DEMONSTRATIO MATHEMATICA
Vol. IX No 1 1976

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ON SOME NON-LINEAR SYSTEM OF INTEGRAL-FUNCTIONAL EQUATIONS

1. We shall consider the following system of $n+1$ integral-functional equations

$$(1)\qquad F(x) = \int_0^{\varphi(x)} G\Big(x,y,\varphi(x),\ \Phi_1\big(f(x,y)\big),\dots,\ \Phi_n\big(f(x,y)\big)\Big)\,dy$$

$$(2)\qquad \Phi_i\Big(f\big(x,\varphi(x)\big)\Big) = H_i\big(x,\varphi(x)\big), \qquad (i=1,\dots,n),$$

where x belongs to an interval I of real numbers. The functions F, G, f, H_i $(i=1,\dots,n)$ are assumed to be given functions, φ and Φ_i $(i=1,\dots,n)$ denote unknown functions.

In the above system of equations the first equation is a non-linear integral-functional equation of Volterra type, the remaining ones are functional equations. Some integral-functional equations of Volterra type have been investigated by C. Popovici [4] and other authors (see M.Ghermanescu [2], p. 83-89). However, in none of those equations the unknown function $\varphi(x)$ appeared in the limits of integration. On the other hand, a system of differential-integral equations of Volterra type in which the unknown function appears in the limits of integration has been considered by H.Movliankulov [3].

The system of integral-functional equations (1), (2) under consideration may be applied to solving boundary-value problems with movable boundary for partial differential equations of parabolic type (see A.Friedman [1], chapt. VIII). We shall reduce this system to an equivalent system of non-linear equations, the first of which is an integral equation of Volterra type and the remaining ones are functional equations. Next we shall prove that the obtained system has a unique solution.

We shall seek a solution of the system (1), (2) in the class of systems of n+1 functions $(\varphi(x), \Phi_1(t), \dots, \Phi_n(t))$ defined for $x \in I$ and t belonging to some interval $I_1 \subset R$[1], whose values lie in a common interval $J \subset R$.

We make the following assumptions.

1°. The real function F(x) is defined and continuous on I.

2°. The real function f(x,y) is defined and continuous at the points (x,y) of the set $\Omega = I \times J$ and takes values in the interval I_1. Moreover, the point y = 0 belongs to the interval J.

3° For every fixed $x \in I$ the function f(x,y) has an inverse with respect to y and the derivative $\frac{\partial f}{\partial y}$ is different from 0.

4°. The real function $G(x,y,z,z_1,\dots,z_n)$ is defined and continuous at points $(x,y,z,z_1,\dots,z_n) \in \Omega_1 = I \times J^{n+2}$.

5° The real functions $H_i(x,v)$ $(i=1,\dots,n)$ are defined and continuous at points $(x,v) \in \Omega$.

2. We shall prove that under the above assumptions the system (1), (2) is equivalent to the system

$$(3)\quad F\left[\psi(t)\right] = \int_{f(\psi(t),0)}^{t} G\big(\psi(t), f_1(\psi(t),u), f_1(\psi(t),t), H_1(\psi(u), f_1(\psi(u),u)), \dots, H_n(\psi(u), f_1(\psi(u),u))\big) \left(\frac{\partial f(x,y)}{\partial y}\Bigg|_{\substack{x=\psi(t)\\ y=f_1(\psi(t),u)}}\right)^{-1} du$$

1) The intervals I and I_1 may be closed, open or one-side open, finite or infinite.

$$(4) \qquad \Phi_i(t) = H_i\big(\psi(t), f_1(\psi(t),t)\big), \qquad (i=1,\dots,n),$$

where $\psi(t)$, $\Phi_i(t)$ $(i=1,\dots,n)$ are unknown functions and $f_1(x,u)$ denotes the inverse function with respect to y of the function $u = f(x,y)$ considered as a function $u = f_x(y)$ of the variable y, i.e.

$$(5) \qquad y = f_1(x,u) \underset{=}{\underline{df}} f^{-1}(x,u).$$

Namely, we are going to prove the following theorem.

T h e o r e m 1. Let the functions $F(x)$, $f(x,y)$, $G(x,y,z, z_1,\dots,z_n)$ and $H_i(x,v)$ $(i=1,\dots,n)$ satisfy the assumptions 1^0-5^0. If the functions $\varphi(x)$ and $\Phi_1(t),\dots,\Phi_n(t)$ satisfy for $x \in I$, $t \in I_1$ the system of equations (1), (2) and the function

$$(6) \qquad t = g(x) \underset{=}{\underline{df}} f(x,\varphi(x)),$$

where $x \in I$ and $t \in I_1$, has an inverse, then the function

$$(7) \qquad \psi(t) \underset{=}{\underline{df}} g^{-1}(t)$$

and the functions $\Phi_i(t)$ $(i=1,\dots,n)$ satisfy for $t \in I_1$ the system of equations (3), (4).

Conversely, if the functions $\psi(t)$, $\Phi_1(t),\dots,\Phi_n(t)$ satisfy for $t \in I_1$ the system of equations (3), (4) and simultaneously the function $\psi(t)$ has an inverse in the interval I_1, then the function

$$(8) \qquad \varphi(x) = f_1\big(x, \psi^{-1}(x)\big)$$

and the functions $\Phi_i(t)$ $(i=1,2,\dots,n)$ satisfy for $x \in I = \psi(I_1)$, $t \in I_1$, the system of integral-functional equations (1), (2).

P r o o f . To prove the first part of the theorem we introduce a new variable into equation (1)

$$(9) \qquad u = f(x,y), \qquad du = \frac{\partial f(x,y)}{\partial y}\, dy.$$

In view of the definition (5), the equation (1) takes the form

$$(10)\qquad F(x) = \int_{f(x,0)}^{f(x,\varphi(x))} G\Big(x, f_1(x,u), \varphi(x), \Phi_1(u),\dots,\Phi_n(u)\Big)\left(\frac{\partial f(x,y)}{\partial y}\bigg|_{y=f_1(x,u)}\right)^{-1} du.$$

Substituting the function $g(x)$ defined by (6) into equations (10), (2), we obtain a system of equations

$$(11)\qquad F(x) = \int_{f(x,0)}^{g(x)} G\Big(x, f_1(x,u), f_1(x,g(x)), \Phi_1(u),\dots,\Phi_n(u)\Big)\left(\frac{\partial f(x,y)}{\partial y}\bigg|_{y=f_1(x,u)}\right)^{-1} du$$

$$(12)\qquad \Phi_i\big(g(x)\big) = H_i\big(x, f_1(x,g(x))\big), \qquad (i=1,\dots,n).$$

Since by assumption the function $g(x)$ has an inverse, then for $t \in I_1$ the following function is well-defined

$$(13)\qquad x = g^{-1}(t) = \psi(t).$$

This function satisfies the system of equations (11), (12), which can be written in the form

$$(14)\qquad F(\psi(t)) = \int_{f(\psi(t),0)}^{t} G\Big(\psi(t), f_1(\psi(t),u), f_1(\psi(t),t), \Phi_1(u),\dots,\Phi_n(u)\Big)\left(\frac{\partial f(x,y)}{\partial y}\bigg|_{\substack{x=\psi(t)\\ y=f_1(\psi(t),u)}}\right)^{-1} du$$

$$(15)\qquad \Phi_i(t) = H_i(\psi(t), f(\psi(t),t)), \qquad (i=1,\dots,n)$$

and next in the form (3), (4). Hence, the first part of Theorem 1 has been proved. In an analogous way we can prove the second part of Theorem 1.

3. To illustrate Theorem 1 we give the following example. For $x < 1$ we consider the system of equations

$$(16)\qquad x(x-1) = \int_0^{\varphi(x)} \left(x - \Phi\left(\frac{y}{x-1}\right) - 1\right) dy$$

$$(17)\qquad \varphi(x) = \Phi\left(\frac{\varphi(x)}{x-1}\right).$$

In this system we have: $F(x) = x(x-1)$, $f(x,y) = \frac{y}{x-1}$, where $x \in I = (-\infty, 1)$, $y \in J = (-\infty, +\infty)$; $G(x,z) = x-z-1$, $H(x,v) = v$, $f_1(x,u) = (x-1)u$, where $z,u,v \in J$.

The system (16), (17) is equivalent to the following one

$$(18)\qquad \psi(t) = \int_0^t (\psi(t) - u(\psi(u) - 1) - 1)\, du$$

$$(19)\qquad \Phi(t) = t(\psi(t) - 1),$$

where $t \in I_1$ (the interval I_1 will be specified in the sequel). The equation (18) is a Volterra linear integral equation of the second kind and can be written in the form

$$(20)\qquad (1-t)\,\psi(t) = \frac{1}{2}t^2 - t - \int_0^t u\,\psi(u)\, du.$$

Differentiating both sides of equation (20) we obtain an equivalent linear differential equation

$$(21)\qquad \psi'(t) - \psi(t) + 1 = 0$$

with the initial condition $\psi(0) = 0$. Since the solution of equation (21) with the condition $\psi(0) = 0$ is the function

$$(22)\qquad \psi(t) = -e^t + 1$$

defined for all $t \in I_1 = (-\infty, +\infty)$, by the second part of the theorem proved above (cf. (8), (15), (22)) we infer that the system (16), (17) has for $x \in I = \psi(I_1) = (-\infty, 1)$ and $t \in I_1$ the solution

$$\varphi(x) = f_1\ x, \psi^{-1}(x) = (x-1)\ \psi^{-1}(x) = (x-1)\ln(1-x)$$

$$\Phi(t) = f_1(\psi(t), t) = (\psi(t) - 1)\ t = -\ te^t.$$

4. To show that the system (3), (4) (and hence the system (1), (2)) has a unique solution we additionally accept the following assumptions:

6° The function $F(x)$ has an inverse for $x \in I$ and the inverse function $F^{-1}(z) = F_1(z)$ satisfies, for $z, \tilde{z} \in J$, the conditions

(23) $$|F_1(z)| \leq M_F + k_F|z|$$

(24) $$|F_1(z) - F_1(\tilde{z})| \leq k_F|z - \tilde{z}|,$$

(k_F and M_F denote some positive constants).

7° The function $G(x,y,z,z_1,\ldots,z_n)$ satisfies the conditions

(25) $$|G(x,y,z,z_1,\ldots,z_n)| \leq M_G + k_G|x| + k'_G|y| + k''_G|z| + l_G \sum_{i=1}^{n}|z_i|$$

(26) $$|G(x,y,z,z_1,\ldots,z_n) - G(\tilde{x},\tilde{y},\tilde{z},\tilde{z}_1,\ldots,\tilde{z}_n)| \leq$$
$$\leq k_G|x-\tilde{x}| + k'_G|y-\tilde{y}| + k''_G|z-\tilde{z}| + l_G \sum_{i=1}^{n}|z_i-\tilde{z}_i|$$

for $(x,y,z,z_1,\ldots,z_n)$, $(\tilde{x},\tilde{y},\tilde{z},\tilde{z}_1,\ldots,\tilde{z}_n) \in \Omega_1$, ($M_G, k_G, k'_G, k''_G$ and L_G - some positive constants).

8° The functions $H_i(x,v)$ $(i=1,\ldots,n)$ satisfy for (x,v), $(\tilde{x},\tilde{v}) \in \Omega$, the conditions

(27) $$|H_i(x,v)| \leq M_H + k_H|x| + l_H|v|.$$

(28) $$|H_i(x,v) - H_i(\tilde{x},\tilde{v})| \leq k_H|x-\tilde{x}| + l_H|v-\tilde{v}|,$$

(k_H, l_H, M_H - some positive constants).

9^0 The function $f(x,y)$ takes values in the interval I_1 of finite length T and for $x,\tilde{x} \in I$ it satisfies Lipschitz's condition

(29) $$|f(x,0) - f(\tilde{x},0)| \leq k_f|x-\tilde{x}|$$

(k_f - some positive constant).

10^0 The function $f_1(x,u)$ satisfies for (x,u), $(\tilde{x},u) \in I \times I_1$ the following conditions

(30) $$|f(x,u)| \leq M_f + k'_f|x|$$

(31) $$|f_1(x,u) - f_1(\tilde{x},u)| \leq k'_f|x-\tilde{x}|,$$

where k'_f and M_f are some positive constants.

11^0 For (x,y), $(\tilde{x},\tilde{y}) \in \Omega = I \times J$ the derivative $f'_y(x,y)$ satisfies Lipschitz's condition

(32) $$|f_y(x,y) - f'_y(\tilde{x},\tilde{y})| \leq k''_f|x-\tilde{x}| + l''_f|y-\tilde{y}|$$

and the inequality

(33) $$|f'_y(x,y)| \geq m_f$$

with some positive constants k''_f, l''_f and m_f.

12^0 There exists a number $\delta \in (0,1)$ such that the constants appearing in assumptions 6^0 - 11^0 satisfy the conditions

(34) $$A \underline{\underline{df}}\ k_F T\left[k_G + k'_f(k'_G + k''_G) + n l_G(k_H + l_H k'_f)\right] < \frac{1}{2} m_f \delta$$

(35) $$k_f < \frac{T}{2}$$

(36) $$k''_f + l''_f k'_f < \frac{m_f}{2}$$

(37) $$B \underline{\underline{df}}\ k_F T \left[M_G + nl_G(l_H M_f + M_H) + M_f(k'_G + k''_G)\right] < (1-\delta)m_f$$

(38) $$M_F < \frac{\delta}{2}$$

Now we shall prove the following theorem.

Theorem 2. Under the assumptions 6° - 12° the integral equation (3) has exactly one solution.

Proof. The proof of Theorem 2 will be carried out with the aid of Banach's theorem on contracting maps. To this aim let us consider the Banach space E consisting of functions $\psi(t)$ defined and continuous in the interval I_1. The norm of a point $\psi \in E$ is defined by the equality

(39) $$\|\psi\| = \sup_{t \in I_1} |\psi(t)|$$

and the distance $\rho(\psi, \tilde{\psi})$ of two points ψ, $\tilde{\psi}$ in the space E is defined by

(40) $$\rho(\psi, \tilde{\psi}) = \|\psi - \tilde{\psi}\| .$$

In this space E let S be the set of all elements of E satisfying for $t \in I_1$ the equality

(41) $$|\psi(t)| \leqslant R,$$

where

(42) $$R = \frac{B + m_f M_F}{m_f - A}$$

and the constants A,B are defined by formulas (34), (37). We transform the set S with the aid of the operation $\chi(t) = \hat{A}\,\psi(t)$ defined by the equality

$$\hat{A}\psi(t) = F^{-1}\left(\int_{f(\psi(t),0)}^{t} G\Big(\psi(t),f_1(\psi(t),u),f_1(\psi(t),t),H_1\big(\psi(u),\right.$$

(43)

$$\left. f_1(\psi(u),u)\big),\dots,H_n\big(\psi(u),f_1(\psi(u),u)\big)\Big)\left(\frac{\partial f(x,y)}{\partial y}\Bigg|_{\substack{x=\psi(t)\\ y=f_1(\psi(t),u)}}\right)^{-1} du\right).$$

The operation (43) associates with each $\psi(t) \in S$ an element $\chi(t)$ of some set S_1. Making use of conditions (23), (25), (27), (33), (30), (41), (34), (37), (42) we obtain the inequality

$$|\chi(t)| \leqslant \frac{k_F T}{m_f}\left[M_G + M_f(k_G'+k_G'') + nl_G(l_H M_f+M_H)\right] + M_F +$$

$$+ R\,\frac{k_F T}{m_f}\left[k_G + k_f'(k_G'+k_G'') + nl_G(k_H+l_H k_f')\right] = \frac{B}{m_f} + M_F + \frac{AR}{m_f} = R$$

which shows that the set S_1 is a subset of S.

Now we are going to prove that the operation defined by (43) is contracting, i.e. it satisfies the condition

(44) $$\varrho(\chi,\tilde{\chi}) \leqslant \alpha\, \varrho(\psi,\tilde{\psi}), \quad \text{where} \quad 0<\alpha<1.$$

Namely, using inequality (24) and definitions (39), (40) we obtain

$$\varrho(\chi,\tilde{\chi}) \leqslant k_F \sup_{t\in I_1}\Bigg| \int_{f(\psi(t),0)}^{t} G\Big(\psi(t),f_1(\psi(t),u),f_1(\psi(t),t),H_1\big(\psi(u),$$

$$f_1(\psi(u),u)\big),\dots,H_n\big(\psi(u),f_1(\psi(u),u)\big)\Big)\left(\frac{\partial f(x,y)}{\partial y}\Bigg|_{\substack{x=\psi(t)\\ y=f_1(\psi(t),u)}}\right)^{-1} du +$$

$$- \int_{f(\tilde{\psi}(t),0)}^{t} G\Big(\tilde{\psi}(t),f_1(\tilde{\psi}(t),u),f_1(\tilde{\psi}(t),t),H_1\big(\tilde{\psi}(u),f_1(\tilde{\psi}(u),u)\big),\dots$$

$$\dots,H_n\big(\tilde{\psi}(u),f_1(\tilde{\psi}(u),u)\big)\Big)\left(\frac{\partial f(x,y)}{\partial y}\Bigg|_{\substack{x=\tilde{\psi}(t)\\ y=f_1(\tilde{\psi}(t),u)}}\right)^{-1} du\Bigg|.$$

We transform the right-hand side of the above inequality using consecutively Lipschitz's conditions 6^o - 11^o, the definition of metric (40) and the definition (41) of the set S.

Then we obtain an estimation

$$\varrho(\chi,\tilde{\chi}) \leq \frac{k_F T}{m_f}\Big\{k_G + k'_f(k'_G+k''_G) + nl_G(k_H+l_H k'_f) + \Big(\frac{k''_f+l''_f k'_f}{m_f} + \frac{k_f}{T}\Big)\cdot$$

$$\cdot\Big[M_G + k_G R + k'_f(k'_G+k''_G)R + nl'_G(k_H R + l_H(k'_f R+M_f) + M_H)\Big\}\varrho(\psi,\tilde{\psi})$$

which coincides with inequality (44), when we put

$$\alpha \underline{\underline{df}} \frac{k_F T}{m_f}\Big\{k_G + k'_f(k'_G+k''_G) + nl_G(k_H + l_H k'_f) + \Big(\frac{k''_f + l''_f k'_f}{m_f} + \frac{k_f}{T}\Big)\cdot$$

$$\cdot\Big[M_G + nl'_G(l_H M_f + M_H) + (k_G + k'_f(k'_G+k''_G) + nl_G(k_H+l_H k'_f))\; R\Big]\Big\}.$$

We shall show that $0 < \alpha < 1$. In fact, from the assumption 12^o it follows that

$$0 < \alpha < \frac{A}{m_f} + \frac{k_F T}{m_f}\Big(\frac{1}{2} + \frac{1}{2}\Big)\Big(\frac{B}{k_F T} + \frac{AR}{k_F T}\Big) =$$

$$= \frac{A+B+AR}{m_f} < \frac{\delta}{2} + (1-\delta) + \frac{\delta}{2} R = 1 - \frac{\delta}{2} + \frac{\delta}{2}\,\frac{R+m_f M_F}{m_f - A} <$$

$$< 1 - \frac{\delta}{2} + \frac{\delta}{2}\,\frac{(1-\delta)m_f + m_f\frac{\delta}{2}}{m_f - \frac{1}{2}m_f\delta} = 1.$$

We see that all the assumptions of Banach s theorem hold. Consequently, equation (3) has exactly one solution. This ends the proof of Theorem 2.,

From the uniqueness of the solution of equation (3) it follows that the solution of system (3), (4) exists and is unique. By the second part of Theorem 1 we obtain the existence and uniqueness of the solution of the system (1), (2).

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Received June 10, 1975.