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Hyers-Ulam stability of quadratic forms in 2-normed spaces

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Abstract: In this paper, we obtain Hyers-Ulam stability of the functional equations

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w),$$

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) + 2f(y, w)$$

and

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w)$$

in 2-Banach spaces. The quadratic forms $ax^2 + bxy + cy^2$, $ax^2 + by^2$ and axy are solutions of the above functional equations, respectively.

Keywords: linear 2-normed space, quadratic form, Hyers-Ulam stability

MSC: 39B52, 39B72

1 Introduction

We introduce some definitions on 2-Banach spaces [1], [2].

Definition 1.1. Let X be a real linear space with $\dim X \geq 2$ and $\|\cdot, \cdot\| : X^2 \rightarrow \mathbb{R}$ be a function. Then $(X, \|\cdot, \cdot\|)$ is called a *linear 2-normed space* if the following conditions hold:

- (a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- (b) $\|x, y\| = \|y, x\|$,
- (c) $\|\alpha x, y\| = |\alpha| \|x, y\|$,
- (d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for all $\alpha \in \mathbb{R}$ and $x, y, z \in X$. In this case, the function $\|\cdot, \cdot\|$ is called a *2-norm* on X .

Definition 1.2. Let $\{x_n\}$ be a sequence in a linear 2-normed space X . The sequence $\{x_n\}$ is said to be *convergent* in X if there exists an element $x \in X$ such that

$$\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$$

for all $y \in X$. In this case, we say that a sequence $\{x_n\}$ converges to the limit x , simply denoted by $\lim_{n \rightarrow \infty} x_n = x$.

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ for all $y \in X$. For convenience, we will write

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$\lim_{m,n \rightarrow \infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$. A 2-Banach space is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space which will be used to prove the stability results.

Lemma 1.4. [3] Let $(X, \|\cdot, \cdot\|)$ be a linear 2-normed space and $x \in X$. Then the following properties hold:

- (a) If $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.
- (b) $\|x, z\| - \|y, z\| \leq \|x - y, z\|$ for all $x, y, z \in X$.
- (c) If a sequence $\{x_n\}$ is convergent in X , then $\lim_{n \rightarrow \infty} \|x_n, y\| = \|\lim_{n \rightarrow \infty} x_n, y\|$ for all $y \in X$.

In 1940, Ulam [4] suggested the stability problem of functional equations concerning the stability of group homomorphisms:

Let a group G and a metric group H with the metric ρ be given. For each $\varepsilon > 0$, the question is whether or not there is a $\delta > 0$ such that if $f : G \rightarrow H$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a group homomorphism $h : G \rightarrow H$ satisfying $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$.

The case of approximately additive mappings was solved by Hyers [5] under the assumption that G and H are Banach spaces. In 1978, Rassias [6] generalized the result of Hyers as follows: Let $f : G \rightarrow H$ be a mapping between Banach spaces and let $0 \leq p < 1$ be fixed. If f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$ and for all $x, y \in G$, then there exists a unique additive mapping $A : G \rightarrow H$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in G$.

In Banach spaces, Bae and Park [7–9] investigated the stability problem of some functional equations:

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x, z) + 2f(y, w), \quad (1)$$

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x, z) + 2f(y, w), \quad (2)$$

and

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x, z) - 2f(y, w). \quad (3)$$

The quadratic forms $f_1, f_2, f_3 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f_1(x, y) := ax^2 + bxy + cy^2$, $f_2(x, y) := ax^2 + by^2$ and $f_3(x, y) := axy$ are solutions of (1), (2) and (3), respectively.

In 2011, Park [10] investigated approximate additive, Jensen and quadratic mappings in 2-Banach spaces. In this paper, we also investigate the stability of the functional equations (1), (2) and (3) in 2-Banach spaces with different assumptions from [10].

2 Results

Throughout this paper, let X be a normed space and Y a 2-Banach space.

Theorem 2.1. Let $p \in (0, 2)$, $\varepsilon > 0$, $\delta, \eta \geq 0$ and let $f : X \times X \rightarrow Y$ be a surjective mapping such that

$$\|f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \varepsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|) \quad (4)$$

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (1) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{\varepsilon}{3} + \frac{2\delta}{4-2^p}(\|x\|^p + \|y\|^p) + \frac{\eta}{3}(\|u\| + \|v\|) \quad (5)$$

for all $x, y, u, v \in X$.

Proof. Letting $y = x$ and $w = z$ in (4), we have

$$\left\| f(x, z) + \frac{1}{4}[f(0, 0) - f(2x, 2z)], f(u, v) \right\| \leq \frac{\varepsilon}{4} + \frac{1}{2}\delta(\|x\|^p + \|z\|^p) + \frac{\eta}{4}(\|u\| + \|v\|)$$

for all $x, z, u, v \in X$. Thus we obtain

$$\left\| \frac{1}{4^j}f(2^j x, 2^j z) - \frac{1}{4^{j+1}}[f(0, 0) + f(2^{j+1}x, 2^{j+1}z)], f(u, v) \right\| \leq \frac{\varepsilon}{4^{j+1}} + 2^{j(p-2)-1}\delta(\|x\|^p + \|z\|^p) + 2^{-2(j+1)}\eta(\|u\| + \|v\|)$$

for all $x, z, u, v \in X$ and all j . Replacing z by y in the above inequality, we see that

$$\left\| \frac{1}{4^j}f(2^j x, 2^j y) - \frac{1}{4^{j+1}}[f(0, 0) + f(2^{j+1}x, 2^{j+1}y)], f(u, v) \right\| \leq \frac{\varepsilon}{4^{j+1}} + 2^{j(p-2)-1}\delta(\|x\|^p + \|y\|^p) + 2^{-2(j+1)}\eta(\|u\| + \|v\|)$$

for all $x, y, u, v \in X$ and all j . For given integers l, m ($0 \leq l < m$), we get

$$\begin{aligned} & \left\| \frac{1}{4^l}f(2^l x, 2^l y) - \frac{1}{4^m}f(2^m x, 2^m y), f(u, v) \right\| \\ & \leq \sum_{j=l}^{m-1} \left[\frac{\varepsilon}{4^{j+1}} + 2^{j(p-2)-1}\delta(\|x\|^p + \|y\|^p) + 2^{-2(j+1)}\eta(\|u\| + \|v\|) \right] \end{aligned} \quad (6)$$

for all $x, y, u, v \in X$. By (6), the sequence $\{\frac{1}{4^j}f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j}f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \rightarrow Y$ by $F(x, y) := \lim_{j \rightarrow \infty} \frac{1}{4^j}f(2^j x, 2^j y)$ for all $x, y \in X$. By (4), we have

$$\begin{aligned} & \left\| \frac{1}{4^j}f(2^j(x+y), 2^j(z+w)) + \frac{1}{4^j}f(2^j(x-y), 2^j(z-w)) - \frac{2}{4^j}f(2^j x, 2^j z) - \frac{2}{4^j}f(2^j y, 2^j w), f(u, v) \right\| \\ & \leq \frac{1}{4^j} [\varepsilon + 2^{jp}\delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|)] \end{aligned}$$

for all $x, y, z, w, u, v \in X$ and all j . Letting $j \rightarrow \infty$, we see that F satisfies (1). Setting $l = 0$ and taking $m \rightarrow \infty$ in (6), one can obtain the inequality (5). If $G : X \times X \rightarrow Y$ is another mapping satisfying (1) and (5), we obtain

$$\begin{aligned} \|F(x, y) - G(x, y), f(u, v)\| &= \frac{1}{4^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\| \\ &\leq \frac{1}{4^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y), f(u, v)\| + \frac{1}{4^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\| \\ &\leq \frac{2}{4^n} \left(\frac{\varepsilon}{3} + \frac{2^{np+1}\delta(\|x\|^p + \|y\|^p)}{4 - 2^p} + \frac{\eta(\|u\| + \|v\|)}{3} \right) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x, y, u, v \in X$. Hence the mapping F is the unique mapping satisfying (1), as desired. $\square$

Corollary 2.2. Let $f : X \times X \rightarrow Y$ be a surjective mapping such that

$$\|f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \varepsilon$$

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (1) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{\varepsilon}{3}$$

for all $x, y, u, v \in X$.

Proof. Taking $\delta = \eta = 0$ in Theorem 2.1, we have the desired result. $\square$

In the case $p > 2$ in Theorem 2.1, one can also obtain a similar result.

Theorem 2.3. Let $p \in (0, 2)$, $\varepsilon > 0$ and $\delta, \eta \geq 0$, and let $f : X \times X \rightarrow Y$ be a surjective mapping such that

$$\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \varepsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|) \quad (7)$$

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (2) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{3}{2}\varepsilon + \frac{\delta}{4-2^p} \left(8\|x\|^p + \frac{17}{2}\|y\|^p \right) + \frac{5}{3}\eta(\|u\| + \|v\|) + \frac{1}{3}\|f(0, 0)\| \quad (8)$$

for all $x, y, u, v \in X$.

Proof. Letting $y = x$ and $w = -z$ in (7), we obtain that

$$\left\| f(x, z) + f(x, -z) - \frac{1}{2} [f(0, 0) + f(2x, 2z)], f(u, v) \right\| \leq \frac{1}{2}\varepsilon + \delta(\|x\|^p + \|z\|^p) + \frac{\eta}{2}(\|u\| + \|v\|) \quad (9)$$

for all $x, z, u, v \in X$. Putting $x = 0$ in (9), we get

$$\left\| f(0, z) + f(0, -z) - \frac{1}{2} [f(0, 0) + f(0, 2z)], f(u, v) \right\| \leq \frac{1}{2}\varepsilon + \delta\|z\|^p + \frac{\eta}{2}(\|u\| + \|v\|)$$

for all $z, u, v \in X$. Replacing z by $-z$ in the above inequality, we have

$$\left\| f(0, -z) + f(0, z) - \frac{1}{2} [f(0, 0) + f(0, -2z)], f(u, v) \right\| \leq \frac{1}{2}\varepsilon + \delta\|z\|^p + \frac{\eta}{2}(\|u\| + \|v\|)$$

for all $z, u, v \in X$. By the above two inequalities, we see that

$$\|f(0, 2z) - f(0, -2z), f(u, v)\| \leq \varepsilon + \delta\|z\|^p + 2\eta(\|u\| + \|v\|) \quad (10)$$

for all $z, u, v \in X$. Setting $y = x$ and $w = z$ in (7), we have

$$\|f(2x, 0) + f(0, 2z) - 4f(x, z), f(u, v)\| \leq \varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|) \quad (11)$$

for all $x, z, u, v \in X$. Replacing z by $-z$ in the above inequality, we see that

$$\|f(2x, 0) + f(0, -2z) - 4f(x, -z), f(u, v)\| \leq \varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|) \quad (12)$$

for all $x, z, u, v \in X$. By (11) and (12), we know that

$$\left\| f(x, z) - f(x, -z) - \frac{1}{4} [f(0, 2z) - f(0, -2z)], f(u, v) \right\| \leq \frac{1}{2} [\varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|)] \quad (13)$$

for all $x, z, u, v \in X$. By (9) and (13), we get

$$\left\| f(x, z) - \frac{1}{8} [f(0, 2z) - f(0, -2z)] - \frac{1}{4} [f(0, 0) + f(2x, 2z)], f(u, v) \right\| \leq \varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|)$$

for all $x, z, u, v \in X$. By (10) and the above inequality, we have

$$\left\| f(x, z) - \frac{1}{4} [f(0, 0) + f(2x, 2z)], f(u, v) \right\| \leq \frac{1}{8} [9\varepsilon + 16\delta\|x\|^p + 17\|z\|^p + 10\eta(\|u\| + \|v\|)]$$

for all $x, z, u, v \in X$. Thus we obtain that

$$\begin{aligned} & \left\| \frac{1}{4^j} f(2^j x, 2^j z) - \frac{1}{4^{j+1}} [f(0, 0) + f(2^{j+1} x, 2^{j+1} z)], f(u, v) \right\| \\ & \leq \frac{1}{8} \left[\frac{9}{4^j} \varepsilon + \frac{\delta}{2^{(2-p)j}} (16\|x\|^p + 17\|z\|^p) + \frac{10\eta}{4^j} (\|u\| + \|v\|) \right] \end{aligned}$$

for all $x, z, u, v \in X$ and all j . Hence, for given integers l, m ($0 \leq l < m$), we see that

$$\left\| \frac{1}{4^l} f(2^l x, 2^l y) - \frac{1}{4^m} f(2^m x, 2^m y) - \frac{1}{3} \left(\frac{1}{4^l} - \frac{1}{4^m} \right) f(0, 0), f(u, v) \right\|$$

$$\leq \frac{1}{8} \sum_{j=l}^{m-1} \left[\frac{9}{4^j} \varepsilon + \frac{\delta}{2^{(2-p)j}} (16\|x\|^p + 17\|z\|^p) + \frac{10\eta}{4^j} (\|u\| + \|v\|) \right] \quad (14)$$

for all $x, y, u, v \in X$. By assumption, the sequence $\{\frac{1}{4^j}f(2^jx, 2^jy)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j}f(2^jx, 2^jy)\}$ converges for all $x, y \in X$. Define $F : X \times X \rightarrow Y$ by $F(x, y) := \lim_{j \rightarrow \infty} \frac{1}{4^j}f(2^jx, 2^jy)$ for all $x, y \in X$. By (7), we have

$$\begin{aligned} & \left\| \frac{1}{4^j}f(2^j(x+y), 2^j(z-w)) + \frac{1}{4^j}f(2^j(x-y), 2^j(z+w)) - \frac{2}{4^j}f(2^jx, 2^jz) - \frac{2}{4^j}f(2^jy, 2^jw), f(u, v) \right\| \\ & \leq \frac{1}{4^j} [\varepsilon + 2^{pj} \delta (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta (\|u\| + \|v\|)] \end{aligned}$$

for all $x, y, z, w, u, v \in X$ and all j . Letting $j \rightarrow \infty$, we see that F satisfies (2). Setting $l = 0$ and taking $m \rightarrow \infty$ in (14), one can obtain the inequality (8). If $G : X \times X \rightarrow Y$ is another mapping satisfying (2) and (8), by Theorem 3 in [8], we obtain that

$$\begin{aligned} \|F(x, y) - G(x, y), f(u, v)\| &= \frac{1}{4^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\| \\ &\leq \frac{1}{4^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y), f(u, v)\| + \frac{1}{4^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\| \\ &\leq \frac{2}{4^n} \left[\frac{3}{2} \varepsilon + \frac{2^{np} \delta}{4 - 2^p} \left(8\|x\|^p + \frac{17}{2} \|y\|^p \right) + \frac{5}{3} \eta (\|u\| + \|v\|) + \frac{1}{3} \|f(0, 0)\| \right] \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x, y, u, v \in X$ and all j . Hence the mapping F is the unique mapping satisfying (2), as desired. $\square$

Corollary 2.4. Let $f : X \times X \rightarrow Y$ be a surjective mapping such that

$$\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) - 2f(y, w), f(u, v)\| \leq \varepsilon$$

for all $x, y, z, w, u, v \in X$ and all $z, w \in X$. Then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (2) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{3}{2} \varepsilon + \frac{1}{3} \|f(0, 0)\|$$

for all $x, y, u, v \in X$.

Proof. Taking $\delta = \eta = 0$ in Theorem 2.3, we have the desired result. $\square$

In the case $p > 2$ in Theorem 2.3, one can also obtain a similar result.

Theorem 2.5. Let $p \in (0, 2)$, $\varepsilon, \delta, \eta \geq 0$ and let $f : X \times X \rightarrow Y$ be a surjective mapping such that

$$\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) + 2f(y, w), f(u, v)\| \leq \varepsilon + \delta (\|x\|^p + \|z\|^p + \|y\|^p + \|w\|^p) + \eta (\|u\| + \|v\|) \quad (15)$$

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (3) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{2}{3} \varepsilon + \frac{3}{4 - 2^p} \delta (\|x\|^p + \|y\|^p) + \frac{2}{3} \eta (\|u\| + \|v\|) \quad (16)$$

for all $x, y, u, v \in X$.

Proof. Letting $y = x$ and $w = -z$ in (15), we gain

$$\|f(2x, 2z) + f(0, 0) - 2f(x, z) + 2f(x, -z), f(u, v)\| \leq \varepsilon + 2\delta (\|x\|^p + \|z\|^p) + \eta (\|u\| + \|v\|) \quad (17)$$

for all $x, z, u, v \in X$. Putting $x = z = \varepsilon = \eta = 0$ in (17), we get $f(0, 0) = 0$. Putting $x = z = 0$ in (15), we get

$$\|f(y, -w) + f(-y, w) + 2f(y, w), f(u, v)\| \leq \varepsilon + \delta (\|y\|^p + \|w\|^p) + \eta (\|u\| + \|v\|)$$

for all $y, w, u, v \in X$. Replacing y by x and w by z in the above inequality, we have

$$\|f(x, -z) + f(-x, z) + 2f(x, z), f(u, v)\| \leq \varepsilon + \delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|) \quad (18)$$

for all $x, z, u, v \in X$. Setting $y = -x$ and $w = z$ in (15), we obtain

$$\|f(2x, 2z) - 2f(x, z) + 2f(-x, z), f(u, v)\| \leq \varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|) \quad (19)$$

for all $x, z, u, v \in X$. By (17) and (18), we gain

$$\|f(2x, 2z) - 4f(x, z) + f(x, -z) - f(-x, z), f(u, v)\| \leq 2\varepsilon + 3\delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|)$$

for all $x, z, u, v \in X$. By (17) and (19), we get

$$\|f(x, -z) - f(-x, z), f(u, v)\| \leq \varepsilon + 2\delta(\|x\|^p + \|z\|^p) + \eta(\|u\| + \|v\|)$$

for all $x, z, u, v \in X$. By (17), (18) and (19), we have

$$\|f(2x, 2z) - 4f(x, z), f(u, v)\| \leq 2\varepsilon + 3\delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|)$$

for all $x, z, u, v \in X$. Replacing x by $2^j x$ and z by $2^j z$ and dividing 4^{j+1} in the above inequality, we obtain that

$$\left\| \frac{1}{4^j} f(2^j x, 2^j z) - \frac{1}{4^{j+1}} f(2^{j+1} x, 2^{j+1} z), f(u, v) \right\| \leq \frac{1}{4^{j+1}} \left[2\varepsilon + 3 \cdot 2^{pj} \delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|) \right]$$

for all $x, z, u, v \in X$ and all $j = 0, 1, 2, \dots$. For given integers l, m ($0 \leq l < m$), we obtain that

$$\left\| \frac{1}{4^l} f(2^l x, 2^l z) - \frac{1}{4^m} f(2^m x, 2^m z), f(u, v) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{4^{j+1}} \left[2\varepsilon + 3 \cdot 2^{pj} \delta(\|x\|^p + \|z\|^p) + 2\eta(\|u\| + \|v\|) \right] \quad (20)$$

for all $x, z, u, v \in X$. By (20), the sequence $\{\frac{1}{4^j} f(2^j x, 2^j z)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j} f(2^j x, 2^j z)\}$ converges for all $x, y \in X$. Define $F : X \times X \rightarrow Y$ by $F(x, y) := \lim_{j \rightarrow \infty} \frac{1}{4^j} f(2^j x, 2^j y)$ for all $x, y \in X$. By (15), we have

$$\begin{aligned} & \left\| \frac{1}{4^j} f(2^j(x+y), 2^j(z-w)) + \frac{1}{4^j} f(2^j(x-y), 2^j(z+w)) - \frac{2}{4^j} f(2^j x, 2^j z) + \frac{2}{4^j} f(2^j y, 2^j w), f(u, v) \right\| \\ & \leq \frac{1}{4^j} [\varepsilon + 2^{pj} \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|)] \end{aligned}$$

for all $x, y, z, w, u, v \in X$ and all $j = 0, 1, 2, \dots$. Letting $j \rightarrow \infty$ in the above inequality, we see that F satisfies (3). Setting $l = 0$ and taking $m \rightarrow \infty$ in (20), one can obtain the inequality (16). If $G : X \times X \rightarrow Y$ is another mapping satisfying (3) and (16), by Theorem 3.1 in [9], we obtain that

$$\begin{aligned} \|F(x, y) - G(x, y), f(u, v)\| &= \frac{1}{4^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\| \\ &\leq \frac{1}{4^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y), f(u, v)\| + \frac{1}{4^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u, v)\| \\ &\leq \frac{1}{4^n} \left[\frac{4}{3} \varepsilon + \frac{6}{4-2^p} \delta(\|x\|^p + \|y\|^p) + \frac{4}{3} \eta(\|u\| + \|v\|) \right] \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x, y, u, v \in X$. Hence the mapping F is the unique mapping satisfying (3), as desired. $\square$

In the case $p > 2$ in Theorem 2.5, one can also obtain a similar result.

Corollary 2.6. *Let $f : X \times X \rightarrow Y$ be a surjective mapping such that*

$$\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) + 2f(y, w), f(u, v)\| \leq \varepsilon$$

for all $x, y, z, w, u, v \in X$. Then there exists a unique mapping $F : X \times X \rightarrow Y$ satisfying (2) such that

$$\|f(x, y) - F(x, y), f(u, v)\| \leq \frac{2}{3}\varepsilon$$

for all $x, y, u, v \in X$.

Proof. Taking $\delta = \eta = 0$ in Theorem 2.5, we have the desired result. $\square$

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