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A simple proof of Pitman–Yor’s Chinese restaurant process from its stick-breaking representation

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Abstract: For a long time, the Dirichlet process has been the gold standard discrete random measure in Bayesian nonparametrics. The Pitman–Yor process provides a simple and mathematically tractable generalization, allowing for a very flexible control of the clustering behaviour. Two commonly used representations of the Pitman–Yor process are the stick-breaking process and the Chinese restaurant process. The former is a constructive representation of the process which turns out very handy for practical implementation, while the latter describes the partition distribution induced. Obtaining one from the other is usually done indirectly with use of measure theory. In contrast, we propose here an elementary proof of Pitman–Yor’s Chinese Restaurant process from its stick-breaking representation.

Keywords: Bayesian nonparametrics, clustering, Pitman–Yor process, partitions, stick-breaking process

MSC: 62F15, 97K50

1 Introduction

The Pitman–Yor process defines a rich and flexible class of random probability measures which was developed by Perman et al. [26] and further investigated by Pitman [28], Pitman and Yor [30]. It is a simple generalization of the Dirichlet process [15], whose mathematical tractability contributed to its popularity in machine learning theory [7], probabilistic models for linguistic applications [35, 37], excursion theory [26, 30], measure-valued diffusions in population genetics [14, 27], combinatorics [19, 36] and statistical physics [11].

Its most prominent role is perhaps in Bayesian nonparametric statistics where it is used as a prior distribution, following the work of [17]. Applications in this setting embrace a variety of inferential problems, including species sampling [3, 12, 24], survival analysis and graphical models in genetics [18, 25], image segmentation [34], curve estimation [6], exchangeable feature allocations [5] and time-series and econometrics [4, 7].

Last but not least, the Pitman–Yor process is also employed in the context of nonparametric mixture modeling, thus generalizing the celebrated Dirichlet process mixture model of [21]. Nonparametric mixture models based on the Pitman–Yor process are characterized by a more flexible parameterization than the Dirichlet process mixture model, thus allowing for a better control of the clustering behaviour [9]. In addition, see [2, 13, 17] for posterior sampling algorithms, [23, 32] for asymptotic properties, and [6, 31] for spike-and-slab extensions. If a standard assumption in multivariate model-based clustering is to model the intra-group distribution with independent Gaussian distribution (diagonal covariance), non independent components have also been proposed in order to better account for complex dependence structures [10, 20].

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It is well known that the Pitman–Yor process admits a stick-breaking representation [26, 30] that generalizes that of the Dirichlet process with iid stick breaks [33] to independent stick breaks. More specifically, let $\mathbf{v}_i \stackrel{\text{iid}}{\sim} \text{Beta}(1 - d, \alpha + id)$ for $i = 1, 2, \dots$ with $d \in (0, 1)$ called the discount parameter and $\alpha > -d$ called the concentration parameter. Let $(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \dots)$ be the successive parts obtained in breaking a stick of length one according to $\boldsymbol{\pi}_1 = \mathbf{v}_1$, and $\boldsymbol{\pi}_j = \mathbf{v}_j \prod_{i=1}^{j-1} (1 - \mathbf{v}_i)$ for $j = 2, 3, \dots$, and let $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots \stackrel{\text{iid}}{\sim} P_0$ where the probability distribution P_0 is called the base distribution. Then the discrete random probability measure

$$\mathbf{P} = \sum_{j=1}^{\infty} \boldsymbol{\pi}_j \delta_{\boldsymbol{\theta}_j} \quad (1)$$

is distributed according to the Pitman–Yor process denoted by $\text{PY}(\alpha, d, P_0)$. Due to its discrete nature (1), the Pitman–Yor process creates ties with positive probability among data $\mathbf{x}_1, \dots, \mathbf{x}_n$ generated from the Bayesian model

$$\begin{aligned} \mathbf{x}_1, \dots, \mathbf{x}_n \mid \mathbf{P} &\stackrel{\text{iid}}{\sim} \mathbf{P}, \\ \mathbf{P} &\sim \text{PY}(\alpha, d, P_0). \end{aligned}$$

Assuming that the base distribution is non-atomic, the distribution on random partitions $\mathbf{C}$ of the first n integers $\{1, \dots, n\}$ induced by the data $\mathbf{x}_1, \dots, \mathbf{x}_n$ is

$$\mathbb{P}(\mathbf{C} = C) = \frac{d^{|C|}}{(\alpha)_{(n)}} \left(\frac{\alpha}{d}\right)_{(|C|)} \prod_{c \in C} (1 - d)_{(|c|-1)}, \quad (2)$$

where $(x)_{(n)} = x(x+1) \dots (x+n-1)$ denotes the rising factorial. The multiplicative factor before the product in (2) is also commonly (and equivalently) written as $(\prod_{i=1}^{|C|-1} \alpha + id)/(\alpha + 1)_{(n-1)}$ in the literature. When the discount parameter d is set to zero, the Pitman–Yor process reduces to the Dirichlet process and the partition distribution (2) boils down to the celebrated Chinese Restaurant process [CRP, see 1]. By abuse of language, we call the partition distribution (2) the Pitman–Yor’s CRP. Under the latter partition distribution, the number of parts in a partition C of n elements, $|C|$, grows to infinity as a power-law of the sample size, n^d [see 29, for details]. This Pitman–Yor power-law growth is more in tune with most of empirical data [8] than the logarithmic growth $\alpha \log n$ induced by the Dirichlet process CRP.

The purpose of this note is to provide a simple proof of Pitman–Yor’s CRP (2) from its stick-breaking representation (1) (Theorem 2.1). This generalizes the derivation by [22] who obtained the Dirichlet process CRP [1] from Sethuraman’s stick-breaking representation [33]. Although the general proof technique for Theorem 2.1 proceeds along similar lines to that of [22], most of the intermediate results required for the proof rely on different techniques or are novel (such as Lemma 3.2). We also provide in Proposition 2.2 the marginal distribution of the latent variables that identify the observations clusters (so-called allocation variables).

A natural extension to the Dirichlet and Pitman–Yor processes are Gibbs-type priors [9]. They are characterized by a simple CRP representation

$$\mathbb{P}(\mathbf{C} = C) = V_{n,k} \prod_{c \in C} (1 - d)_{(|c|-1)},$$

where the triangular array of nonnegative parameters $V_{n,k}$, $1 \leq k \leq n$, satisfy the backward recurrence relation

$$V_{n,k} = (n - dk)V_{n+1,k} + V_{n+1,k+1},$$

with $V_{1,1} = 1$. Like the Dirichlet and Pitman–Yor processes, Gibbs-type priors have an explicit stick-breaking representation. See for instance Theorem 14.23 of [16] for details. But in contrast to the Dirichlet and Pitman–Yor cases, the latter representation is obtained by breaking sequentially proportions of a stick of length one in a *non independent* manner. Due to this loss of independence, we have not been able to extend the present proof to the class of Gibbs-type priors, which remains an open problem.

2 Partition distribution from stick-breaking

We denote the set of positive integers $\{1, 2, \dots\}$ by $\mathbb{N}$. For $n \in \mathbb{N}$, we say that integers $z_1, \dots, z_n \in \mathbb{N}$ induce a partition of $\{1, \dots, n\}$ denoted by C into disjoint and non-empty parts $c_1, \dots, c_k$ if C is induced by the equivalence relation $z_i = z_j$, for any $i, j \in \{1, \dots, n\}$. We denote by k the number of parts $|C|$ in the partition C and by $|c_j|$ the number of elements in part c_j . Bold font is used to represent random variables. We denote by Γ the gamma function and by B the beta function.

Theorem 2.1. Suppose

$$\begin{aligned} \mathbf{v}_i &\stackrel{\text{ind}}{\sim} \text{Beta}(1-d, \alpha + id) \text{ for } i = 1, 2, \dots, \\ \pi_j &= \mathbf{v}_j \prod_{i=1}^{j-1} (1 - \mathbf{v}_i) \text{ for } j = 1, 2, \dots \end{aligned}$$

Let allocation variables be defined by

$$\mathbf{z}_1, \dots, \mathbf{z}_n | \boldsymbol{\pi} = \boldsymbol{\pi} \stackrel{\text{iid}}{\sim} \boldsymbol{\pi}, \text{ meaning, } \mathbb{P}(\mathbf{z}_i = j | \boldsymbol{\pi}) = \pi_j, \quad (3)$$

and $\mathbf{C}$ denote the random partition of $\{1, \dots, n\}$ induced by $\mathbf{z}_1, \dots, \mathbf{z}_n$. Then

$$\mathbb{P}(\mathbf{C} = C) = \frac{d^{|C|}}{(\alpha)_{(n)}} \left(\frac{\alpha}{d}\right)_{(|C|)} \prod_{c \in C} (1-d)_{(|c|-1)}.$$

The proof of Theorem 2.1 follows the lines of [22]’s derivation. We need the next two technical results, which we will prove in Section 3. Let C_z denote the partition $\{1, \dots, n\}$ induced by z for any $z \in \mathbb{N}^n$. Let k be the number of parts in the partition. We define $m(z) = \max\{z_1, \dots, z_n\}$, and $g_j(z) = \#\{i : z_i \geq j\}$.

Proposition 2.2. For any $n \in \mathbb{N}$, any $z \in \mathbb{N}^n$, the marginal distribution of the allocation variables vector $\mathbf{z} = (z_1, \dots, z_n)$ is given by

$$\mathbb{P}(\mathbf{z} = z) = \frac{1}{(\alpha)_{(n)}} \prod_{c \in C_z} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \prod_{j=1}^{m(z)} \frac{\alpha + (j-1)d}{g_j(z) + \alpha + (j-1)d}.$$

Lemma 2.3. For any $n \in \mathbb{N}$, any partition C of $\{1, \dots, n\}$,

$$\sum_{z \in \mathbb{N}^n} \mathbb{1}(C_z = C) \prod_{j=1}^{m(z)} \frac{\alpha + (j-1)d}{g_j(z) + \alpha + (j-1)d} = \frac{d^{|C|}}{\prod_{c \in C} (|c| - d)} \left(\frac{\alpha}{d}\right)_{(|C|)}.$$

Proof of Theorem 2.1.

$$\begin{aligned} \mathbb{P}(\mathbf{C} = C) &= \sum_{z \in \mathbb{N}^n} \mathbb{P}(\mathbf{C} = C | \mathbf{z} = z) \mathbb{P}(\mathbf{z} = z) \\ &\stackrel{(a)}{=} \sum_{z \in \mathbb{N}^n} \mathbb{1}(C_z = C) \frac{1}{(\alpha)_{(n)}} \left(\prod_{c \in C_z} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \right) \prod_{j=1}^{m(z)} \frac{\alpha + (j-1)d}{g_j(z) + \alpha + (j-1)d} \\ &= \frac{1}{(\alpha)_{(n)}} \left(\prod_{c \in C} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \right) \sum_{z \in \mathbb{N}^n} \mathbb{1}(C_z = C) \prod_{j=1}^{m(z)} \frac{\alpha + (j-1)d}{g_j(z) + \alpha + (j-1)d} \\ &\stackrel{(b)}{=} \frac{1}{(\alpha)_{(n)}} \left(\prod_{c \in C} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \right) \frac{d^{|C|}}{\prod_{c \in C} (|c| - d)} \left(\frac{\alpha}{d}\right)_{(|C|)} \\ &\stackrel{(c)}{=} \frac{1}{(\alpha)_{(n)}} \left(\prod_{c \in C} (1-d)_{(|c|-1)} \right) \left(\prod_{c \in C} (|c| - d) \right) \frac{d^{|C|}}{\prod_{c \in C} (|c| - d)} \left(\frac{\alpha}{d}\right)_{(|C|)} \end{aligned}$$

$$= \frac{d^{|C|}}{(\alpha)_{(n)}} \left(\frac{\alpha}{d} \right)_{(|C|)} \prod_{c \in C} (1 - d)_{(|c|-1)},$$

where (a) is by Proposition 2.2, (b) is by Lemma 2.3, and (c) is since $\Gamma(|c| + 1 - d) = (|c| - d)\Gamma(|c| - d)$. $\square$

3 Proofs of the technical results

3.1 Additional lemmas

We require the following additional lemmas. Let S_k denote the set of $k!$ permutations of $[k]$. The following lemma is key for proving Lemma 2.3.

Lemma 3.1. *For any $n_1, \dots, n_k \in \mathbb{N}$,*

$$\sum_{\sigma \in S_k} \prod_{i=1}^k \frac{1}{a_i(\sigma) - (k-i+1)d} = \frac{1}{\prod_{i=1}^k (n_i - d)}$$

where $a_i(\sigma) = n_{\sigma_i} + n_{\sigma_{i+1}} + \dots + n_{\sigma_k}$.

The proof of Lemma 3.1 follows similar lines to that of Lemma 3.1 by [22], relying on an urn argument for simplifying sums over permutations in S_k . The latter lemma is stated and proven by [22] for integer-valued ball sizes $n_1, \dots, n_k \in \mathbb{N}$, however it is straightforward to check that it carries over to any positive real-valued ball sizes. Application of this result to ball sizes $n_1 - d, \dots, n_k - d$, which are all positive since $d < 1$ and $n_j \geq 1$ for any $j = 1, \dots, k$, yields the statement of Lemma 3.1.

Lemma 3.2. *Let $b_0 = 0$ and introduce the notation $\bar{b}_i = b_0 + b_1 + \dots + b_i$ for positive integers b_j . Then*

$$\sum_{b_1 \in \mathbb{N}} \frac{\left(\frac{\alpha}{d}\right)_{(b_1)}}{\left(\frac{\alpha_1 + \alpha}{d}\right)_{(b_1)}} \left(\sum_{b_2 \in \mathbb{N}} \frac{\left(\frac{\alpha}{d} + b_1\right)_{(b_2)}}{\left(\frac{\alpha_2 + \alpha}{d} + b_1\right)_{(b_2)}} \left(\dots \left(\sum_{b_k \in \mathbb{N}} \frac{\left(\frac{\alpha}{d} + \bar{b}_{k-1}\right)_{(b_k)}}{\left(\frac{\alpha_k + \alpha}{d} + \bar{b}_{k-1}\right)_{(b_k)}} \right) \dots \right) \right) = \frac{\left(\frac{\alpha}{d}\right)_{(k)}}{\prod_{i=1}^k \left(\frac{\alpha_i}{d} - (k-i+1)\right)}.$$

Proof. For $j \in \{1, \dots, k\}$, let A_j denote the function from $\mathbb{N}^{j-1}$ to $\mathbb{R}_+$ defined by the partial product of the $k-j+1$ inner terms in the left-hand side of the lemma's statement,

$$A_j(b_1, \dots, b_{j-1}) = \sum_{b_j \in \mathbb{N}} \frac{\left(\frac{\alpha}{d} + \bar{b}_{j-1}\right)_{(b_j)}}{\left(\frac{\alpha_j + \alpha}{d} + \bar{b}_{j-1}\right)_{(b_j)}} \left(\dots \left(\sum_{b_k \in \mathbb{N}} \frac{\left(\frac{\alpha}{d} + \bar{b}_{k-1}\right)_{(b_k)}}{\left(\frac{\alpha_k + \alpha}{d} + \bar{b}_{k-1}\right)_{(b_k)}} \right) \dots \right). \quad (4)$$

We show by induction decreasing from $j = k$ to $j = 1$ that the relation

$$A_j(b_1, \dots, b_{j-1}) = \frac{\left(\frac{\alpha}{d} + \bar{b}_{j-1}\right)_{(k-j+1)}}{\prod_{i=j}^k \left(\frac{\alpha_i}{d} - (k-i+1)\right)} \quad (5)$$

holds for any $j \in \{1, \dots, k\}$, and that the relations obtained with $j = 1$ corresponds to the statement of the lemma. When $j = k$, defining $X \sim \text{Beta}\left(\frac{\alpha}{d} + \bar{b}_{k-1}, \frac{\alpha_k}{d}\right)$ yields

$$\begin{aligned} A_k(b_1, \dots, b_{k-1}) &= \sum_{b_k \in \mathbb{N}} \frac{\left(\frac{\alpha}{d} + \bar{b}_{k-1}\right)_{(b_k)}}{\left(\frac{\alpha_k + \alpha}{d} + \bar{b}_{k-1}\right)_{(b_k)}} \stackrel{(a)}{=} \sum_{b_k \in \mathbb{N}} \mathbb{E}[X^{b_k}] \\ &= \mathbb{E}\left[\sum_{b_k \in \mathbb{N}} X^{b_k}\right] \stackrel{(b)}{=} \mathbb{E}\left[\frac{X}{1-X}\right] \stackrel{(a)}{=} \frac{\alpha + d\bar{b}_{k-1}}{\alpha_k - d} = \frac{\left(\frac{\alpha}{d} + \bar{b}_{k-1}\right)_{(1)}}{\frac{\alpha_k}{d} - 1}, \end{aligned}$$

where both steps (a) are due to the fact that if $X \sim \text{Beta}(a, b)$, for $a, b > 0$, then for $c > -a$, $d > -b$,

$$\mathbb{E}[X^c(1-X)^d] = \frac{B(a+c, b+d)}{B(a, b)}, \quad (6)$$

while step (b) is due to the identity

$$(1-X)(1+X+X^2+\dots) = 1 \text{ when } X \in [0, 1) \text{ almost surely.} \quad (7)$$

This proves the initialization for (5) with $j = k$. We now consider the case of an arbitrary $j \in \{1, \dots, k-1\}$, and assume by induction hypothesis that Equation (5) holds for $j+1$, that is

$$A_{j+1}(b_1, \dots, b_j) = \frac{(\frac{\alpha}{d} + \bar{b}_j)_{(k-j)}}{\prod_{i=j+1}^k (\frac{a_i}{d} - (k+1-i))}.$$

Therefore, by definition of function A_j in (4) and by induction hypothesis,

$$A_j(b_1, \dots, b_{j-1}) = \sum_{b_j \in \mathbb{N}} \frac{(\frac{\alpha}{d} + \bar{b}_{j-1})_{(b_j)}}{(\frac{a_j+\alpha}{d} + \bar{b}_{j-1})_{(b_j)}} A_{j+1}(b_1, \dots, b_j) = \sum_{b_j \in \mathbb{N}} \frac{(\frac{\alpha}{d} + \bar{b}_{j-1})_{(b_j)}}{(\frac{a_j+\alpha}{d} + \bar{b}_{j-1})_{(b_j)}} \frac{(\frac{\alpha}{d} + \bar{b}_j)_{(k-j)}}{\prod_{i=j+1}^k (\frac{a_i}{d} - (k+1-i))}.$$

Rearranging the rising factorials in the numerator, we can write

$$\begin{aligned} \left(\frac{\alpha}{d} + \bar{b}_{j-1}\right)_{(b_j)} \left(\frac{\alpha}{d} + \bar{b}_j\right)_{(k-j)} &= \left(\frac{\alpha}{d} + \bar{b}_{j-1}\right)_{(b_j)} \left(\frac{\alpha}{d} + \bar{b}_{j-1} + b_j\right)_{(k-j)} \\ &= \left(\frac{\alpha}{d} + \bar{b}_{j-1}\right)_{(b_j+k-j)} \\ &= \left(\frac{\alpha}{d} + \bar{b}_{j-1}\right)_{(k-j)} \left(\frac{\alpha}{d} + \bar{b}_{j-1} + k-j\right)_{(b_j)}, \end{aligned}$$

and thus factorize the terms independent of b_j in order to obtain

$$A_j(b_1, \dots, b_{j-1}) = \frac{(\frac{\alpha}{d} + \bar{b}_{j-1})_{(k-j)}}{\prod_{i=j+1}^k (\frac{a_i}{d} - (k+1-i))} \sum_{b_j \in \mathbb{N}} \frac{(\frac{\alpha}{d} + \bar{b}_{j-1} + k-j)_{(b_j)}}{(\frac{a_j+\alpha}{d} + \bar{b}_{j-1})_{(b_j)}}.$$

The sum above can be rewritten, using $X \sim \text{Beta}(\frac{\alpha}{d} + \bar{b}_{j-1} + k-j, \frac{a_j}{d} - (k-j))$ and expressions (6) and (7), as

$$\sum_{b_j \in \mathbb{N}} \mathbb{E}[X^{b_j}] = \mathbb{E}\left[\frac{X}{1-X}\right] = \frac{\frac{\alpha}{d} + \bar{b}_{j-1} + k-j}{\frac{a_j}{d} - (k+1-j)}.$$

Putting this all together,

$$A_j(b_1, \dots, b_{j-1}) = \frac{(\frac{\alpha}{d} + \bar{b}_{j-1})_{(k-j)}}{\prod_{i=j+1}^k (\frac{a_i}{d} - (k+1-i))} \frac{\frac{\alpha}{d} + \bar{b}_{j-1} + k-j}{\frac{a_j}{d} - (k+1-j)} = \frac{(\frac{\alpha}{d} + \bar{b}_{j-1})_{(k-j+1)}}{\prod_{i=j}^k (\frac{a_i}{d} - (k+1-i))}$$

which coincides with (5) and proves the desired result for j . By induction, this result is true for all $j \in \{1, \dots, k\}$. Letting $j = 1$ gives the result stated in the lemma, since $\bar{b}_0 = b_0 = 0$. $\square$

3.2 Proof of Proposition 2.2 and Lemma 2.3

Proof of Proposition 2.2. For simplicity, we fix the allocation variable vector to a value z and denote $m(z)$ by m and $g_j(z)$ by g_j . We have

$$\mathbb{P}(z = z | \pi_1, \dots, \pi_m) = \prod_{i=1}^n \pi_{z_i} = \prod_{j=1}^m \pi_j^{e_j}$$

where $e_j = \#\{i : z_i = j\}$. Thus,

$$\mathbb{P}(z = z | v_1, \dots, v_m) = \prod_{j=1}^m \left(v_j \prod_{i=1}^{j-1} (1-v_i)\right)^{e_j} = \prod_{j=1}^m v_j^{e_j} (1-v_j)^{f_j}$$

where $f_j = \#\{i : z_i > j\}$. Therefore,

$$\begin{aligned}
 \mathbb{P}(\mathbf{z} = \mathbf{z}) &= \int \mathbb{P}(\mathbf{z} = \mathbf{z} | v_1, \dots, v_m) p(v_1, \dots, v_m) dv_1 \cdots dv_m \\
 &= \int \left(\prod_{j=1}^m v_j^{e_j} (1 - v_j)^{f_j} \right) p_1(v_1) \cdots p_m(v_m) dv_1 \cdots dv_m \\
 &= \prod_{j=1}^m \int v_j^{e_j} (1 - v_j)^{f_j} p_j(v_j) dv_j \\
 &\stackrel{(a)}{=} \prod_{j=1}^m \frac{B(e_j + 1 - d, f_j + \alpha + jd)}{B(1 - d, \alpha + jd)} \\
 &= \prod_{j=1}^m \frac{\Gamma(e_j + 1 - d) \Gamma(f_j + \alpha + jd)}{\Gamma(e_j + f_j + \alpha + (j - 1)d + 1)} \frac{\Gamma(\alpha + (j - 1)d + 1)}{\Gamma(1 - d) \Gamma(\alpha + jd)} \\
 &\stackrel{(b)}{=} \left(\prod_{j=1}^m \frac{\Gamma(e_j + 1 - d)}{\Gamma(1 - d)} \right) \left(\prod_{j=1}^m \frac{\Gamma(g_{j+1} + \alpha + jd)}{\Gamma(g_j + \alpha + (j - 1)d + 1)} \right) \left(\prod_{j=1}^m \frac{\Gamma(\alpha + (j - 1)d + 1)}{\Gamma(\alpha + jd)} \right) \\
 &\stackrel{(c)}{=} \left(\prod_{j=1}^m \frac{\Gamma(e_j + 1 - d)}{\Gamma(1 - d)} \right) \left(\prod_{j=1}^m \frac{\alpha + (j - 1)d}{g_j + \alpha + (j - 1)d} \right) \frac{\Gamma(g_{m+1} + \alpha + md) \Gamma(\alpha)}{\Gamma(g_1 + \alpha) \Gamma(\alpha + md)} \\
 &\stackrel{(d)}{=} \frac{\Gamma(\alpha)}{\Gamma(n + \alpha)} \left(\prod_{c \in C_z} \frac{\Gamma(|c| + 1 - d)}{\Gamma(1 - d)} \right) \left(\prod_{j=1}^m \frac{\alpha + (j - 1)d}{g_j + \alpha + (j - 1)d} \right)
 \end{aligned}$$

where step (a) follows from Equation 6, step (b) since $f_j = g_{j+1}$ and $g_j = e_j + f_j$, step (c) since $\Gamma(x + 1) = x\Gamma(x)$, and step (d) since $g_1 = n$ and $g_{m+1} = 0$. $\square$

Proof of Lemma 2.3. For given n and C , let $k = |C|$, and denote the parts of C by $c_1, \dots, c_k$. We denote the distinct values taken on by $z_1, \dots, z_n$ by $j_1 < \dots < j_k$. We define $j_0 = b_0 = 0$, $b_i = j_i - j_{i-1}$, and $\bar{b}_i = b_0 + \dots + b_i$ for $i \in \{1, \dots, k\}$. We use the notation $a_i(\sigma) = |c_{\sigma_i}| + \dots + |c_{\sigma_k}|$, where σ is the permutation of $[k]$ such that $c_{\sigma_i} = \{\ell : z_\ell = j_i\}$. Then for any $\mathbf{z} \in \mathbb{N}^n$ such that $C_z = C$,

$$\prod_{j=1}^{m(\mathbf{z})} \frac{\alpha + (j - 1)d}{g_j(\mathbf{z}) + \alpha + (j - 1)d} = \prod_{j=1}^{m(\mathbf{z})} \frac{\frac{\alpha}{d} + j - 1}{\frac{g_j(\mathbf{z}) + \alpha}{d} + j - 1} = \prod_{i=1}^k \prod_{j=\bar{b}_{i-1}+1}^{\bar{b}_i} \frac{\frac{\alpha}{d} + j - 1}{\frac{g_j(\mathbf{z}) + \alpha}{d} + j - 1} = \prod_{i=1}^k \frac{(\frac{\alpha}{d} + \bar{b}_{i-1})_{(b_i)}}{(\frac{\alpha + a_i(\sigma)}{d} + \bar{b}_{i-1})_{(b_i)}},$$

because $g_j(\mathbf{z}) = a_i(\sigma)$ for $\bar{b}_{i-1} < j \leq \bar{b}_i$. It follows from the definition of $b = (b_1, \dots, b_k)$ and σ that there is a one-to-one correspondence between $\{\mathbf{z} \in \mathbb{N}^n : C_z = C\}$ and $\{(\sigma, b) : \sigma \in S_k, b \in \mathbb{N}^k\}$. Therefore,

$$\begin{aligned}
 &\sum_{\mathbf{z} \in \mathbb{N}^n} \mathbb{1}(C_z = C) \prod_{j=1}^{m(\mathbf{z})} \frac{\alpha + (j - 1)d}{g_j(\mathbf{z}) + \alpha + (j - 1)d} \\
 &= \sum_{\sigma \in S_k} \left(\sum_{b_1 \in \mathbb{N}} \frac{(\frac{\alpha}{d} + \bar{b}_0)_{(b_1)}}{(\frac{\alpha_1 + \alpha}{d} + \bar{b}_0)_{(b_1)}} \left(\cdots \left(\sum_{b_k \in \mathbb{N}} \frac{(\frac{\alpha}{d} + \bar{b}_{k-1})_{(b_k)}}{(\frac{\alpha_k + \alpha}{d} + \bar{b}_{k-1})_{(b_k)}} \right) \cdots \right) \right) \\
 &\stackrel{(a)}{=} \sum_{\sigma \in S_k} \prod_{i=1}^k \frac{(\frac{\alpha}{d})_{(k)}}{a_i(\sigma) - (k - i + 1)} \\
 &= d^k \left(\frac{\alpha}{d} \right)_{(k)} \sum_{\sigma \in S_k} \prod_{i=1}^k \frac{1}{a_i(\sigma) - (k - i + 1)} \\
 &\stackrel{(b)}{=} \frac{d^k}{\prod_{c \in C} (|c| - d)} \left(\frac{\alpha}{d} \right)_{(k)},
 \end{aligned}$$

where step (a) follows from Lemma 3.2 and step (b) from Lemma 3.1. $\square$

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