕ , of mappings such that $f : X \rightarrow Y$, $\phi : A \rightarrow B$. Let G_A and H_B be soft subsets of $S(X_A)$ and $S(Y_B)$, respectively. Then the image of G_A and pre-image of H_B are defined by:

(i) $f_\phi(G_A) = (f_\phi(G))_B$ is a soft subset of $S(Y_B)$ such that

$$f_\phi(G)(b) = \begin{cases} \bigcup_{a \in \phi^{-1}(b)} f(G(a)) & : \phi^{-1}(b) \neq \emptyset \\ \emptyset & : \phi^{-1}(b) = \emptyset \end{cases}$$

for each $b \in B$.

(ii) $f_\phi^{-1}(H_B) = (f_\phi^{-1}(H))_A$ is a soft subset of $S(X_A)$ such that $f_\phi^{-1}(H)(a) = f^{-1}(H(\phi(a)))$ for each $a \in A$.

Definition 2.10. [8] A soft mapping $f_\phi : S(X_A) \rightarrow S(Y_B)$ is said to be injective (resp. surjective, bijective) if f and ϕ are injective (resp. surjective, bijective).

Definition 2.11. [1] A soft set x_E over X is defined by $x(e) = \{x\}$ for each $e \in E$.

Definition 2.12. [1] The collection τ of soft sets over X under a parameters set E is said to be a soft topology on X if the following three axioms hold:

- (i) $\widetilde{X}$ and $\widetilde{\Phi}$ belong to τ .
- (ii) The union of an arbitrary family of soft sets in τ belongs to τ .
- (iii) The intersection of a finite family of soft sets in τ belongs to τ .

The triple (X, τ, E) is called a soft topological space. Every member of τ is called a soft open set and its relative complement is called a soft closed set.

Definition 2.13. [1] For a soft set G_E over X and $x \in X$, we say that:

- (i) $x \in G_E$ if $x \in G(e)$ for each $e \in E$.
- (ii) $x \notin G_E$ if $x \notin G(e)$ for some $e \in E$.

Since there are other types of belong and non belong relations (see, Definition below), the notations $\in$ and $\notin$ are called respectively natural belong and natural non belong relations.

Definition 2.14. [14] Let G_E be a soft set over X and $x \in X$. We say that:

- (i) $x \in \subseteq G_E$, reading as x partially belongs to a soft set G_E , if $x \in G(e)$ for some $e \in E$.
- (ii) $x \notin \subseteq G_E$, reading as x does not totally belong to a soft set G_E , if $x \notin G(e)$ for each $e \in E$.

Definition 2.15. [14] A soft set G_E over X is said to be stable if there exists a subset S of X such that $G(e) = S$ for each $e \in E$.

Proposition 2.16. [14] Consider a soft mapping $f_\phi : S(X_A) \rightarrow S(Y_B)$ and let G_A and H_B be soft sets in $S(X_A)$ and $S(Y_B)$, respectively. Then the following statements hold.

- (i) If f is injective and $x \notin G_A$, then $f(x) \notin f_\phi(G_A)$.
- (ii) If ϕ is surjective and $x \in G_A$, then $f(x) \in f_\phi(G_A)$.
- (iii) If f_ϕ is injective and $x \notin G_A$, then $f(x) \notin f_\phi(G_A)$.
- (iv) If $y \notin H_B$, then $x \notin f_\phi^{-1}(H_B)$ for each $x \in f^{-1}(y)$.
- (v) If $y \in H_B$, then $x \in f_\phi^{-1}(H_B)$ for each $x \in f^{-1}(y)$.
- (vi) If ϕ is surjective and $y \notin H_B$, then $x \notin f_\phi^{-1}(H_B)$ for each $x \in f^{-1}(y)$.

Definition 2.17. [2] The collection μ of soft sets over X under a parameters set E is said to be a supra soft topology on X if it satisfies the first and second axioms of Definition (2.12).

The triple (X, μ, E) is called a supra soft topological space. Every member of μ is called a supra soft open set and its relative complement is called a supra soft closed set.

Definition 2.18. [2] For a soft subset H_E of (X, μ, E) , we define the following:

- (i) $\text{Int}_\mu(H_E)$ is the union of all supra soft open sets contained in H_E .
- (ii) $\text{Cl}_\mu(H_E)$ is the intersection of all supra soft closed sets containing H_E .

Theorem 2.19. [26] Let H_E and F_E be two soft subsets of (X, μ, E) . Then:

- (i) If $H_E \subseteq F_E$, then $\text{Cl}_\mu(H_E) \subseteq \text{Cl}_\mu(F_E)$.
- (ii) $P_e^x \in \text{Cl}_\mu(H_E)$ if and only if $G_E \cap H_E \neq \emptyset$ for each supra soft open set G_E containing P_e^x .

Definition 2.20. [26] Let (X, μ, E) be a supra soft topological space and Y be a non-empty subset of X . Then $\mu_Y = \{\tilde{Y} \cap G_E : G_E \in \mu\}$ is called a supra soft relative topology on Y and (Y, μ_Y, E) is called a supra soft subspace of (X, μ, E) .

Theorem 2.21. [26] Let (Y, μ_Y, E) be a supra soft subspace of (X, μ, E) . Then H_E is a supra soft closed subset of (Y, μ_Y, E) if and only if there exists a supra soft closed subset F_E of (X, μ, E) such that $H_E = \tilde{Y} \cap F_E$.

Definition 2.22. [30] A supra soft topological space (X, μ, E) is called supra soft normal if for every two disjoint supra soft closed sets H_{1_E} and H_{2_E} , there exist two disjoint supra soft open sets G_{1_E} and G_{2_E} such that $H_{1_E} \subseteq G_{1_E}$ and $H_{2_E} \subseteq G_{2_E}$.

3 Supra soft T_i -spaces ($i = 0, 1, 2, 3, 4$)

In this section, we explain the notions of supra soft T_i -spaces ($i = 0, 1, 2, 3, 4$) and investigate their main properties. Also, we construct various examples to show the relationships between them and to clarify some obtained results.

Definition 3.1. A supra soft topological space (X, μ, E) is said to be:

- (i) a supra soft T_0 -space if for every $x \neq y \in X$, there exists a supra soft open set G_E such that $x \in G_E$, $y \notin G_E$ or $y \in G_E$, $x \notin G_E$.

- (ii) a supra soft T_1 -space if for every $x \neq y \in X$, there exist supra soft open sets G_E and F_E such that $x \in G_E, y \notin G_E$ and $y \in F_E, x \notin F_E$.
- (iii) a supra soft T_2 -space if for every $x \neq y \in X$, there exist disjoint supra soft open sets G_E and F_E such that $x \in G_E, y \notin G_E$ and $y \in F_E, x \notin F_E$.

Proposition 3.2. Every supra soft T_i -space is a supra soft T_{i-1} -space, for $i = 1, 2$.

Proof. Straightforward. □

The converse of the above proposition is not necessarily true as illustrated in the following example.

Example 3.3. Let $X = \{x, y\}$ be the universe set and $E = \{e_1, e_2\}$ be a fixed set of parameters. We define the three soft sets F_E, G_E, H_E over X as follows:

$$\begin{aligned} F_E &= \{(e_1, \{x\}), (e_2, X)\}; \\ G_E &= \{(e_1, \{y\}), (e_2, \emptyset)\} \text{ and} \\ H_E &= \{(e_1, X), (e_2, \{y\})\}. \end{aligned}$$

Then $\mu_1 = \{\tilde{\Phi}, \tilde{X}, F_E, G_E\}$ and $\mu_2 = \{\tilde{\Phi}, \tilde{X}, F_E, H_E\}$ are two supra soft topologies on X . Obviously, (X, μ_1, E) is a supra soft T_0 -space and (X, μ_2, E) is a supra soft T_1 -space. On the other hand, $x \neq y$ and there does not exist a supra soft open subset of (X, μ_1, E) containing y such that x does not belong to it. So (X, μ_1, E) is not supra soft T_1 . Also, there do not exist two disjoint supra soft open subsets of (X, μ_2, E) such that one of them contains x and the other contains y . So (X, μ_2, E) is not supra soft T_2 .

Proposition 3.4. Let (X, μ, E) be a supra soft topological space. Then the collection $\mu_e = \{G(e) : G_E \in \mu\}$ forms a supra topology on X for each $e \in E$.

Proof. Straightforward. □

Henceforth, we use the term a parametric supra topology for μ_e which given in the above proposition.

Proposition 3.5. If (X, μ, E) is a supra soft T_2 -space, then a parametric supra topological space (X, μ_e) is supra T_2 .

Proof. Let $x \neq y \in X$. Since (X, μ, E) is a supra soft T_2 -space, then there exist disjoint supra soft open sets F_E and G_E containing x and y , respectively. Therefore, $F(e)$ and $G(e)$ are disjoint supra soft open sets such that $x \in F(e)$ and $y \in G(e)$. This shows that (X, μ_e) is a supra T_2 -space. □

To see that the converse of the above proposition is not necessarily true, we give the following example.

Example 3.6. Assume that X and E are the same as in Example (3.3). Let the two soft sets F_E, G_E over X be given as follows:

$$\begin{aligned} F_E &= \{(e_1, \{x\}), (e_2, \{y\})\} \text{ and} \\ G_E &= \{(e_1, \{y\}), (e_2, \{x\})\}. \end{aligned}$$

Then $\mu = \{\tilde{\Phi}, \tilde{X}, F_E, G_E\}$ is a supra soft topology on X . Since there does not exist a proper supra soft open subset of $\tilde{X}$ containing x or y , then (X, μ, E) is not a supra soft T_0 -space. On the other hand, the two parametric supra topological spaces (X, μ_{e_1}) and (X, μ_{e_2}) are supra T_2 .

We draw the attention of the reader to the fact that there is no a relationship between supra soft T_i -spaces and their parametric supra topological spaces in the case of $i = 0, 1$. This matter can be clarified in the following example.

Example 3.7. One of the implications is illustrated in the above example.

In Example (3.3), it can be noted that $(X, \mu_{1_{e_2}})$ is not supra T_0 in spite of (X, μ_1, E) being supra soft T_0 . And $(X, \mu_{2_{e_1}})$ and $(X, \mu_{2_{e_2}})$ are not supra T_1 in spite of (X, μ_2, E) being supra soft T_1 .

Definition 3.8. A supra soft topological space (X, μ, E) is said to be stable if every supra soft open set is stable.

Proposition 3.9. If (X, μ, E) is stable, then it is supra soft T_i if and only if a parametric supra topological space (X, μ_e) is supra T_i for each $i = 0, 1, 2$.

Proof. We only prove the proposition in the case of $i = 0$. The other cases follow similar lines.

Necessity: Suppose that $x \neq y$ in a supra soft T_0 -space (X, μ, E) . Then there exists a supra soft open subset G_E of (X, μ, E) such that $x \in G_E, y \notin G_E$ or $y \in G_E, x \notin G_E$. Say, $x \in G_E, y \notin G_E$. Obviously, $G(e)$ is a supra open subset of (X, μ_e) and $x \in G(e)$ for each $e \in E$. Since (X, μ, E) is stable, then $y \notin G(e)$ for each $e \in E$. Thus, (X, μ_e) is supra T_0 .

Sufficiency: Let $x \neq y$ in a supra T_0 -space (X, μ_e) . Then there exists a supra open subset G of (X, μ_e) such that $x \in G, y \notin G$ or $y \in G, x \notin G$. Say, $x \in G, y \notin G$. By the stability of (X, μ, E) , there is a supra soft open subset M_E of (X, μ, E) such that $M(e) = G$ for each $e \in E$. Hence, (X, μ, E) is supra soft T_0 . $\square$

The following result can be proved easily.

Proposition 3.10. If (X, μ, E) is a stable supra soft topological space, then $\mu_{e_i} = \mu_{e_j}$ for each $e_i \neq e_j \in E$.

Example (3.6) clarifies that the converse of the above proposition is not necessarily true.

Proposition 3.11. If x_E is a supra soft closed subset of (X, μ, E) for each $x \in X$, then (X, μ, E) is a supra soft T_1 -space.

Proof. Let $x \neq y \in X$ and let the given condition be satisfied. Then x_E^c and y_E^c are supra soft open sets such that $y \in x_E^c$ and $x \in y_E^c$. Clearly, $x \notin x_E^c$ and $y \notin y_E^c$. Hence, (X, μ, E) is a supra soft T_1 -space. $\square$

In the next example, we point out that the converse of the above proposition need not be true in general.

Example 3.12. Assume that (X, μ_2, E) is the same as in Example (3.3). It can be seen that x_E and y_E are not supra soft closed sets in spite of (X, μ, E) being a supra soft T_1 -space.

Proposition 3.13. If (X, μ, E) is a supra soft T_2 -space, then x_E is supra soft closed for each $x \in X$.

Proof. For each $y_i \in X \setminus \{x\}$, it follows that there exist disjoint supra soft open sets G_{iE} and F_{iE} such that $x \in G_{iE}$ and $y_i \in F_{iE}$. Therefore, $X \setminus \{x\} = \bigcup_{i \in I} F_i(e)$ for each $e \in E$. Thus, $\bigcup F_{iE} = L_E$ is a supra soft open set, where $L(e) = X \setminus \{x\}$ for each $e \in E$. Hence, x_E is supra soft closed. $\square$

The converse of the above proposition fails as demonstrated in the next example.

Example 3.14. Assume that $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$; and let the three soft sets F_E, G_E, H_E over X be given as follows:

$$\begin{aligned} F_E &= \{(e_1, \{x, y\}), (e_2, \{x, y\})\}; \\ G_E &= \{(e_1, \{x, z\}), (e_2, \{x, z\})\} \text{ and} \\ H_E &= \{(e_1, \{y, z\}), (e_2, \{y, z\})\}. \end{aligned}$$

Then $\mu = \{\tilde{\Phi}, \tilde{X}, F_E, G_E, H_E\}$ is a supra soft topology on X . Obviously, x_E, y_E and z_E are supra soft closed sets. On the other hand, (X, μ, E) is not a supra soft T_2 -space.

Theorem 3.15. (X, μ, E) is a supra soft T_2 -space if and only if for each $x \neq y$, there exists a supra soft open set G_E such that $x \in G_E$ and $y \notin Cl_\mu(G_E)$.

Proof. Necessity: For each $x \neq y$ in a supra soft T_2 -space, there exist supra soft open sets G_E and F_E such that $x \in G_E$, $y \in F_E$ and $G_E \widetilde{\cap} F_E = \emptyset$. Now, $G_E \widetilde{\subseteq} F_E^c$ implies that $Cl_\mu(G_E) \widetilde{\subseteq} F_E^c$. Since $y \notin F^c(e)$ for each $e \in E$, then $y \notin Cl_\mu(G_E)$.

Sufficiency: Let $x \neq y \in X$ and G_E be a supra soft open set such that $x \in G_E$ and $y \notin Cl_\mu(G_E)$. Put $Cl_\mu(G_E) = H_E$. Now, $y \in H_E^c$ and H_E^c is supra soft open. Then $Cl_\mu(G_E) \widetilde{\cap} H_E^c = \emptyset$. This shows that (X, μ, E) is a supra soft T_2 -space. $\square$

Hereafter, we define supra soft regular spaces and characterize them.

Definition 3.16. A supra soft topological space (X, μ, E) is said to be supra soft regular if for every supra soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint supra soft open sets G_E and F_E such that $H_E \widetilde{\subseteq} G_E$ and $x \in F_E$.

Proposition 3.17. For each supra soft open or supra soft closed subset G_E of a supra soft regular space (X, μ, E) , we have $x \in G_E$ if and only if $x \in G_E$.

Proof. Necessity: In view of the Definition (2.13) and Definition (2.14), we find that $x \in G_E$ implies $x \in G_E$ for any soft set G_E .

Sufficiency: Consider G_E is a supra soft open set and let $x \in G_E$. Then we have two cases:

Case 1 $x \in G(e)$ for each $e \in E$. Thus, the result holds.

Case 2 There exists $e \in E$ such that $x \notin G(e)$. Now, $x \notin G_E^c$. Since (X, μ, E) is supra soft regular, then there exist disjoint supra soft open sets V_E and W_E such that $G_E^c \widetilde{\subseteq} V_E$ and $x \in W_E$. Since $x \notin G(e)$, we obtain $V(e) \cap W(e) \neq \emptyset$. But this contradicts the disjointness of V_E and W_E . Thus, it must be that $x \in G(e)$ for each $e \in E$.

The sufficient condition in the case of G_E being a supra soft closed set can be proven in a similar way. $\square$

Corollary 3.18. Every supra soft regular space (X, μ, E) is stable.

Corollary 3.19. If (X, μ, E) is a supra soft regular space, then the parametric supra topologies μ_e are equal for each $e \in E$.

Corollary 3.20. A finite supra soft regular space (X, μ, E) contains at most $2^{|X|}$ supra soft open sets.

Example (3.14) clarifies that the converse of Corollary (3.18) and Corollary (3.19) is not always true; and Example (3.6) clarifies that the converse of Corollary (3.20) is not always true.

Proposition 3.21. If (X, μ, E) is a supra soft regular space, then for each $P_e^x \in X$ and supra soft open set F_E containing P_e^x , there exists a supra soft open set G_E such that $x \in G_E \widetilde{\subseteq} Cl_\mu(G_E) \widetilde{\subseteq} F_E$.

Proof. Let $x \in X$ and F_E be a supra soft open set containing P_e^x . Then F_E^c is supra soft closed and $x \notin F_E^c$. Therefore, there are disjoint supra soft open sets W_E and G_E such that $F_E^c \widetilde{\subseteq} W_E$ and $x \in G_E$. Therefore, $G_E \widetilde{\subseteq} W_E^c \widetilde{\subseteq} F_E$. Thus, $Cl_\mu(G_E) \widetilde{\subseteq} W_E^c \widetilde{\subseteq} F_E$. $\square$

The converse of the above proposition need not be true in general as shown in the next example.

Example 3.22. Assume that (X, μ, E) is the same as in Example (3.6). Then it can be seen that for each supra soft open set F_E containing x , there exists a supra soft open set G_E such that $x \in G_E \widetilde{\subseteq} Cl_\mu(G_E) \widetilde{\subseteq} F_E$. On the other hand, (X, μ, E) is not a supra soft regular space.

Theorem 3.23. For any supra soft regular space (X, μ, E) , the following three statements are equivalent:

- (i) (X, μ, E) is a supra soft T_2 -space.
- (ii) (X, μ, E) is a supra soft T_1 -space.

(iii) (X, μ, E) is a supra soft T_0 -space.

Proof. Obviously, (i) $\rightarrow$ (ii) $\rightarrow$ (iii).

(iii) $\rightarrow$ (i): Let $x \neq y \in X$ and (X, μ, E) be a supra soft T_0 -space. Then there exists a supra soft open set G_E such that $x \in G_E$ and $y \notin G_E$, or $y \in G_E$ and $x \notin G_E$. Say, $x \in G_E$ and $y \notin G_E$. By Proposition (3.17), we find that $y \notin G_E$. Now, we have $x \notin G_E^c$ and $y \in G_E^c$. Since (X, μ, E) is supra soft regular, then there exist two disjoint supra soft open sets W_{1_E} and W_{2_E} such that $x \in W_{1_E}$ and $y \in G_E^c \subseteq W_{2_E}$. This completes the proof. $\square$

Definition 3.24. A supra soft topological space (X, μ, E) is said to be:

(i) a supra soft T_3 if it is both supra soft regular and supra soft T_1 .

(ii) a supra soft T_4 if it is both supra soft normal and supra soft T_1 .

Proposition 3.25. Every supra soft T_3 -space is supra soft T_2 .

Proof. It follows directly from Theorem (3.23). $\square$

Corollary 3.26. If (X, μ, E) is a supra soft T_3 -space, then x_E is supra soft closed for all $x \in X$.

The converse of the above proposition need not true in general as shown in the next example.

Example 3.27. Assume that $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$; and let the soft sets $\{F_{iE} : i = 1, 2, \dots, 6\}$ over X be given as follows:

$$\begin{aligned} F_{1E} &= \{(e_1, \{x\}), (e_2, \{x\})\}; \\ F_{2E} &= \{(e_1, \{y\}), (e_2, \{y\})\}; \\ F_{3E} &= \{(e_1, \{x, y\}), (e_2, \{x, y\})\}; \\ F_{4E} &= \{(e_1, \{x, z\}), (e_2, \{x, z\})\}; \\ F_{5E} &= \{(e_1, \{y, z\}), (e_2, \{y, z\})\} \text{ and} \\ F_{6E} &= \{(e_1, \{x, y\}), (e_2, X)\}. \end{aligned}$$

Then $\mu = \{\tilde{\Phi}, \tilde{X}, F_{iE} : i = 1, 2, \dots, 6\}$ is a supra soft topology on X . Obviously, (X, μ, E) is a supra soft T_2 -space. On the other hand, F_{6E}^c is a supra soft closed set such that $z \notin F_{6E}^c$. Since there are no two disjoint supra soft open sets such that one of them contains z and the other contains F_{6E}^c , then (X, μ, E) is not a supra soft regular space. Hence, it is not a supra soft T_3 -space.

Theorem 3.28. If (X, μ, E) is a supra soft T_3 -space, then a parametric supra topological space (X, μ_e) is supra T_3 for each $e \in E$.

Proof. Since (X, μ, E) is supra soft T_3 , then it is stable. So, from Proposition (3.9), we find that (X, μ_e) is supra T_1 . It remains to prove the supra regularity of (X, μ_e) . Suppose that H is a supra closed subset of X such that $x \notin H$. By Corollary (3.18), there exists a supra soft closed set F_E such that $F(e) = H$ for each $e \in E$. Obviously, $x \notin F_E$. By hypothesis, there exist two disjoint supra soft open sets G_E and W_E such that $x \in G_E$ and $F_E \subseteq W_E$. This means that $G(e)$ and $W(e)$ are two disjoint supra open subsets of (X, μ_e) such that $x \in G(e)$ and $F(e) = H \subseteq W(e)$. Thus, (X, μ_e) is supra regular. Hence, (X, μ_e) is supra T_3 . $\square$

The converse of the above theorem is not necessarily true as illustrated in the following example.

Example 3.29. Assume that (X, μ, E) is the same as in Example (3.27). Then the two parametric supra topologies $\mu_{e_1} = \mu_{e_2} = \{\emptyset, X, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ are supra T_3 -spaces. On the other hand, we demonstrate in Example (3.27) that (X, μ, E) is not a supra soft T_3 -space.

Remark 3.30. We prove that the concepts of supra soft T_3 -spaces and supra soft T_4 -spaces are independent of each other, using the two examples below.

Example 3.31. Assume that (X, μ, E) is the same as in Example (3.27). We clarify that (X, μ, E) is supra soft T_2 , but it is not supra soft T_3 . On the other hand, the collection of all supra soft closed subsets of (X, μ, E) is $\{\tilde{\Phi}, \tilde{X}, F_{1E}, F_{2E}, F_{3E}^c, F_{4E}, F_{5E}, F_{6E}^c\}$. It can be shown that (X, μ, E) is supra soft normal. Moreover, it is supra soft T_4 .

Example 3.32. Assume that $X = \{v, w, x, y, z\}$ and $E = \{e_1, e_2\}$; and let the soft sets $\{F_{iE} : i = 1, 2, \dots, 23\}$ over X be given as follows:

$$\begin{aligned} F_{1E} &= \{(e_1, \{v, x\}), (e_2, \{v, x\})\}; \\ F_{2E} &= \{(e_1, \{v, y\}), (e_2, \{v, y\})\}; \\ F_{3E} &= \{(e_1, \{v, z\}), (e_2, \{v, z\})\}; \\ F_{4E} &= \{(e_1, \{w, x\}), (e_2, \{w, x\})\}; \\ F_{5E} &= \{(e_1, \{w, y\}), (e_2, \{w, y\})\}; \\ F_{6E} &= \{(e_1, \{w, z\}), (e_2, \{w, z\})\}; \\ F_{7E} &= \{(e_1, \{x, y\}), (e_2, \{x, y\})\}; \\ F_{8E} &= \{(e_1, \{x, z\}), (e_2, \{x, z\})\}; \\ F_{9E} &= \{(e_1, \{v, w, x\}), (e_2, \{v, w, x\})\}; \\ F_{10E} &= \{(e_1, \{v, w, y\}), (e_2, \{v, w, y\})\}; \\ F_{11E} &= \{(e_1, \{v, w, z\}), (e_2, \{v, w, z\})\}; \\ F_{12E} &= \{(e_1, \{v, x, y\}), (e_2, \{v, x, y\})\}; \\ F_{13E} &= \{(e_1, \{v, x, z\}), (e_2, \{v, x, z\})\}; \\ F_{14E} &= \{(e_1, \{v, y, z\}), (e_2, \{v, y, z\})\}; \\ F_{15E} &= \{(e_1, \{w, x, y\}), (e_2, \{w, x, y\})\}; \\ F_{16E} &= \{(e_1, \{w, x, z\}), (e_2, \{w, x, z\})\}; \\ F_{17E} &= \{(e_1, \{w, y, z\}), (e_2, \{w, y, z\})\}; \\ F_{18E} &= \{(e_1, \{x, y, z\}), (e_2, \{x, y, z\})\}; \\ F_{19E} &= \{(e_1, \{v, w, x, y\}), (e_2, \{v, w, x, y\})\}; \\ F_{20E} &= \{(e_1, \{v, w, x, z\}), (e_2, \{v, w, x, z\})\}; \\ F_{21E} &= \{(e_1, \{v, w, y, z\}), (e_2, \{v, w, y, z\})\}; \\ F_{22E} &= \{(e_1, \{v, x, y, z\}), (e_2, \{v, x, y, z\})\} \text{ and} \\ F_{23E} &= \{(e_1, \{w, x, y, z\}), (e_2, \{w, x, y, z\})\}. \end{aligned}$$

Then $\mu = \{\tilde{\Phi}, \tilde{X}, F_{iE} : i = 1, 2, \dots, 23\}$ is a supra soft topology on X . It can be shown that (X, μ, E) is a supra soft T_3 -space. On the other hand, the two soft sets F_{9E}^c and F_{18E}^c are two disjoint supra soft closed sets. Since every supra soft open set containing F_{9E}^c intersects any supra soft open set containing F_{18E}^c , then (X, μ, E) is not a supra soft normal space. Hence, it is not a supra soft T_4 -space.

Theorem 3.33. If (X, μ, E) is stable, then (X, μ, E) is a supra soft T_4 -space if and only if the parametric supra topological space (X, μ_e) is supra T_4 for each $e \in E$.

Proof. Suppose that (X, μ, E) is a stable supra soft T_4 -space. Then it follows from Proposition (3.9), that (X, μ_e) is supra soft T_1 . Let F and H be two disjoint supra closed subsets of (X, μ_e) . Since (X, μ, E) is stable, then there exist two disjoint supra soft closed subsets U_E and V_E of (X, μ, E) such that $U(e) = F$ and $V(e) = H$ for each $e \in E$. By hypothesis, there exist two disjoint supra soft open subsets G_E and W_E of (X, μ, E) such that $U_E \subseteq G_E$ and $V_E \subseteq W_E$. This implies that $G(e)$ and $W(e)$ are disjoint supra open subsets of (X, μ_e) containing F and H , respectively. Thus, (X, μ_e) is supra normal. Hence, it is supra T_4 .

To prove the converse, let $x \neq y \in X$. Since (X, μ_e) is supra T_4 , then there are two supra open sets G and W containing x and y , respectively, such that $x \notin W$ and $y \notin G$; and since (X, μ, E) is stable, then there are two supra soft open sets M_E and N_E such that $M(e) = G$ and $N(e) = W$ for each $e \in E$. This means that (X, μ, E) is supra soft T_1 . Suppose that U_E and V_E are two non-null disjoint supra soft closed subsets of (X, μ, E) . By the stability of (X, μ, E) , it follows that $U(e)$ and $V(e)$ are two non-empty disjoint supra closed subsets of (X, μ_e) for each $e \in E$. Since (X, μ_e) is supra normal, then there are two disjoint supra open sets G and W containing $U(e)$ and $V(e)$, respectively. From Corollary (3.19), it follows that the soft sets M_E and N_E , which are defined as

$M(e) = G$ and $N(e) = W$ for each $e \in E$, are disjoint supra soft open sets containing U_E and V_E , respectively. Thus, (X, μ, E) is supra soft normal. Hence, it is supra soft T_4 . $\square$

Theorem 3.34. Every soft subspace (Y, μ_Y, E) of a supra soft T_i -space (X, μ, E) is a supra soft T_i -space, for $i = 0, 1, 2, 3$.

Proof. We prove the theorem in the case of $i = 3$ and the other cases follow similar lines.

To prove that (Y, μ_Y, E) is supra soft T_1 , let $x \neq y \in Y$. Since (X, μ, E) is supra soft T_1 , then there exist supra soft open sets G_E and F_E such that $x \in G_E, y \notin G_E$ and $y \in F_E, x \notin F_E$. Therefore, $x \in U_E = \widetilde{Y} \cap G_E$ and $y \in V_E = \widetilde{Y} \cap F_E$. Since $y \notin G_E$ and $x \notin F_E$, then $y \notin U_E$ and $x \notin V_E$. Thus, (Y, μ_Y, E) is supra soft T_1 .

To prove the supra soft regularity of (Y, μ_Y, E) , let $y \in Y$ and L_E be a supra soft closed subset of (Y, μ_Y, E) such that $y \notin L_E$. Then there exists a supra soft closed subset H_E of (X, μ, E) such that $L_E = \widetilde{Y} \cap H_E$. Since $y \notin H_E$, then there exist disjoint supra soft open sets G_E and F_E such that $H_E \subseteq G_E$ and $y \in F_E$. Now, we find that $L_E \subseteq W_{1_E} = \widetilde{Y} \cap G_E, y \in W_{2_E} = \widetilde{Y} \cap F_E$ and $W_{1_E} \cap W_{2_E} = \widetilde{\emptyset}$. Thus, (Y, μ_Y, E) is supra soft regular.

Hence, (Y, μ_Y, E) is supra soft T_3 . $\square$

Theorem 3.35. Every supra soft closed subspace (Y, μ_Y, E) of a supra soft T_4 -space (X, μ, E) is a supra soft T_4 -space.

Proof. By the above theorem, a supra soft closed subspace (Y, μ_Y, E) of a supra soft T_4 -space (X, μ, E) is supra soft T_1 . To prove the supra soft normality of (Y, μ_Y, E) , let H_{1_E} and H_{2_E} be two disjoint supra soft closed subsets of (Y, μ_Y, E) . Since (Y, μ_Y, E) is a supra soft closed subspace of (X, μ, E) , then H_{1_E} and H_{2_E} are two disjoint supra soft closed subsets of (X, μ, E) as well. Since (X, μ, E) is supra soft normal, then there are two disjoint supra soft open subsets G_{1_E} and G_{2_E} of (X, μ, E) such that $H_{1_E} \subseteq G_{1_E}$ and $H_{2_E} \subseteq G_{2_E}$. Obviously, $\widetilde{Y} \cap G_{1_E}$ and $\widetilde{Y} \cap G_{2_E}$ are disjoint supra soft open subsets of (Y, μ_Y, E) containing H_{1_E} and H_{2_E} , respectively. This completes the proof. $\square$

Definition 3.36. Let $\{(X_i, \mu_i, A_i) : i = 1, 2, \dots, n\}$ be the collection of supra soft topological spaces. Then $\prod_{i=1}^n \mu_i = \{\prod_{i=1}^n G_{A_i} : G_{A_i} \in \mu_i\}$ defines a supra soft topology on $\prod_{i=1}^n X_i$ under a parameters set $\prod_{i=1}^n A_i$. We call $\prod_{i=1}^n \mu_i$ a finite product supra soft topology and $(\prod_{i=1}^n X_i, \prod_{i=1}^n \mu_i, \prod_{i=1}^n A_i)$ a finite product supra soft space.

Lemma 3.37. If $H_{A_1 \times A_2}$ is a supra soft closed subset of a product supra soft space $(X \times Y, \mu_1 \times \mu_2, A_1 \times A_2)$, then $H_{A_1 \times A_2} = [(G_{A_1})^c \times \widetilde{Y}] \cup [\widetilde{X} \times (U_{A_2})^c]$ for some $G_{A_1} \in \mu_1$ and $U_{A_2} \in \mu_2$.

Theorem 3.38. The finite product of supra soft T_i -spaces (X_r, μ_r, A_r) is a supra soft T_i -space, for $i = 0, 1, 2, 3$.

Proof. We prove the theorem for two supra soft topological spaces in the case of $i = 0, 3$. The other cases follow similar lines.

(i) Consider two supra soft T_0 -spaces (X_1, μ_1, A_1) and (X_2, μ_2, A_2) and let $(x_1, y_1) \neq (x_2, y_2)$ in $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$. Without lose of generality, let $x_1 \neq x_2$. Then there exists a supra soft open subset G_{A_1} of (X_1, μ_1, A_1) such that $x_1 \in G_{A_1}$ and $x_2 \notin G_{A_1}$ or $x_2 \in G_{A_1}$ and $x_1 \notin G_{A_1}$. Say, $x_1 \in G_{A_1}$ and $x_2 \notin G_{A_1}$. Therefore, $(x_1, y_1) \in G_{A_1} \times \widetilde{X}_2$ and $(x_2, y_2) \notin G_{A_1} \times \widetilde{X}_2$. Thus, $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$ is a supra soft T_0 -space.

(ii) Let $H_{A_1 \times A_2}$ be a supra soft closed subset of a supra soft space $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$. Then $H_{A_1 \times A_2} = [(G_{A_1})^c \times \widetilde{X}_2] \cup [\widetilde{X}_1 \times (U_{A_2})^c]$ for some $G_{A_1} \in \mu_1$ and $U_{A_2} \in \mu_2$. For every $(x, y) \notin H_{A_1 \times A_2}$, we have $(x, y) \notin (G_{A_1})^c \times \widetilde{X}_2$ and $(x, y) \notin \widetilde{X}_1 \times (U_{A_2})^c$. So $x \notin (G_{A_1})^c$ and $y \notin (U_{A_2})^c$. Since (X_1, μ_1, A_1) and (X_2, μ_2, A_2) are supra soft regular, then there exist disjoint supra soft open sets $F_{1_{A_1}}$ and $F_{2_{A_1}}$ containing x and $(G_{A_1})^c$, respectively, and there exist disjoint supra soft open sets $F_{3_{A_2}}$ and $F_{4_{A_2}}$ containing y and $(U_{A_2})^c$, respectively. Thus, $H_{A_1 \times A_2} \subseteq [F_{2_{A_1}} \times \widetilde{X}_2] \cup [\widetilde{X}_1 \times F_{4_{A_2}}]$ and $(x, y) \in [F_{1_{A_1}} \times F_{3_{A_2}}]$. Since $[F_{1_{A_1}} \times F_{3_{A_2}}] \cap \{[F_{2_{A_1}} \times \widetilde{X}_2] \cup [\widetilde{X}_1 \times F_{4_{A_2}}]\} = \widetilde{\emptyset}$, then $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$ is supra soft regular. From (i) above and Theorem (3.23), the desired result is proved. $\square$

Remark 3.39. It is well known that a supra soft topological space (X, μ, E) is supra topological space if a set of parameters E is a singleton. We also know, from general topology, that a Sorgenfrey Line space is normal, however, the product of two Sorgenfrey Line spaces is not normal. This explains why supra soft T_4 -spaces are excluded from the above theorem.

Definition 3.40. A soft mapping $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ is said to be:

- (i) Soft S^* -continuous if the inverse image of each supra soft open subset of (Y, θ, B) is a supra soft open subset of (X, μ, A) .
- (ii) Soft S^* -open (resp. Soft S^* -closed) if the image of each supra soft open (resp. supra soft closed) subset of (X, μ, A) is a supra soft open (resp. supra soft closed) subset of (Y, θ, B) .
- (iii) Soft S^* -homeomorphism if it is bijective, soft S^* -continuous and soft S^* -open.

Definition 3.41. A property is said to be a supra soft topological property if the property is preserved by a soft S^* -homeomorphism mapping.

Proposition 3.42. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a soft S^* -continuous mapping such that f is an injective mapping. If (Y, θ, B) is supra soft T_i , then (X, μ, A) is supra soft T_i for $i = 0, 1, 2$.

Proof. We only prove the proposition for $i = 2$.

Consider two distinct points a and b in X . By injectivity of f , there are two distinct points x and y in Y such that $f(a) = x$ and $f(b) = y$. Since (Y, θ, B) is a supra soft T_2 -space, then there are two disjoint supra soft open sets G_B and F_B such that $x \in G_B$ and $y \in F_B$. From Proposition (2.16), we obtain $a \in f_\phi^{-1}(G_B)$ and $b \in f_\phi^{-1}(F_B)$. Obviously, $f_\phi^{-1}(G_B) \cap f_\phi^{-1}(F_B) = \emptyset$. Thus, (X, μ, A) is a supra soft T_2 -space. $\square$

For the sake of brevity, we omit the proofs of the following four results.

Proposition 3.43. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a bijective soft S^* -continuous mapping. If (Y, θ, B) is supra soft T_i , then (X, μ, A) is supra soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 3.44. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a bijective soft S^* -open mapping. If (X, μ, A) is supra soft T_i , then (Y, θ, B) is supra soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 3.45. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a soft S^* -homeomorphism mapping. Then (X, μ, A) is supra soft T_i if and only if (Y, θ, B) is supra soft T_i for $i = 0, 1, 2, 3, 4$.

Corollary 3.46. The property of being supra soft T_i is a supra soft topological property, for $i = 0, 1, 2, 3, 4$.

4 Supra p-soft T_i -spaces ($i = 0, 1, 2, 3, 4$)

The main purpose of this section is to define and study the concepts of supra p-soft T_i -spaces ($i = 0, 1, 2, 3, 4$) by using natural belong and total non belong relations. With the help of examples, the relationships between them, as well as between them and supra soft T_i -spaces introduced in the previous section are illustrated.

Definition 4.1. A supra soft topological space (X, μ, E) is said to be:

- (i) a supra p-soft T_0 -space if for every $x \neq y \in X$, there exists a supra soft open set G_E such that $x \in G_E, y \notin G_E$ or $y \in G_E, x \notin G_E$.
- (ii) a supra p-soft T_1 -space if for every $x \neq y \in X$, there exist supra soft open sets G_E and F_E such that $x \in G_E, y \notin G_E$ and $y \in F_E, x \notin F_E$.

(iii) a supra p -soft T_2 -space if for every $x \neq y \in X$, there exist two disjoint supra soft open sets G_E and F_E such that $x \in G_E$, $y \notin G_E$ and $y \in F_E$, $x \notin F_E$.

Remark 4.2. For any two disjoint soft sets G_E and F_E such that $x \in G_E$ and $y \in F_E$, we find that $y \notin G_E$ and $x \notin F_E$ if and only if $y \notin G_E$ and $x \notin F_E$. So the definitions of supra p -soft T_2 -spaces and supra soft T_2 -spaces are equivalent. Hence, the properties of a supra soft T_2 -spaces are still valid for a supra p -soft T_2 -spaces.

Proposition 4.3. Every supra p -soft T_i -space is a supra soft T_i -space, for $i = 0, 1$.

Proof. It follows from the fact that a total non belong relation $\notin$ implies a non belong relation $\not\subseteq$. $\square$

The next example points out that the converse of the above proposition is not true.

Example 4.4. Assume that (X, μ_2, E) is the same as in Example (3.3). We mentioned that (X, μ_2, E) is supra soft T_1 . On the other hand, $x \neq y$ and there does not exist a supra soft open subset of (X, μ_2, E) which contains one of the points x or y such that the other point does not totally belong to it. So (X, μ_2, E) is not supra p -soft T_0 .

Proposition 4.5. If (X, μ, E) is a supra p -soft T_i -space, then a parametric supra topological space (X, μ_e) is supra T_i for $i = 0, 1, 2$.

Proof. In the case of $i = 2$, the proof follows from Proposition (3.5) and Remark (4.2).

In the case of $i = 1$, let $x \neq y \in X$. Since (X, μ, E) is a supra soft T_1 -space, then there exist two supra soft open sets F_E and G_E containing x and y , respectively, such that $y \notin F_E$ and $x \notin G_E$. Thus, $F(e)$ and $G(e)$ are non-empty proper supra soft open subsets of (X, μ_e) containing x and y , respectively, such that $y \notin F(e)$ and $x \notin G(e)$. This shows that (X, μ_e) is a supra T_2 -space.

One can similarly prove the proposition in the case of $i = 0$. $\square$

Example (3.6) obviously shows that the converse of the above proposition does not hold.

We discuss in the following results some properties related to a p -soft T_0 -space.

Lemma 4.6. Let H_E be a soft subset of (X, μ, E) and $x \in X$. Then $x \notin Cl_\mu(H_E)$ if and only if there exists a supra soft open set F_E containing x such that $H_E \cap F_E = \tilde{\Phi}$.

Proof. $\Rightarrow$: Let $x \notin Cl_\mu(H_E)$. Then $x \in (Cl_\mu(H_E))^c = F_E$. Thus, $H_E \cap F_E = \tilde{\Phi}$.

$\Leftarrow$: Let F_E be a supra soft open set such that $x \in F_E$ and $H_E \cap F_E = \tilde{\Phi}$. Then $H_E \subseteq F_E^c$. Therefore, $Cl_\mu(H_E) \subseteq F_E^c$. Since $x \notin F_E^c$, then $x \notin Cl_\mu(H_E)$. $\square$

Theorem 4.7. If (X, μ, E) is a supra p -soft T_0 -space, then $Cl_\mu(x_E) \neq Cl_\mu(y_E)$ for every $x \neq y \in X$.

Proof. Let x, y be two distinct points in a supra p -soft T_0 -space. Then there is a supra soft open set G_E such that $x \in G_E$ and $y \notin G_E$ or $y \in G_E$ and $x \notin G_E$. Say, $x \in G_E$ and $y \notin G_E$. Now, $y_E \cap G_E = \tilde{\Phi}$. So, by the above lemma, $x \notin Cl_\mu(y_E)$. But $x \in Cl_\mu(x_E)$. This shows that $Cl_\mu(x_E) \neq Cl_\mu(y_E)$. $\square$

Corollary 4.8. If (X, μ, E) is a supra p -soft T_0 -space, then $Cl_\mu(P_\alpha^x) \neq Cl_\mu(P_\beta^y)$ for each $x \neq y \in X$ and $\alpha, \beta \in E$.

Proof. From the proof of Theorem(4.7), we have $x \notin Cl_\mu(y_E)$ or $y \notin Cl_\mu(x_E)$ for each $x \neq y$. Say, $x \notin Cl_\mu(y_E)$. As $Cl_\mu(P_\beta^y) \subseteq Cl_\mu(y_E)$, then $x \notin Cl_\mu(P_\beta^y)$. Obviously, $x \in Cl_\mu(P_\alpha^x)$. Hence, $Cl_\mu(P_\alpha^x) \neq Cl_\mu(P_\beta^y)$, as required. $\square$

It can be seen from the next example that the converse of the above theorem fails.

Example 4.9. Assume that (X, μ_1, E) is the same as in Example (3.3). Then $Cl_\mu(x_E) \neq Cl_\mu(y_E)$. On the other hand, we show in Example (4.4) that (X, μ_1, E) is not a supra p -soft T_0 -space.

In the following results, we characterize a supra p -soft T_1 -space and investigate some of its features.

Theorem 4.10. (X, μ, E) is a supra p -soft T_1 -space if and only if x_E is a supra soft closed set for all $x \in X$.

Proof. Necessity: For each $y_i \in X \setminus \{x\}$, there is a supra soft open set G_{iE} such that $y_i \in G_{iE}$ and $x \notin G_{iE}$. Therefore, $X \setminus \{x\} = \bigcup_{i \in I} G_{iE}(e)$ and $x \notin \bigcup_{i \in I} G_{iE}(e)$ for each $e \in E$. Thus, $\bigcup_{i \in I} G_{iE} = \widetilde{X \setminus \{x\}}$ is a supra soft open set. Hence, x_E is supra soft closed.

Sufficiency: For each $x \neq y$, we have x_E and y_E are supra soft closed sets. Now, $(x_E)^c$ and $(y_E)^c$ are two supra soft open sets containing y and x , respectively. Since $x \notin (X \setminus \{x\})_E$ and $y \notin (X \setminus \{y\})_E$, then (X, μ, E) is a supra p -soft T_1 -space. $\square$

Corollary 4.11. A finite supra soft topological space (X, μ, E) with $|X| \geq 2$ contains at least $|X| + 2$ supra soft open sets.

Proof. Let X be the universe set with $|X| \geq 2$ and E be a set of parameters. Then the collection $\theta = \{\tilde{\Phi}, \tilde{X}, G_{iE}$ such that $G_i(e) = X \setminus \{x_i\}$, for each $e \in E : i = 1, 2, \dots, |X|\}$ forms a supra soft topology on X . The number of all non-null proper supra soft open subsets of (X, θ, E) is $\binom{|X|}{|X|-1} = |X|$. So $|\theta| = |X| + 2$. Take arbitrary distinct points $x_i, x_j \in X$. The soft sets F_E and H_E defined as follows:

$$F_E = \{(e, X \setminus \{x_j\}) : e \in E; x_j \in X\} \text{ and}$$

$$H_E = \{(e_1, X \setminus \{x_i\}) : e \in E; x_i \in X\}$$

are supra soft open sets containing x_i and x_j , respectively, such that $x_i \notin H_E$ and $x_j \notin F_E$. This proves that (X, θ, E) is a supra soft T_1 -space. To prove that it is the smallest supra soft T_1 , remove any non-null proper supra soft open set. Say, G_{1E} . Then x_{1E} is not a supra soft closed set. This contradicts that (X, θ, E) is supra soft T_1 . Hence, the desired result is proved. $\square$

Proposition 4.12. If (X, μ, E) is a supra p -soft T_1 -space, then $x_E = \bigcap \{G_E : x \in G_E \in \mu\}$ for all $x \in X$.

Proof. For each $y \in X$ such that $x \neq y$, there exists a supra soft open set G_E such that $x \in G_E$ and $y \notin G_E$. So $x \in \bigcap G_E$ and $y \notin \bigcap G_E$. Hence, the desired result is proved. $\square$

Definition 4.13. The collection Λ of soft sets over X with a parameters set E is called enriched if it contains all soft sets G_E which are defined as $G(e) = X$ or $\emptyset$ for each $e \in E$.

Theorem 4.14. If (X, μ, E) is an enriched supra p -soft T_1 -space, then P_e^x is a supra soft closed set for all $P_e^x \in \tilde{X}$.

Proof. From Theorem (4.10), we get $\widetilde{X \setminus \{x\}}$ is supra soft open. As (X, μ, E) is enriched, then a soft set H_E , defining as $H(e) = \emptyset$ and $H(\alpha) = X$ for each $\alpha \neq e$, is supra soft open. Therefore, $\widetilde{X \setminus \{x\}} \bigcup H_E$ is supra soft open. Thus, $(\widetilde{X \setminus \{x\}} \bigcup H_E)^c = P_e^x$ is a supra soft closed set. $\square$

Corollary 4.15. If (X, μ, E) is an enriched supra p -soft T_1 -space, then the intersection of all supra soft open sets containing U_E is exactly U_E for each $U_E \subseteq \tilde{X}$.

Proof. Let U_E be a soft subset of $\tilde{X}$ and $P_e^x \in U_E^c$. Since P_e^x is supra soft closed, then $\tilde{X} \setminus P_e^x$ is a supra soft open set containing U_E . By doing similarly for each $P_e^x \in U_E^c$, the corollary holds. $\square$

The following two results demonstrate that the condition of supra soft regularity guarantees the equivalence between the introduced supra soft separation axioms for $i = 0, 1, 2$.

Theorem 4.16. Let (X, μ, E) be a supra soft regular space. Then (X, μ, E) is supra p -soft T_2 if and only if it is supra soft T_0 .

Proof. The proof of the "if" part follows directly from Proposition(3.2) and Remark(4.2).

The proof of the "only if" part follows directly from Theorem(3.23) and Remark(4.2). $\square$

Corollary 4.17. Let (X, μ, E) be a supra soft regular space. Then the following five statements are equivalent.

- (i) (X, μ, E) is supra soft T_1 .
- (ii) (X, μ, E) is supra soft T_0 .
- (iii) (X, μ, E) is supra p -soft T_2 .
- (iv) (X, μ, E) is supra p -soft T_1 .
- (v) (X, μ, E) is supra p -soft T_0 .

Remark 4.18. In the definition of supra soft regular space in the previous section, we point out, in Corollary (3.18), that every supra soft open or supra soft closed subset of a supra soft regular space must be stable. To avoid this strict condition, we introduce a concept of a supra p -soft regular space by using a total non belong relation instead of a natural non belong relation.

Definition 4.19. A supra soft topological space (X, μ, E) is said to be supra p -soft regular if for every supra soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint supra soft open sets G_E and F_E such that $H_E \subseteq G_E$ and $x \in F_E$.

Proposition 4.20. Every supra soft regular space is supra p -soft regular.

Proof. The proof follows from the fact total non belong relation $\not\subseteq$ obviously implies natural non belong relation $\not\subset$. $\square$

The next example elucidates that the converse of the above proposition fails.

Example 4.21. The supra soft topological space (X, μ, E) which given in Example (3.27) is supra p -soft regular, because for any supra soft closed set and each point does not totally belong to it, there exist two disjoint supra soft open sets, one of them contains a supra soft closed set and the other contains a point. On the other hand, we elucidate in Example (3.27) that (X, μ, E) is not supra soft regular.

Theorem 4.22. (X, μ, E) is supra p -soft regular if and only if for each $x \in X$ and every supra soft open set F_E containing x , there exists a supra soft open set G_E such that $x \in G_E \subseteq Cl_\mu(G_E) \subseteq F_E$.

Proof. To prove the necessary part, let $x \in X$ and F_E be a supra soft open set containing x . Then F_E^c is supra soft closed and $x \notin F_E^c$. Therefore, there are disjoint supra soft open sets W_E and G_E such that $F_E^c \subseteq W_E$ and $x \in G_E$. So $G_E \subseteq W_E^c \subseteq F_E$. Thus, $Cl_\mu(G_E) \subseteq W_E^c \subseteq F_E$.

To prove the sufficient part, let F_E^c be a supra soft closed set. Then for each $x \notin F_E^c$, we have $x \in F_E$. By hypothesis, there is a supra soft open set G_E containing x such that $Cl_\mu(G_E) \subseteq F_E$. Therefore, $F_E^c \subseteq (Cl_\mu(G_E))^c$ and $G_E \cap (Cl_\mu(G_E))^c = \emptyset$. Thus, (X, μ, E) is supra p -soft regular, as required. $\square$

Theorem 4.23. For any supra p -soft regular space (X, μ, E) , the following three statements are equivalent:

- (i) (X, μ, E) is a supra p -soft T_2 -space.
- (ii) (X, μ, E) is a supra p -soft T_1 -space.
- (iii) (X, μ, E) is a supra p -soft T_0 -space.

Proof. Obviously, (i) $\rightarrow$ (ii) $\rightarrow$ (iii).

(iii) $\rightarrow$ (i): Let $x \neq y \in X$ and (X, μ, E) be a supra p -soft T_0 -space. Then there exists a supra soft open set G_E such that $x \in G_E$ and $y \notin G_E$ or $y \in G_E$ and $x \notin G_E$. Say, $x \in G_E$ and $y \notin G_E$. So $x \notin G_E^c$ and $y \in G_E^c$. Since (X, μ, E) is supra p -soft regular, then there exist two disjoint supra soft open sets W_{1_E} and W_{2_E} such that $x \in W_{1_E}$ and $y \in G_E^c \subseteq W_{2_E}$. This finishes the proof. $\square$

Definition 4.24. The collection Λ of supra soft open subsets of (X, μ, E) is called a supra soft basis for $\tilde{X}$ if every supra soft open set can be expressed as a soft union of members of Λ .

Proposition 4.25. *The collection Λ is a supra soft basis for (X, μ, E) if and only if for each supra soft open set G_E such that $x \in G_E$, there is a supra soft open set H_E such that $x \in H_E \subseteq G_E$.*

Proof. Straightforward. $\square$

Theorem 4.26. *If (X, μ, E) has a supra soft basis of supra soft clopen sets, then (X, μ, E) is supra p -soft regular.*

Proof. Let Λ be a supra soft basis for $\tilde{X}$ consisting of supra soft clopen sets. Take an arbitrary point $x \in X$ and a supra soft closed set H_E such that $x \notin H_E$. Then $x \in H_E^c$ which is a supra soft open set, and therefore there exists a supra soft clopen set F_E in Λ such that $x \in F_E \subseteq H_E^c$. Obviously, $H_E \subseteq F_E^c$. Thus, F_E and F_E^c are supra soft open sets separating x and H_E . Hence, (X, μ, E) is supra p -soft regular, as required. $\square$

Definition 4.27. *A supra soft topological space (X, μ, E) is said to be:*

- (i) *a supra p -soft T_3 -space if it is both supra p -soft regular and supra p -soft T_1 .*
- (ii) *a supra p -soft T_4 -space if it is both supra soft normal and supra p -soft T_1 .*

Proposition 4.28. *Every supra soft T_3 -space is a supra p -soft T_3 -space.*

Proof. One can obtain the proof from Corollary (4.17) and Proposition (4.20). $\square$

It can be seen from Example (3.27) that the converse of the above proposition need not be true in general.

Proposition 4.29. *Every supra p -soft T_4 -space is a supra soft T_4 -space.*

Proof. Straightforward. $\square$

To show that the converse of the above proposition fails, we give the next example.

Example 4.30. *Let $E = \{e_1, e_2, e_3\}$ and $\mu = \{\tilde{\Phi}, \tilde{X}, G_{i_E} : i = 1, 2, \dots, 5\}$ be a supra soft topology on $X = \{x, y\}$, where*

$$\begin{aligned} G_{1_E} &= \{(e_1, X), (e_2, \{x\}), (e_3, X)\}; \\ G_{2_E} &= \{(e_1, X), (e_2, \{y\}), (e_3, X)\}; \\ G_{3_E} &= \{(e_1, \emptyset), (e_2, \{x\}), (e_3, \emptyset)\}; \\ G_{4_E} &= \{(e_1, \emptyset), (e_2, \{y\}), (e_3, \emptyset)\} \text{ and} \\ G_{5_E} &= \{(e_1, \emptyset), (e_2, X), (e_3, \emptyset)\}. \end{aligned}$$

Then (X, μ, E) is a supra soft T_4 -space. On the other hand, there does not exist a supra soft open set containing y such that x does not totally belong to it. So (X, μ, E) is not a supra p -soft T_1 -space, hence, it is not a supra p -soft T_4 -space.

Now, we elucidate a relationship between supra p -soft T_i -spaces and deduce some results which associate them with some soft topological notions such as supra soft subspaces and product supra soft spaces.

Proposition 4.31. *Every supra p -soft T_i -space is supra p -soft T_{i-1} , for $i = 1, 2, 3, 4$.*

Proof. We prove the proposition in the case of $i = 3, 4$. The other cases follow similar lines.

For $i = 3$, let x, y be two distinct points in a supra p -soft T_3 -space (X, μ, E) . Then (X, μ, E) is supra p -soft T_1 . So x_E is supra soft closed. Since $y \notin x_E$ and (X, μ, E) is supra p -soft regular, then there are disjoint supra soft open sets G_E and F_E such that $x_E \subseteq G_E$ and $y \in F_E$. Therefore, (X, μ, E) is a supra p -soft T_2 -space.

For $i = 4$, let $x \in X$ and H_E be a supra soft closed set such that $x \notin H_E$. Since (X, μ, E) is supra p -soft T_1 , then x_E is supra soft closed. Since $x_E \cap H_E = \tilde{\Phi}$ and (X, μ, E) is supra soft normal, then there are disjoint supra soft open sets G_E and F_E such that $H_E \subseteq G_E$ and $x_E \subseteq F_E$. Thus, (X, μ, E) is supra soft regular. Hence, it is a supra p -soft T_3 -space. $\square$

The following examples and remark show that the converse of the above proposition need not be true in general.

Example 4.32. Let $\mu = \{\tilde{\Phi}, \tilde{X}, \{(e_1, \{x\}), (e_2, \{x\})\}\}$ be a supra soft topology on $X = \{x, y\}$ under a parameters set $E = \{e_1, e_2\}$. Obviously, (X, μ, E) is a supra p -soft T_0 -space, but it is not a supra p -soft T_1 -space.

Example 4.33. Assume that $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$; and let the soft sets $\{F_{iE} : i = 1, 2, \dots, 8\}$ over X be given as follows:

$$\begin{aligned} F_{1E} &= \{(e_1, \{x\}), (e_2, \{x\})\}; \\ F_{2E} &= \{(e_1, \{y\}), (e_2, \{y\})\}; \\ F_{3E} &= \{(e_1, \{x, y\}), (e_2, \{x, y\})\}; \\ F_{4E} &= \{(e_1, \{x, z\}), (e_2, \{x, z\})\}; \\ F_{5E} &= \{(e_1, \{y, z\}), (e_2, \{y, z\})\}; \\ F_{6E} &= \{(e_1, \{z\}), (e_2, X)\}; \\ F_{7E} &= \{(e_1, \{x, z\}), (e_2, X)\} \text{ and} \\ F_{8E} &= \{(e_1, \{y, z\}), (e_2, X)\}. \end{aligned}$$

Then $\mu = \{\tilde{\Phi}, \tilde{X}, F_{iE} : i = 1, 2, \dots, 8\}$ is a supra soft topology on X . Obviously, (X, μ, E) is a supra p -soft T_2 -space. On the other hand, F_{6E}^c is a supra soft closed set such that $z \notin F_{6E}^c$. Since there are no two disjoint supra soft open sets such that one of them contains z and the other contains F_{6E}^c , then (X, μ, E) is not a supra p -soft regular space. Hence, it is not a supra p -soft T_3 -space.

Remark 4.34. The supra soft topological space given in Example (3.14) is supra p -soft T_1 , but is not supra p -soft T_2 . Also, the supra soft topological space given in Example (3.32) is supra p -soft T_3 , but it is not supra p -soft T_4 .

Theorem 4.35. The finite product of supra p -soft T_i -spaces (X_r, μ_r, A_r) is a supra p -soft T_i -space, for $i = 0, 1, 2, 3$.

Proof. We prove the theorem for two supra soft topological spaces in the case of $i = 0, 3$. The other cases follow similar lines.

(i) Consider two supra p -soft T_0 -spaces (X_1, μ_1, A_1) and (X_2, μ_2, A_2) and let $(x_1, y_1) \neq (x_2, y_2)$ in $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$. Without loss of generality, let $x_1 \neq x_2$. Then there exists a supra soft open subset G_{A_1} of (X_1, μ_1, A_1) such that $x_1 \in G_{A_1}$ and $x_2 \notin G_{A_1}$ or $x_2 \in G_{A_1}$ and $x_1 \notin G_{A_1}$. Say, $x_1 \in G_{A_1}$ and $x_2 \notin G_{A_1}$. Therefore, $(x_1, y_1) \in G_{A_1} \times \tilde{X}_2$ and $(x_2, y_2) \notin G_{A_1} \times \tilde{X}_2$. Thus, $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$ is a supra p -soft T_0 -space.

(ii) Let $H_{A_1 \times A_2}$ be a supra soft closed subset of a supra soft space $(X_1 \times X_2, \mu_1 \times \mu_2, A_1 \times A_2)$. Then $H_{A_1 \times A_2} = [(G_{A_1})^c \times \tilde{X}_2] \cup [\tilde{X}_1 \times (U_{A_2})^c]$ for some $G_{A_1} \in \mu_1$ and $U_{A_2} \in \mu_2$. For every $(x, y) \notin H_{A_1 \times A_2}$, we have $(x, y) \notin (G_{A_1})^c \times \tilde{X}_2$ and $(x, y) \notin \tilde{X}_1 \times (U_{A_2})^c$. So $x \notin (G_{A_1})^c$ and $y \notin (U_{A_2})^c$. Since (X_1, μ_1, A_1) and (X_2, μ_2, A_2) are supra p -soft regular, then there exist disjoint supra soft open sets F_{1A_1} and F_{2A_1} containing x and $(G_{A_1})^c$, respectively, and there exist disjoint supra soft open sets F_{3A_2} and F_{4A_2} containing y and $(U_{A_2})^c$, respectively. Thus, $H_{A_1 \times A_2} \subseteq [F_{2A_1} \times \tilde{X}_2] \cup [\tilde{X}_1 \times F_{4A_2}]$, $(x, y) \in [F_{1A_1} \times F_{3A_2}]$ and $[F_{1A_1} \times F_{3A_2}] \cap \{[F_{2A_1} \times \tilde{X}_2] \cup [\tilde{X}_1 \times F_{4A_2}]\} = \emptyset$. From (i) above and Theorem (4.23), the desired result is proved. $\square$

Remark 4.36. Since the two relations $\not\in$ and $\notin$ are equivalent if E is a singleton, then Remark (3.39) explains the excluding of supra p -soft T_4 -spaces from the above theorem.

Theorem 4.37. Every soft subspace (Y, μ_Y, E) of a supra p -soft T_i -space (X, μ, E) is a supra p -soft T_i -space, for $i = 0, 1, 2, 3$.

Proof. We prove the theorem in the case of $i = 3$ and the other cases follow similar lines.

To prove that (Y, μ_Y, E) is supra p-soft T_1 , let $x \neq y \in Y$. Since (X, μ, E) is a supra p-soft T_1 -space, then there exist supra soft open sets G_E and F_E such that $x \in G_E, y \notin G_E, y \in F_E$, and $x \notin F_E$. Therefore, $x \in U_E = \widetilde{Y} \cap G_E$ and $y \in V_E = \widetilde{Y} \cap F_E$. Since $y \notin G_E$ and $x \notin F_E$, then $y \notin U_E$ and $x \notin V_E$. Thus, (Y, μ_Y, E) is supra p-soft T_1 .

To prove the supra p-soft regularity of (Y, μ_Y, E) , let $y \in Y$ and L_E be a supra soft closed subset of (Y, μ_Y, E) such that $y \notin L_E$. Then there exists a supra soft closed subset H_E of (X, μ, E) such that $L_E = \widetilde{Y} \cap H_E$ and $y \notin H_E$. Therefore, there exist disjoint supra soft open sets G_E and F_E such that $H_E \subseteq G_E$ and $y \in F_E$. Now, we find that $L_E \subseteq W_{1_E} = \widetilde{Y} \cap G_E, y \in W_{2_E} = \widetilde{Y} \cap F_E$ and $W_{1_E} \cap W_{2_E} = \emptyset$. Thus, (Y, μ_Y, E) is supra p-soft regular. Hence, (Y, μ_Y, E) is supra p-soft T_3 . $\square$

Theorem 4.38. Every supra soft closed subspace (Y, μ_Y, E) of a supra p-soft T_4 -space (X, μ, E) is a supra p-soft T_4 -space.

Proof. The proof is similar to that of Theorem (3.35). $\square$

Proposition 4.39. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a soft S^* -continuous mapping such that f is an injective mapping. If (Y, θ, B) is supra p-soft T_i , then (X, μ, A) is supra p-soft T_i for $i = 0, 1, 2$.

Proof. We only prove the proposition in the case of $i = 1$. The other cases follow similar lines.

Let $x \neq y \in X$. Since f is injective, then $f(x) \neq f(y) \in Y$. By hypothesis, there are two supra soft open subsets G_B and W_B of (Y, θ, B) such that $f(x) \in G_B, f(y) \notin G_B$ and $f(y) \in W_B, f(x) \notin W_B$. It follows from Proposition (2.16), that $x \in f_\phi^{-1}(G_B), y \notin f_\phi^{-1}(G_B)$ and $y \in f_\phi^{-1}(W_B), x \notin f_\phi^{-1}(W_B)$. Since f_ϕ is soft S^* -continuous, then $f_\phi^{-1}(G_B)$ and $f_\phi^{-1}(W_B)$ are two supra soft open subsets of (X, μ, A) . Hence, (X, μ, A) is supra p-soft T_1 . $\square$

For the sake of brevity, we omit the proofs of the following four results.

Proposition 4.40. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a bijective soft S^* -continuous mapping. If (Y, θ, B) is supra p-soft T_i , then (X, μ, A) is supra p-soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 4.41. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a bijective soft S^* -open mapping. If (X, μ, A) is supra p-soft T_i , then (Y, θ, B) is supra p-soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 4.42. Let $f_\phi : (X, \mu, A) \rightarrow (Y, \theta, B)$ be a soft S^* -homeomorphism mapping. Then (X, μ, A) is supra p-soft T_i if and only if (Y, θ, B) is supra p-soft T_i for $i = 0, 1, 2, 3, 4$.

Corollary 4.43. The property of being supra p-soft T_i is a supra soft topological property, for $i = 0, 1, 2, 3, 4$.

In the end of this work, we summarize the relationships between the two introduced supra soft separation axioms in the following diagram.

5 Conclusion and future work

Soft separation axioms are among the most widespread, important and interesting concepts in soft topology. They can be utilized to construct more restricted classes of soft topological spaces. This, as well as the reasons mentioned in the penultimate paragraph of the introduction section, prompt us to study separation axioms of supra soft topologies in this paper. So we define the concepts of supra soft T_i -spaces ($i = 0, 1, 2, 3, 4$). The fact that we cannot adapt all claims concerning supra topologies to supra soft topologies by using supra soft T_i -spaces (see, Example (3.12), Example (3.22), Remark (3.30)) leads us to define supra p-soft T_i -spaces ($i = 0, 1, 2, 3, 4$) which preserve more properties of supra topologies on supra soft topologies (see, Theorem (4.10), Theorem (4.22), Proposition (4.31)). With the help of examples, we explore the relationships between supra

Figure 1: The relationships among some types of supra soft separation spaces.

soft T_i and between supra p-soft T_i ; and we study the relationships between supra soft T_i and supra p-soft T_i . Also, we explain the relationships between them and their parametric supra topologies. We further discuss some essential properties of the initiated supra soft axioms. In particular, we characterize the concepts of supra p-soft T_i -spaces ($i = 1, 2$) and supra p-soft regular spaces. We define new notions such as enriched supra soft topologies, supra soft basis and soft S^* -continuous mappings; and conclude some results which relate them to some supra soft separation axioms initiated herein.

Eventually, we plan to do the following in the upcoming papers:

- (i) Define a concept of supra soft topological ordered spaces in a similar way to the concept of soft topological ordered spaces [38].
- (ii) Study a concept of soft lattices [39] with respect to the partial order relations given in [38].
- (iii) Investigate the possibility for applications of these two types of supra soft separation axioms in digital and approximation spaces and decision making problems.
- (iv) Define a notion soft somewhere dense sets [40] via supra soft topological spaces and generalize supra soft separation axioms initiated herein with respect to this.

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