DEMONSTRATIO MATHEMATICA
Vol. IX No 1 1976

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A GENERALIZATION OF THE THEOREM ON DIRECTIONAL DERIVATIVE FOR THE MAXIMUM FUNCTIONAL AND ITS APPLICATION

1. In the present paper we generalize two theorems [3] about directional derivative for the maximum functional. With the aid of directional derivative one constructs cones of decreasing directions and cones of admissible directions which are used in the theory of Dubowicki-Miliutin [3]. When determining the maximum of a functional it is important to know the above-mentioned cones. The generalization proposed in our paper extends the class of functionals for which these cones can be constructed. The paper also gives an application of the generalized theorem on directional derivative in the theory of functional equations developed in [1].

2. We recall certain definitions and theorems to be used in the paper.

Let X, Y be complete metric spaces.

Definition 1. A point-to-set transformation $F : X \to 2^Y$ is said to be semicontinuous from below at a point $x_0 \in X$ provided that for every sequence $\{x_n\} \subset X$, $x_n \to x_0$ and every $y_0 \in F(x_0)$ there exists a sequence $\{y_n\} \subset Y$ such that $y_n \in F(x_n)$ and $y_n \to y_0$.

Definition 2. A point-to-set transformation $F : X \to 2^Y$ is said to be closed at a point $x_0 \in X$ provided the following implication holds

$$(\{x_n\} \subset X,\ x_n \to x_0,\ y_n \in F(x_n),\ y_n \to y_0) \Rightarrow (y_0 \in F(x_0)).$$

D e f i n i t i o n 3. A point-to-set transformation $F: X \to 2^Y$ is said to be uniformly compact in the neighbourhood of a point $x_0 \in X$ if there exists a neighbourhood $V(x_0)$ of the point x_0 such that the closure $\overline{\bigcup_{x \in V(x_0)} F(x)}$ is a compact set in Y.

D e f i n i t i o n 4. A point-to-set transformation $F: X \to 2^Y$ that is semicontinuous from below and closed at a point $x_0 \in X$ is said to be continuous at x_0.

D e f i n i t i o n 5. A point-to-set transformation $F: X \to 2^Y$ is said to be semicontinuous from above at a point $x_0 \in X$ provided that for every open set U containing the set $F(x_0)$ there exists a neighbourhood $V(x_0)$ of x_0 such that $x \in V(x_0)$ implies $F(x) \subset U$.

T h e o r e m 1. If the transformation $F: X \to 2^Y$ is continuous at a point $x_0 \in X$ and uniformly compact in the neighbourhood of x_0 and if $p: X \times Y \to R^1$ is continuous on $\{x_0\} \times F(x_0)$, then $m(x) = \max_{y \in F(x)} p(x,y)$ is continuous at x_0.

T h e o r e m 2. If the transformation $F: X \to 2^Y$ is continuous at a point $x_0 \in X$ and $p: X \times Y \to R^1$ is continuous on $\{x_0\} \times F(x_0)$, then the transformation $\tilde{F}: X \to 2^Y$ defined by $\tilde{F}(x) = \{y \in F(x) \mid p(x,y) = \sup_{z \in F(x)} p(x,z)\}$ is closed at x_0.

Definitions 1 - 4 and Theorems 1 and 2 can be found in [4], where the transformation semicontinuous from below is called open. The notion of a transformation semicontinuous from above is given by Berge [2].

T h e o r e m 3. Let X be a linear space, and let $p_i: X \to R^1$, $i = 1,\dots,n$, be functionals possessing a derivative at a point $x_0 \in X$ in the direction l. Then $m(x) = \max_{1 \le i \le n} p_i(x)$ has a derivative at the point x_0 in the direction l, where $m'(x_0,l) = \max_{i \in I} p_i'(x_0,l)$ with $I = \{i : p_i(x_0) = m(x_0)\}$.

T h e o r e m 4. Let $r(u,y)$ be a function defined and continuous on $R^n \times [0,T]$ differentiable with respect to u, with $r'_u(u,y)$ continuous on $R^n \times [0,T]$. Then the function $m(x) = \max_{y\in[0,T]} r(x(y),y)$, where $x \in C^n[0,T]$ ($C^n[0,T]$ denotes the space of n-dimensional functions continuous on $[0,T]$), has a derivative at any point $x_0 \in C^n[0,T]$ in any direction 1, where $m'(x_0,l) = \max_{y\in R_0} (r'_u(x_0(y),y),l(y))$ and $R_0 = \{y \in [0,T] \mid m(x_0) = r(x_0(y),y)\}$.

Theorems 3 and 4 togehter with proofs can be found in [3].

T h e o r e m 5. If A, B are any numbers and $a,b \in (0,1)$, then the sequence $c_n = \max(A + ac_{n-1}, B + bc_{n-1})$, $n = 2,3,\ldots$ with $c_1 = \max(A,B)$ is convergent and $\lim_{n\to\infty} c_n = \max\left(\frac{A}{1-a}, \frac{B}{1-b}\right)$.

The proof of this theorem can be found in [5].

T h e o r e m 6. If functions g and h are continuous on $[0, +\infty)$, $g(0) = h(0) = 0$, and if they have derivatives at 0, then the solution $f(x)$ of the equation $f(x) = \max_{y\in[0,x]} [g(y) + h(x-y) + f(ay+b(x-y))]$, where a and b are given numbers in $(0,1)$, also possesses a derivative at 0 and $f'(0) = \max\left(\frac{g'(0)}{1-a}, \frac{h'(0)}{1-b}\right)$. This theorem is also proved in [5].

3. We shall formulate and prove three lemmas and a theorem with a corollary being a generalization of Theorems 3 and 4.

L e m m a 1. Let Y be a compact metric space with metric ρ, and let A be a closed subset of Y. Then the transformation $\bar{R} : [0, +\infty) \to 2^Y$ defined by $\bar{R}(q) = \{y \in Y \mid \min_{z\in A} \rho(z,y) \le q\}$ is continuous at the point $q = 0$ and uniformly compact in the neighbourhood of 0.

P r o o f . Let $d_A(y)$ denote the distance between y and A. It is a continuous function on Y. Hence the function $\varphi(y,q) = d_A(y)-q$ is continuous on the product $Y \times [0,+\infty)$. We have $\bar{R}(q) = \{y \in Y \mid \varphi(y,q) \le 0\}$. Since φ is continuous, $\bar{R}$ is closed, and the compactness of Y implies that $\bar{R}$ is uniformly compact in the neighbourhood of 0. We are going to show that $\bar{R}$ is semicontinuous from below at $q = 0$. To this aim we have to show that for every sequence $\{q_n\} \subset [0,+\infty)$, $q_n \to 0$ and any $y_0 \in \bar{R}(0) = A$ there exists a sequence $\{y_n\} \subset Y$ such that

$y_n \in \bar{R}(q_n)$ and $y_n \to y_0$. Accordingly, let $\{q_n\}$ satisfy the hypothesis above and $y_0 \in \bar{R}(0) = A$. Since $0 \leqslant q_1 \leqslant q_2$ implies that $\bar{R}(q_1) \subset \bar{R}(q_2)$, we see that $y_0 \in \bar{R}(q)$ for every $q \geqslant 0$. It suffices to take $y_n = y_0$ for $n = 1,2,\ldots$

Lemma 2. Let Y be a compact metric space and $a : [0,\varepsilon_0] \to 2^Y$ a constant function defined by $a(\varepsilon) = Y$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Let $\varphi : [0,\varepsilon_0] \times Y \to R^1$ be a continuous functional on the product $\{0\} \times Y$, with the derivative with respect to ε at $\varepsilon = 0$ continuous with respect to $y \in Y$ and satisfying the condition

$$(*) \qquad \lim_{\substack{\varepsilon \to 0+ \\ y \to y_0}} \frac{\varphi(\varepsilon,y)-\varphi(0,y)}{\varepsilon} = \varphi'_\varepsilon(0,y_0) \text{ for every } y_0 \in Y.$$

Then $\varphi(\varepsilon,y) = \varphi(0,y) + \varepsilon\varphi'_\varepsilon(0,y) + \varepsilon e(\varepsilon,y)$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $y \in Y$, where $e(\varepsilon,y)$ is a function defined on the product $[0,\varepsilon_0] \times Y$ and continuous on $\{0\} \times Y$ with $\lim_{\varepsilon \to 0+} \max_{y \in Y} e(\varepsilon,y) = \lim_{\varepsilon \to 0+} \min_{y \in Y} e(\varepsilon,y) = 0$.

Proof. The existence of $\varphi'_\varepsilon(0,y)$ implies that there exists a funtion $e(\varepsilon,y)$ defined on $[0,\varepsilon_0] \times Y$ and satisfying the conditions:

$$(1) \qquad e(0,y) = 0 \text{ for every } y \in Y$$

$$(2) \qquad \lim_{\varepsilon \to 0^+} e(\varepsilon,y) = 0 \text{ for every } y \in Y$$

$$(3) \qquad \varphi(\varepsilon,y) = \varphi(0,y) + \varepsilon\varphi'_\varepsilon(0,y) + \varepsilon e(\varepsilon,y).$$

From (3) and (*) it follows that $\lim_{\substack{\varepsilon \to 0^+ \\ y \to y_0}} e(\varepsilon,y) = 0$ for every $y_0 \in Y$ which shows that $e(\varepsilon,y)$ is continuous on $\{0\} \times Y$. In view of the fact that $a(\varepsilon)$ is continuous at 0 and uniformly compact in the neighbourhood of 0, this implies that the functions $E_1(\varepsilon) = \max_{y \in Y} e(\varepsilon,y)$ and $E_2(\varepsilon) = \min_{y \in Y} e(\varepsilon,y)$ are continuous at $\varepsilon = 0$, in accordance with Theorem 1. Hence and from $E_1(0) = E_2(0) = 0$ we get $\lim_{\varepsilon \to 0+} E_1(\varepsilon) = \lim_{\varepsilon \to 0+} E_2(\varepsilon) = 0$, which ends the proof of Lemma 2.

L e m m a 3 . Let Y be a compact metric space with metric ϱ, and let $\tilde{a} : [0,+\infty) \to 2^Y$ be a closed map. Then there exists a function $\alpha : [0,+\infty) \to R^1$ having the limit at 0 equal to 0 and satisfying the condition

$$\tilde{a}(\varepsilon) \subset R(q) = \{ y \in Y \mid \min_{z \in \tilde{a}(0)} \varrho(y,z) < q \} \text{ for } \varepsilon \geq 0 \text{ and } q = \alpha(\varepsilon).$$

P r o o f . Since $\tilde{a}$ is closed and Y compact, it follows that $\tilde{a}$ is semicontinuous from above (see [2]). Since for every $q > 0$ the set $R(q)$ is open in Y and $\tilde{a}(0) \subset R(q)$, this implies that for every $q > 0$ there exists a number $\beta(q) > 0$ such that

1^o $\tilde{a}(\varepsilon) \subset R(q)$ for $0 \leq \varepsilon \leq \beta(q)$,

2^o $\beta(q_1) \leq \beta(q_2)$ whenever $0 < q_1 \leq q_2$.

In fact, 2^o follows from the inclusion

$$(\tilde{a}(\varepsilon) \subset R(q_1),\ 0 < q_1 \leq q_2) \Rightarrow \tilde{a}(\varepsilon) \subset R(q_2).$$

Let $\tilde{\alpha}(\varepsilon) = \inf \{ q > 0 \mid \beta(q) \geq \varepsilon \}, \varepsilon \geq 0$. From 2^o it follows that $\tilde{\alpha}$ is monotonic, and consequently the limit $\lim_{\varepsilon \to 0^+} \tilde{\alpha}(\varepsilon) = \bar{q} \geq 0$ exists. We are going to show that $\bar{q} = 0$. Suppose that $\bar{q} > 0$. Then 1^o implies that $\tilde{a}(\varepsilon) \subset R(\frac{\bar{q}}{2})$ for $0 \leq \varepsilon \leq \beta(\frac{\bar{q}}{2})$. Since $\frac{\bar{q}}{2} \in \{ q > 0 : \beta(q) \geq \varepsilon \}$ for $0 \leq \varepsilon \leq \beta(\frac{\bar{q}}{2})$, it follows that $\tilde{\alpha}(\varepsilon) \leq \frac{\bar{q}}{2} < \bar{q}$ for $0 \leq \varepsilon \leq \beta(\frac{\bar{q}}{2})$ which contradicts the obvious inequality $\tilde{\alpha}(\varepsilon) \geq \bar{q}$ for every $\varepsilon \geq 0$. Hence $\bar{q} = 0$. From the definition of $\tilde{\alpha}$ it follows that for every $\varepsilon \geq 0$ there exists $\alpha(\varepsilon) > 0$ such that $\alpha(\varepsilon) = \tilde{\alpha}(\varepsilon) + \varepsilon$ and $\beta[\alpha(\varepsilon)] \geq \varepsilon$. It is easy to see that $\lim_{\varepsilon \to 0^+} \alpha(\varepsilon) = 0$ and that condition 1^o holds for $\varepsilon \geq 0$ and $q = \alpha(\varepsilon)$. This ends the proof of Lemma 3.

T h e o r e m 7. Let Y be a compact metric space and let $\varphi : [0, \varepsilon_0] \times Y \to R^1$ be a functional continuous on $\{0\} \times Y$ with derivative $\varphi'_\varepsilon(0,y)$ continuous on Y and satisfying condition (*). Then the functional $\Phi(\varepsilon) = \max_{y \in Y} \varphi(\varepsilon, y)$ also possesses a derivative at 0, and $\Phi'(0) = \max_{y \in \tilde{Y}(0)} \varphi'_\varepsilon(0,y)$, where $\tilde{Y}(\varepsilon) = \{ y \in Y \mid \varphi(\varepsilon, y) = \Phi(\varepsilon) \}$.

P r o o f . Putting $a(\varepsilon) = Y$ for every $0 \leq \varepsilon \leq \varepsilon_0$ and using Lemma 2 we have $\varphi(\varepsilon,y) = \varphi(0,y) + \varepsilon\varphi'_{\varepsilon}(0,y) + \varepsilon e(\varepsilon,y)$ for $0 \leq \varepsilon \leq \varepsilon_0$ and $y \in Y$, where $e(\varepsilon,y)$ has been defined above. From this representation of $\varphi(\varepsilon,y)$ we infer that for $0 \leq \varepsilon \leq \varepsilon_0$ and $y \in Y$ we have

$$\Phi(\varepsilon) = \max_{y\in Y}\varphi(\varepsilon,y) \geq \max_{y\in\tilde{Y}(0)}\varphi(\varepsilon,y) \geq \max_{y\in\tilde{Y}(0)}\left[\varphi(0,y) + \varepsilon\varphi'_{\varepsilon}(0,y) + \varepsilon e(\varepsilon,y)\right] \geq$$

$$\geq \Phi(0) + \varepsilon\max_{y\in\tilde{Y}(0)}\left[\varphi'_{\varepsilon}(0,y) + e(\varepsilon,y)\right] \geq \Phi(0) + \varepsilon\max_{y\in\tilde{Y}(0)}\varphi'_{\varepsilon}(0,y) + \varepsilon\min_{y\in Y} e(\varepsilon,y).$$

For any $q > 0$ let $R(q) = \{y \in Y \mid \min_{z\in\tilde{Y}(0)}\varrho(z,y) < q\}$, where ϱ is the metric in Y. By Lemma 2, denoting the closure of $R(q)$ by $\bar{R}(q)$, we have $\tilde{Y}(\varepsilon) \subset \bar{R}(q)$ for $0 \leq \varepsilon \leq \varepsilon_0$ and $q = \alpha(\varepsilon)$, where α is some function tending to 0 together with ε. Consequently, for $0 \leq \varepsilon \leq \varepsilon_0$ and $q = \alpha(\varepsilon)$ we have

$$\Phi(\varepsilon) = \max_{y\in Y}\varphi(\varepsilon,y) \leq \max_{y\in\tilde{Y}(\varepsilon)}\varphi(\varepsilon,y) \leq \max_{y\in\bar{R}(q)}\varphi(\varepsilon,y) \leq \max_{y\in\bar{R}(q)}\left[\varphi(0,y) + \right.$$

$$\left. + \varepsilon\varphi'_{\varepsilon}(0,y) + \varepsilon e(\varepsilon,y)\right] \leq \max_{y\in\bar{R}(q)}\varphi(0,y) + \varepsilon\max_{y\in\bar{R}(q)}\left[\varphi'_{\varepsilon}(0,y) + e(\varepsilon,y)\right] \leq$$

$$\leq \Phi(0) + \varepsilon\max_{y\in\bar{R}(q)}\varphi'_{\varepsilon}(0,y) + \varepsilon\max_{y\in\bar{R}(q)} e(\varepsilon,y) \leq \Phi(0) + \varepsilon\max_{y\in\bar{R}(q)}\varphi'_{\varepsilon}(0,y) +$$

$$+ \varepsilon\max_{y\in Y} e(\varepsilon,y).$$

Recapitulating we can write the inequality

$$\min_{y\in Y} e(\varepsilon,y) + \max_{y\in\tilde{Y}(0)}\varphi'_{\varepsilon}(0,y) \leq \frac{\Phi(\varepsilon)-\Phi(0)}{\varepsilon} \leq \max_{y\in\bar{R}(q)}\varphi'_{\varepsilon}(0,y) + \max_{y\in Y} e(\varepsilon,y)$$

which holds for $0 \leq \varepsilon \leq \varepsilon_0$ and $q = \alpha(\varepsilon)$. Taking limit with $\varepsilon \to 0^+$ we obtain the thesis, because

$$\lim_{\varepsilon\to 0^+} \max_{y\in Y} e(\varepsilon,y) = \lim_{\varepsilon\to 0^+} \min_{y\in Y} e(\varepsilon,y) = 0$$

and

$$\lim_{\varepsilon\to 0^+} \max_{y\in \bar{R}(\alpha(\varepsilon))} \varphi'_{\varepsilon}(0,y) = \max_{y\in\tilde{Y}(0)} \varphi'_{\varepsilon}(0,y)$$

by means of Lemmas 1 and 2 as well as Theorem 1.

C o r o l l a r y 1 . Let X be a linear space, Y a compact metric space and let $F : X \to 2^Y$ be a constant map, i.e. $F(x) = Y$ for every $x\in X$. Let $p : X\times Y \to R^1$ be a functional whose derivative in the direction l at a point $x_0\in X$ exists for every $y\in Y$ and is continuous with respect to y. Let $p(x_0 + \varepsilon l,y)$ be a function defined on $[0, \varepsilon_0]\times Y$ and continuous on $\{0\}\times Y$. Assume that the following condition holds

$$(**)\quad \lim_{\substack{\varepsilon\to 0^+\\ y\to y_0}} \frac{p(x_0+\varepsilon l,y)-p(x_0,y)}{\varepsilon} = p'(x_0,l;y_0) \text{ for every } y_0\in Y.$$

Then there exists a derivative of the function $m(x) = \max_{y\in F(x)} p(x,y)$ in the direction l at x_0 and $m'(x_0,l) = \max_{y\in \tilde{F}(x_0)} p'(x_0,l;y)$, where

$$\tilde{F}(x_0) = \{ y\in F(x_0) \mid m(x_0) = p(x_0,y)\}.$$

P r o o f . Let $a(\varepsilon) = F(x_0 + \varepsilon l)$, $\tilde{a}(\varepsilon) = \tilde{F}(x_0 + \varepsilon l)$, $\varphi(\varepsilon,y) = p(x_0 + \varepsilon l,y)$, $\phi(\varepsilon) = m(x_0 + \varepsilon l)$. Then it is easy to see that our corollary follows directly from Theorem 7.

Observe that Corollary 1 is a generalization of Theorem 3. Namely, let $Y = \{1,2,\dots,n\}$, $F(x) = Y$ for every $x\in X$, $p(x,y) = p_i(x)$ $(i = 1,\dots,n)$, $\tilde{F}(x_0) = I$. Condition (**) is equivalent to the existence of the derivative $p'_i(x_0,l)$ for $i = 1,2,\dots,n$.

We shall show that Corollary 1 generalizes Theorem 4 as well. Let $X = C^n[0,T]$, $Y = [0,T]$, $F(x) = [0,T]$ for every $x\in X$, $p(x,y) = r(x(y),y)$, $\tilde{F}(x_0) = R_0$. Condition (**) follows from the continuity of $r(u,y)$ and $r'_u(u,y)$ with respect to the variables u,y.

The following example shows that condition (* *) is essential for the validity of Corollary 1.

E x a m p l e 1 . Let $X = R^1$, $Y = [0,1]$, $x_o = 0$, $l = 1$.

We define $p(x,y) = \begin{cases} 0 & \text{for } y \geq x \\ \sqrt{y(x-y)} & \text{for } y < x \end{cases}$ if $y \neq 0$ and $p(x,0)=0$.

Then

$$m(x) = \max_{y \in F(x)} p(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{x}{2} & \text{for } 0 \leq x \leq 2 \\ \sqrt{x-1} & \text{for } x \geq 2. \end{cases}$$

The derivative $p'(0,1;y_o)$ exists for every $y_o \in [0,1]$ and equals 0. On the other hand $m'(0,1) = \frac{1}{2}$. We easily check that condition (* *) does not hold.

The assumption of Theorems 3 and 4 and of Corollary 1 that F is constant cannot be dropped as shown by the following example.

E x a m p l e 2. Let $X = R^1$, $Y = [0,2]$, $x_o = 1$, $p(x,y) = y^2 + (x-y)^3$. We define

$$F(x) = \begin{cases} \{0\} & \text{for } x < 0 \\ [0,x] & \text{for } 0 \leq x \leq 2 \\ [0,2] & \text{for } x > 2. \end{cases}$$

Then we have (a)

$$\text{(a)} \quad m(x) = \max_{y \in F(x)} \left[y^2 + (x-y)^3\right] = \max\left[x^2, x^3\right] = \begin{cases} x^3 & \text{for } x \leq 0 \\ x^2 & \text{for } 0 \leq x \leq 1 \\ x^3 & \text{for } x > 1. \end{cases}$$

This implies

$$\tilde{F}(x) = \begin{cases} \{0\} & \text{for } x \leq 0 \\ \{x\} & \text{for } 0 \leq x < 1 \\ \{0,1\} & \text{for } x = 1 \\ \{0\} & \text{for } x > 1. \end{cases}$$

It is easy to see that the transformation $\tilde{F}$ is not continuous at $x_0 = 1$. Computing $m'(1,l)$ by means of (a) we obtain

$$m'(1,l) = \begin{cases} 2l & \text{for } l < 0 \\ 3l & \text{for } l > 0. \end{cases}$$

On the other hand, in view of the thesis of Corollary 1 we have

$$m'(1,l) = \max_{y\in\{0,1\}}\left[3(1-y)^2\ l\right] = \begin{cases} 3l & \text{for } l > 0 \\ 0 & \text{for } l < 0. \end{cases}$$

Hence for $l < 0$ we obtain a contradiction.

In turns out that in many cases the transformation that is not constant can be turn into a constant one by means of a suitable substitution. This fact allows us to extend the class of functionals to which Theorem 7 applies.

4. Now we are going to indicate some applications of Theorem 7. We consider the equation

$$f(x) = \max_{y\in[0,x]}\left[g(y) + h(x-y) + f(ay+b(x-y))\right],$$

where $g(x)$ and $h(x)$ are given functions continuous for $x \geqslant 0$, where $g(0) = h(0) = 0$, a and b are given numbers in the interval $(0,1)$, and $f(x)$ is an unknown function. Substituting $y = tx$ we obtain the equation

$$f(x) = \max_{t\in[0,1]}\left[g(tx) + h(x-tx) + f(atx+bx(1-t))\right].$$

We consider a sequence of successive approximations

$$f_1(x) = \max_{t\in[0,1]}\left[g(tx) + h(x-tx)\right]$$

$$f_n(x) = \max_{t\in[0,1]}\left[g(tx) + h(x-tx) + f_{n-1}(atx+bx(1-t))\right], \quad n=2,3,\ldots$$

If we assume that the functions g and h have derivatives at 0, then from Theorem 7 it follows that f_n $(n=1,2,\ldots)$ have derivatives at 0 and

$$f_1'(0) = \max_{t\in[0,1]}\left[tg'(0) + (1-t)h'(0)\right] = \max\left[g'(0),h'(0)\right]$$

$$f_2'(0) = \max_{t\in[0,1]}\left[tg'(0) + (1-t)h'(0)+(at+b(1-t))f_1'(0)\right] =$$

$$= \max\left[g'(0)+af_1'(0),\ h'(0)+bf_1'(0)\right],$$

and generally for $n \geqslant 2$

$$f_n'(0) = \max\left[g'(0)+af_{n-1}'(0),\ h'(0)+bf_{n-1}'(0)\right].$$

Note that the hypotheses of Corollary 1 hold, in particular assumption (**) i.e.

$$\lim_{\substack{\varepsilon\to 0^+\\ t\to t_0}} \frac{p_n(0+\varepsilon,t) - p_n(0,t)}{\varepsilon} = p_n'(0,1;t_0) \qquad \text{for every } t_0\in[0,1] \text{ and } n=1,2,\ldots,$$

where $p_n(x,t) = g(tx) + h(x-tx) + f_{n-1}(atx+bx(1-t))$ for $n \geqslant 2$ and $p_1(x,t) = g(tx) + h(x-tx)$.

In fact, e.g. for $n = 1$ and $t_0 \in (0,1)$ we have

$$\lim_{\substack{\varepsilon\to 0^+\\ t\to t_0}} \frac{g(t\varepsilon)+h((1-t)\varepsilon)-g(0)-h(0)}{\varepsilon} = \lim_{\substack{\varepsilon\to 0^+\\ t\to t_0}}\left[\frac{g(t\varepsilon)t}{t\varepsilon} + \frac{h((1-t)\varepsilon)}{(1-t)\varepsilon}(1-t)\right] =$$

$$= g'(0)t_0 + h'(0)(1-t_0),$$

and for $t_0 = 0$ and $t_0 = 1$ the above limit equals $h'(0)$ and $g'(0)$, respectively.

From Theorem 5 it follows that $\lim_{n\to\infty} f'_n(0) = f'(0)$, i.e.

$$f'(0) = \max\left[g'(0) + af'(0),\ h'(0) + bf'(0)\right] = \max\left[\frac{g'(0)}{1-a}\ \frac{h'(0)}{1-b}\right].$$

Hence we have obtained the thesis of Theorem 6.

Now we consider another functional equation, the so-called equation with not complete separation

$$f(x) = \max_{\substack{y_1+y_2\leq x\\ y_1\geq 0,\, y_2\geq 0}} \left[g(y_1) + h(y_2) + f(ay_1+by_2+x-y_1-y_2)\right],$$

where the assumption about the functions g and h remain as before. Suppose that the functions g and h have non-negative derivatives at 0. Consider the sequence of successive approximations

$$f_n(x) = \max_{\substack{y_1+y_2\leq x\\ y_1\geq 0,\, y_2\geq 0}} \left[g(y_1) + h(y_2) + f_{n-1}(ay_1+by_2+x-y_1-y_2)\right],\quad n=2,3,\ldots,$$

where $f_1(x) = \max_{\substack{y_1+y_2\leq x\\ y_1\geq 0,\, y_2\geq 0}} \left[g(y_1) + h(y_2)\right]$.

Substituting $y_1 = t_1x$, $y_2 = t_2x$ we obtain

$$f_n(x) = \max_{\substack{t_1+t_2\leq 1\\ t_1\geq 0,\, t_2\geq 0}} \left[g(t_1x) + h(t_2x) + f_{n-1}(at_1x+bt_2x+x-t_1x-t_2x)\right] \quad \text{for } n=2,3,\ldots$$

$$f_1(x) = \max_{\substack{t_1+t_2\leq 1\\ t_1\geq 0,\, t_2\geq 0}} \left[g(t_1x) + h(t_2x)\right].$$

Applying Theorem 7 and taking into account that $g'(0)$ and $h'(0)$ are non-negative we get

$$f'_1(0) = \max_{\substack{t_1+t_2 \leq 1 \\ t_1 \geq 0, t_2 \geq 0}} \left[t_1 g'(0) + t_2 h'(0)\right] = \max\left[g'(0), h'(0), 0\right] =$$

$$= \max\left[g'(0), h'(0)\right]$$

$$f'_2(0) = \max_{\substack{t_1+t_2 \leq 1 \\ t_1 \geq 0, t_2 \geq 0}} \left[t_1 g'(0) + t_2 h'(0) + (at_1+bt_2+1-t_1-t_2) f'_1(0)\right] =$$

$$= \max\left[g'(0) + af'_1(0),\ h'(0) + bf'_1(0),\ f'_1(0)\right] =$$

$$= \max\left[g'(0) + af'_1(0),\ h'(0) + bf'_1(0)\right]$$

and generally

$$f'_n(0) = \max\left[g'(0) + af'_{n-1}(0),\ h'(0) + bf'_{n-1}(0)\right].$$

By Theorem 5 we have

$$\lim_{n \to \infty} f'_n(0) = \max\left(\frac{g'(0)}{1-a}, \frac{h'(0)}{1-b}\right).$$

Note that also in this case the assumption (*) of Theorem 7 holds.

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Received February 10, 1975.