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THE METHOD OF REDUCING THE ORDER OF LINEAR OPERATOR EQUATIONS

In the theory of ordinary differential equations it is well known that it is possible to reduce the order of a linear equation, if one knows a special solution of the corresponding homogeneous equation (cf. [4]). The reason for the success of this method is the property

$$D(uv) = (Du)v + uDv \tag{1}$$

of the ordinary derivative $D = d/dt$. A similar method is known for difference equations (cf. [3]) and functional equations (cf. [5]) with a basic operator H, defined by $Hu(t) = u(t+1)$ and $Hu(t) = u(\varphi(t))$ respectively, which has the property

$$H(uv) = (Hu)Hv. \tag{2}$$

Here we want to generalize this method to other operator equations. Since by applying this method to an equation of order one we get an equation of order zero, we give the following precise definition of the method. Let

$$Ay + ay = f \tag{3}$$

be a given equation and x a non-vanishing solution of the homogeneous equation

$$Ax + ax = 0. \tag{4}$$

Then we consider the factorization

$$y = xz \tag{5}$$

and search for two operators B,C depending only on A, so that z is to be determined from the equation

$$(Bx)Cz = f, \tag{6}$$

which is an equation of order zero with respect to Cz.

For simplicity we assume that all lower case letters denote elements from a field $\mathcal{R}$ and all capitals denote operators from $\mathcal{R}$ into $\mathcal{R}$. However, it would be possible to consider the case, where the solution, the coefficient and the right-hand side of (3) belong to different sets which need not be fields. The unit element of $\mathcal{R}$ we denote by e.

Our result can be formulated as follows: There are only two possibilities namely A is expressible either by a derivative D with (1) or by a homomorphism H with (2).

Theorem 1. If (3) is equivalent to (6), then A has the property

$$A(uv) = (Au)v + (Bu)Cv \tag{7}$$

for all $u,v \in \mathcal{R}$ with $u \neq 0$ and conversely.

Proof. Let x,z be two arbitrary elements of $\mathcal{R}$ with $x \neq 0$. We define a by (4) and f by (3) and (5). Substituting (5) into (3) and considering (4) we get

$$A(xz) - (Ax)z = f, \tag{8}$$

and by comparing with (6) we see that (7) holds. If (7) is fulfilled and y is a solution of (3), we define z by $z = y/x$ and from (8) we obtain (6). If (7) is fulfilled and (6) holds, then (6) may be written in the form (8) and from (4) it follows that (3) with (5) holds.

Theorem 2. If the operator A with the property (7) is no element of $\mathcal{R}$ and

$$Ae = \alpha, \tag{9}$$

then

$$D = A - \alpha \tag{10}$$

has for all non-vanishing $u, v \in \mathcal{R}$ the property

$$D(uv) = (Du)v + uDv + \varepsilon(Du)Dv \tag{11}$$

with a certain $\varepsilon \in \mathcal{R}$.

P r o o f . From (7), by setting $u = e$, it follows that

$$\beta Cv = Av - \alpha v$$

with $\beta = Be$. In the case $\beta = 0$ we would have $A = \alpha$ contrary to our assumption. So we have $\beta \neq 0$ and without loss of generality we can put $\beta = e$, hence (7) takes the form

$$A(uv) = (Au)v + (Bu)(Av - \alpha v).$$

From this, by substitution (10), it follows that

$$D(uv) = (Du)v + (Bu)Dv$$

and, in view of $uv = vu$, we obtain

$$(Du)(v - Bv) = (u - Bu)Dv.$$

From $A \neq \alpha$ it follows that there exists a special element v with $Dv \neq 0$, hence we obtain

$$Bu = u + \varepsilon Du$$

with a special element $\varepsilon \in \mathcal{R}$. So we have proved (11).

C o r o l l a r y . Using the operator D from (10) our equation (3) takes the form

$$Dy + by = f \tag{12}$$

with $b = a + \alpha$, and (6) takes the form

$$cDz = f \tag{13}$$

with $c = x + \varepsilon Dx \neq 0$.

P r o o f . We have to show only that $c \neq 0$. This is evident for $\varepsilon = 0$. If $c = 0$ and $\varepsilon \neq 0$, then $\varepsilon^{-1} = b$ and from

(11) with $u = x$, $v = y/x$ it follows that $Dy = -by$, i.e. $D = -b \in \mathcal{R}$, which contradicts our assumption $A \notin \mathcal{R}$.

Henceforth we assume that D is an additive operator which until now has been not necessary. In view of (9) and (10) we have $De = 0$, so that D is not invertible, but D has always a right inverse operator T (cf. [6]), however, T may be non-additive (cf. [1]). In this case the general solution of (13) is as follows

$$z = z_0 + Tc^{-1}f,$$

and the general solution of (12) takes the form

$$y = z_0 x + xTc^{-1}f,$$

where z_0 is an arbitrary element of the kernel of D with $z_0 = (e - TD)z$.

Now it is easy to see that there are only two possibilities mentioned above: we have either $\varepsilon = 0$ and D is a derivative with (1) or we have $\varepsilon \neq 0$ and

(14) $$H = \varepsilon D + e$$

is a homomorphism with (2) (cf. [2]).

T h e o r e m 3. If D is an additive operator with the property (1) and $DT = e$, then $y = xTz$ is a solution of the equation

(15) $$\sum_{i=0}^{n} a_i D^i y = f,$$

where x is a nontrivial solution of the corresponding homogeneous equation and z is a solution of the equation

$$\sum_{j=0}^{n-1} \Big(\sum_{i=j}^{n-1} a_{i+1} \binom{i+1}{j+1} (D^{i-j}x) \Big) D^j z = f.$$

If H is an additive operator with (2) and $(H-e)R = e$, then $y = xRz$ is a solution of the equation

(16) $$\sum_{i=0}^{n} b_i H^i y = f,$$

where x is a nontrivial solution of the corresponding homogeneous equation and z a solution of the equation

$$\sum_{j=o}^{n-1}\Big(\sum_{i=j}^{n-1} b_{i+1}\Big)H^j z = f.$$

P r o o f . The first assertion follows immediately by substituting Leibniz's rule

$$D^i(xTz) = (D^i x)Tz + \sum_{j=o}^{i-1} \binom{i}{j+1}(D^{i-j-1})D^j z$$

into (15) and changing the order of summation. The second assertion follows analogously from the formula

$$H^i R = R + \sum_{j=o}^{i-1} H^j.$$

If in the case (14) the equation is given in the form (15), we can always go over to the form (16) by substituting (14) into (15). Since the pair of equations (12), (13) after the substitution (14) takes the form

$$Hy + b'y = f', \; Hz - z = c^{-1}f',$$

with $b' = \varepsilon b - e$, $f' = \varepsilon f$, we see that the general equation of order one is transformed into the special equation with $b' = -e$.

R e m a r k . It would be possible to generalize the property (7) by the functional equation

$$A(uv) = (Au)v + \sum_{i=1}^{n} (B_i u)(C_i v)$$

(cf. [2]). Then (6) is replaced by

$$\sum_{i=1}^{n} (B_i x)C_i z = f,$$

however, the advantage of such a procedure is generally speaking very difficult to recognize.

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