Research Article

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Students' Concepts of the Trapezoid at the End of Lower Secondary Level Education

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Abstract: Understanding basic geometric concepts is important for the development of students' thinking and geometric imagination, both of which facilitate their progress in mathematics. Factors that may affect students' understanding include the teacher of mathematics (in particular his/her teaching style) and the textbooks used in mathematics lessons. In our research, we focused on the conceptual understanding of trapezoids among Czech students at the end of the 2nd level of education (ISCED 2). As part of the framework for testing the understanding of geometrical concepts, we assigned a task in which students had to choose trapezoids from six figures (four trapezoids and two other quadrilaterals). In total, 437 students from the 9th grade of lower secondary schools and corresponding years of grammar schools participated in the test. The gathered data were subjected to a qualitative analysis. We found that less than half of the students commanded an adequate conceptual understanding of trapezoids. It turned out that some students did not recognise the non-model of a trapezoid, or vice versa, considering a trapezoid in an untypical position to be a non-model. Our research was supplemented by analysis of geometry textbooks usually used in the Czech Republic which revealed that the trapezoid is predominantly presented in a prototypical position. Moreover, we investigated how pre-service teachers define a trapezoid. Finally, we present some recommendations for mathematics teaching and pre-service teacher training that could help.

Keywords: conceptual maps, conceptual understanding, hierarchy of quadrilaterals, non-models, trapezoid.

1 Introduction

Based on our experience in teaching students of upper secondary schools and pre-service teachers of mathematics we observed that some students' misconceptions in geometry persist among upper secondary school and university students. Therefore we decided to concentrate on this problem in our research. Understanding basic concepts is a prerequisite for teaching and learning mathematics. This enables students to master other subject matter and develop thinking, which contributes to their further progress in mathematics. As a result, our long-term research focused on how Czech students understand different geometrical concepts such as a line segment, a half-line (Moravcová et al., 2018), a triangle (Robová et al., 2019), or an axial symmetry (Moravcová et al., 2019). In this paper, we deal with the perceptions of trapezoids held among students at the end of lower secondary schools, and how this concept is understood by pre-service mathematics teachers in their first year of university studies.

In many European countries, the trapezoid is usually introduced at lower secondary schools. In this paper, we use the term trapezoid ("lichoběžník" in Czech) in accordance with Czech mathematics textbooks, i.e. a quadrilateral that has only one pair of parallel sides. According to the Czech mathematics curriculum, a student at the end of the lower secondary school should be able to "characterise and classify basic two-dimensional figures" (MŠMT, 2017, p. 36). Students learn not only to recognise a trapezoid and determine its properties, including its perimeter and area, but also to gradually classify these planar figures together with other quadrilaterals. This classification is important because it helps students to better understand the characteristics of each type of quadrilateral and the links between them, thereby not only enhancing their conceptual knowledge, but also their structural knowledge of the concept. While at lower secondary school students gradually build classification of quadrilaterals, they should be able to classify them at upper secondary school independently.

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Figure 1: The classification of quadrilaterals in accordance with the exclusive definition of trapezoid (used in the Czech Republic, Slovakia, Croatia, etc.)

Figure 2: The classification of quadrilaterals in accordance with the inclusive definition of trapezoid (used in the United Kingdom, Germany, Hungary, Poland, Turkey, etc.)

The classification of quadrilaterals is based on the definitions and characteristics of particular types of quadrilateral. Within the context of this classification, we can find two possible definitions of a trapezoid in mathematics textbooks – exclusive (Figure 1) and inclusive (Figure 2). The exclusive definition (used in the Czech Republic, Slovakia, Croatia, etc.) introduces a trapezoid as a quadrilateral with only one pair of parallel sides, whereas the inclusive definition as a quadrilateral having at least one pair of parallel sides. The latter definition (used in the United Kingdom¹, Germany, Hungary,

Poland, Turkey, etc.) makes a parallelogram a special type of trapezoid, i.e. the type of definition influences the classification of quadrilaterals. The inclusive definition of a trapezoid is consistent with its use in calculus. In Czech textbooks, for lower and upper secondary schools, the exclusive definition is used, whereby students are taught that parallelograms are not trapezoids. Unless stated otherwise, we use the exclusive classification in the following.

In this paper we deal with how 9th grade students understand the concept of a trapezoid and what is the conceptual and structural knowledge of a trapezoid of pre-service mathematics teachers.

¹ In the United Kingdom, a quadrilateral having at least one pair of parallel sides is called a trapezium (see Merriam-Webster dictionary, <https://www.merriam-webster.com/dictionary/trapezium>). In this paper, we use the term “trapezium” for a quadrilateral with no pair of parallel sides.

2 Theoretical framework

Many scientific disciplines, including psychology and pedagogy, address the issue of student knowledge and conceptual understanding. Here, we make the distinction between these basic types of knowledge: declarative/conceptual, procedural, structural, contextual and metacognitive (Mareš, 2011). The basis for good quality learning is a deeper understanding of the concepts and relationships between them. In our research, we focused on the conceptual knowledge of students at the end of lower secondary school (9th grade, 15 years old) and the conceptual and structural knowledge of pre-service mathematics teachers in relation to the concept of the trapezoid.

Analyses of students' conceptual knowledge are mostly based on cognitive theories that deal with how students develop concepts. Vinner and HersHKowitz (1980) and Tall and Vinner (1981) dealt with the relationship between subjective conceptualisations and formal definitions of mathematical concepts. They introduced the terms *concept image* and *concept definition* to mathematics learning. Concept image describes “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes”. Concept definition is “a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole”. (Tall & Vinner, 1981, p. 152). “The concept image may also include a (personal) concept definition as an individual reconstruction of the mathematical one.” (Rösken & Rolka, 2007, p. 184).

We can also analyse the understanding of mathematical concepts using the *Theory of Generic Models* (Hejný, 2012): students get to abstract knowledge from motivation, through experience with isolated models and through the creation of general models. During this process, it is also important to present students with non-models, i.e. objects that are not models of the given concept.

An inadequate notion of a mathematical concept can be related to a student's personal concept definition (Tall & Vinner, 1981). Formal student knowledge, which is retained only by memory and for which the student has not created adequate models, also plays a role. In geometry, this formalism can be observed when identifying figures: students usually only recognise a typical model of a given entity, which is referred to as a prototype (HersHKowitz, 1989; Vinner & HersHKowitz, 1980; Monaghan, 2000); from the perspective of Hejný (2012) it is a case of an

isolated model of the concept. For example, pre-school children do not have a problem designating a prototypical triangle (an equilateral triangle with one horizontal side) as a triangle, while they will hesitate in the case of an obtuse triangle (Clements et al., 1999; Tirosh & Tsamir et al., 2011). Pre-school children also usually recognise a prototypical square in a standard position with horizontal and vertical sides (Budínová, 2017a, 2017b; Gunčaga et al., 2017; Žilková, 2018).

We can also investigate a student's conceptual understanding of geometry on the basis of the Van Hiele model. According to this model, learners advance through five levels of thought in geometry: *Visualisation*, *Analytic*, *Abstraction*, *Deduction*, *Rigor* (Van Hiele, 1999; Fujita & Jones, 2007). This model is generally considered to be useful for describing learners' behaviours in geometry. The Van Hiele model is widely used in research into the conceptual understanding of geometry among primary school students and pre-service primary teachers, e.g. (Fujita & Jones, 2006; Gunčaga et al., 2017).

A lot of research studies have addressed the conceptual understanding of trapezoids in both secondary school students and pre-service mathematics teachers. Teachers play an important role in the education process and can significantly influence students (Martínková, Goldhaber & Erosheva, 2018). Monaghan (2000) examined approximately 2,000 students, two-thirds of whom were not native speakers, at upper secondary school level. The students answered five questions about the differences between specific types of quadrilaterals; the last one asked about the difference between a trapezoid and a parallelogram. The students were largely unable to look at a trapezoid (the inclusive definition of the trapezoid) as a superior term for parallelogram. Gunčaga et al. (2017) examined the understanding of geometrical concepts among pre-school children and 4th grade students aged 10. In one task, the students were asked to recognise the non-models of squares among which was also a trapezoid. 81.25% of the 4th grade students correctly identified the trapezoid as a non-model, but 45% incorrectly did the same for a square rotated by 45°. “Most of the correct answers were given for those shapes that corresponded to the prototype of these shapes.” (Gunčaga et al., 2017, p. 513). Jirotková (2017) found that Czech students can substitute a trapezoid for a parallelogram on the basis of the argument that a parallelogram also has a pair of parallel sides. Türnüklü (2014) investigated the concept images of a trapezoid among 156 middle school students (10–13 years old) and 36 pre-service teachers; he used the qualitative approach and semi-structured interview method; during the interview, individuals were asked to

define a trapezoid and draw one. A trapezoid is defined in course books as “a quadrilateral with straight sides that has a pair of opposite parallel sides”. There is no definitive judgement with regards to whether the remaining two sides are parallel or not. Türnüklü found that many of the definitions acquired were personal rather than formal; students used a trapezoid prototype in their drawings. An inaccurate definition in textbooks and a prototypical perception might have caused the participants to create an inaccurate concept image of a trapezoid. In total, 22 of the 36 pre-service mathematics teachers (i.e. 62%) submitted an erroneous definition of a trapezoid, or could not define it at all.

Okazaki and Fujita (2007) investigated how students understand the relationships between quadrilaterals. They collected data from 234 students attending the 9th grade of a Japanese public junior high school and 111 Scottish pre-service primary teachers in their first year of university studies. The Japanese students strongly used the prototype for squares and rectangles, which proved to be an obstacle to the correct understanding of the rectangle/parallelogram and square/rectangle relationships. In contrast, Scottish pre-service primary teachers had relatively flexible images of parallelograms; the strongest prototype phenomenon appeared for squares.

Fujita and Jones (2006) conducted a study examining two groups of respondents in the UK: 158 pre-service primary teachers in their first year of university studies and 124 pre-service primary teachers in their third year of university studies. The first group of students were asked to answer several questions concerning quadrilaterals (e.g. “Is a square a trapezium?”, “Is a parallelogram a trapezium?”)². The second group was asked to identify quadrilaterals in a given figure, including drawing arrows between them, i.e. their understanding of the hierarchical classification of quadrilaterals was investigated. In the first group, 8.9% of students correctly answered the question that a square is a trapezoid, and 18.4% that a parallelogram is a trapezoid. The analysis of the data from the second group revealed “the majority of students at least draw correct images of quadrilaterals (with the exception of a trapezium), but far less were able to provide definitions”. (Fujita & Jones, 2006). This means that the students' *personal figural concepts* influenced the quadrilateral classification.

In 2007, Fujita and Jones presented another research study in which they examined the conceptual

understanding of parallelograms of 105 pre-service primary teachers in their second year of university studies. On the basis of the Van Hiele model, the researchers found that “... almost half of the learners still regard parallelograms in terms of limited images despite the fact that they have a fair understanding of the concept definition. Thus, there is a ‘gap’ between their formal and personal figural concepts of parallelograms.” (Fujita & Jones, 2007, p. 11).

The fact that pre-service teachers have difficulty defining a trapezoid and quadrilateral classifications has also been proven in other research. Erdogan and Dur (2014) investigated the ability of 57 pre-service mathematics teachers with regards to definitions of trapezoids (besides other types of quadrilaterals), their recognition of trapezoids among 15 figures, and their ability to hierarchically arrange the quadrilaterals in a quadrilateral classification. 46% of them provided a correct definition of a trapezoid, but only four students drew it correctly in the quadrilateral classification. Another study by Butuner and Filiz (2017) examined 33 mathematics teachers in practice, only one of them correctly included a trapezoid in the quadrilateral classification.

A tool that can be used to analyse the conceptual and structural knowledge of school and university students in mathematics is a conceptual map (Chinnappan & Lawson, 2005; Rösken & Rolka, 2007). This is a planar graphical representation of the relationships between concepts; the individual terms are suitably spaced and the relationships between them are captured by lines. These maps may have hierarchical or non-hierarchical forms (Davies, 2011). In our research, we used the hierarchical ones. Conceptual maps began to be used in the USA in the 1970s when researching children's knowledge structures, especially in the field of key science concepts (Mareš, 2011). They have gradually spread not only to other disciplines, but are also commonly used in the teaching and learning process today. The use of conceptual maps promotes so-called meaningful learning by school and university students and can contribute to improving the quality of teaching (Hay et al., 2008; Mareš, 2011).

Conceptual maps make it possible to visualise students' ideas about concepts and relationships between them, thereby serving as a diagnostic tool for teachers or researchers to capture the cognitive structure of concepts (Rösken & Rolka, 2007). The fact that the students/respondents make their ideas about concepts and the relationships between them visible, reveals their way of thinking and their previous learning. Using conceptual maps also makes it possible to identify a student's misconceptions. There are plenty of methodologies

² The term “trapezium” here means a trapezoid according to the inclusive definition and UK terminology (at least one pair of parallel sides).

for evaluating conceptual maps that focus on different aspects. For example, we can evaluate the number of concepts and links between them, i.e. so-called structural evaluation (Vaňková, 2014), or the correctness and completeness of the drawn relationships, i.e. so-called relationship evaluation (Chinnappan & Lawson, 2005). Alternatively, we can focus on a qualitative diagnosis of learners' knowledge (Gouli et al., 2004).

To identify trapezoid and quadrilateral classification knowledge, we used the conceptual map method with pre-service mathematics teachers. We investigated whether these students understood the hierarchical classification of quadrilaterals. The hierarchical classification of quadrilaterals is usually preferred in mathematics and school curricula, in particular, at the lower secondary school level (Fujitsa & Jones, 2007). However, this classification is not explicitly mentioned in the Czech curriculum. The situation in textbooks is better: in two upper secondary school textbooks (Vondra, J., 2013a; Molnár, J., 2011) a hierarchical classification of quadrilaterals is presented.

3 Materials and methods

As part of our research, we analysed several Czech textbooks dealing with trapezoids. We subsequently drafted a test task aimed at recognising trapezoids located in different positions, including non-models.

In view of the unfavourable results of 9th grade students in the test, we were interested in how pre-service mathematics teachers define a trapezoid and whether they correctly integrate it into the classification of quadrilaterals. As a result, in March 2019, we asked 28 students in their first year of university studies to take the test.

3.1 Textbook analysis

As we have already mentioned, one part of our research was the analysis of textbooks. First, we analysed those textbooks that are used at the lower secondary schools where we performed our testing, and (in our experience) are among the most commonly used (Binterová et al., 2008; Herman et al., 2006; Molnár et al., 1999; Odvárko & Kadleček, 2012; Šarounová et al., 1998). Further, we analysed all the upper secondary school textbooks introducing the concept of trapezoid (Molnár, 2011; Pomykalová, 1993; Pomykalová et al., 2012; Vondra, 2013a, 2013b).

We focused on the following questions about these textbooks:

- trapezoid definition: whether the phrase “with only one pair of parallel sides” or the more explicit phrase “one pair of parallel sides and one pair of non-parallel sides” appears in the trapezoid definition;
- trapezoid representation: whether there are only prototypes in the text, or only trapezoids with horizontal bases; in addition, the appearance of an obtuse trapezoid;
- the appearance of non-models: whether there appear non-models, counterexamples, and trapezoid recognition tasks in the textbook;
- quadrilaterals classification: whether it is described only verbally or a classification scheme is attached; moreover, the types of trapezoids listed, or their classification;
- emphasis on context: whether the concept of trapezoid is included in the context of other quadrilaterals and whether their common characteristics and differences are highlighted.

3.2 Test preparation, assignment and evaluation – lower secondary schools

Before testing the lower secondary school students we assessed the comprehensibility of the test and the time limit for taking the test through a pre-test conducted on a small group of 30 students of the same age (approximately 15 years of age). The sample set of students were acquired based on availability. The contents of the test were based on the Czech national curriculum (MŠMT, 2017).

Further, for the purposes of our research, we personally conducted an anonymous written test among 437 students in the last grade at lower secondary school level (typically 15 years of age). Of these, 180 attended the last grade of lower secondary school, and 257 attended the corresponding grade of grammar school (lower secondary school with entrance examination). Students undertook the test (printed on paper) without any additional instructions and without any supporting materials. We tested the students from April to June 2018.

In the test, there was one task to choose trapezoids from six given (pre-drawn) quadrilaterals (see Figure 3). The task was: “Select trapezoids from the drawn quadrilaterals. Circle them.”

The tests were then assessed qualitatively. Each student's answer to each task was given a code. The tests from every class were independently coded by different pairs of researchers and the data were entered into

Figure 3: Test question regarding trapezoids

tables. Any differences in the assessment of specific tasks were discussed among the whole research team until a collective consensus was reached.

Correct answers according to the exclusive definition (i.e. *bcde*) were given the code *OK* (students correctly chose all the trapezoids). Any other set of chosen quadrilaterals was coded with a sequence of letters to mark the chosen quadrilaterals. Codes with a low occurrence were consolidated under the code “other answer” (*OA*). From this data, the absolute and relative frequencies of the codes were determined. Out of the test, a contingency table of the absolute frequencies was made for the choices *b* and *c*, *b* and *d*, *d* and *e*, *a* and *f*.

During the testing, we differentiated the students and students according to their gender. However, after a preliminary statistical analysis, it appeared there were no significant differences between both groups. As a result, this is not taken into any further consideration in this article.

3.3 Test preparation, assignment and evaluation – pre-service mathematics teachers

In March 2019, a short test was conducted among 28 pre-service mathematics teachers – first-year students at the Faculty of Mathematics and Physics of Charles University – to determine what they knew about trapezoids. The test was conducted before they discussed the classification of quadrilaterals in their university course. The task was to define a trapezoid and to hierarchically classify (represented by a conceptual map) the following quadrilaterals: square, rhombus, rectangle, rhomboid,

parallelogram, trapezoid, trapezium, deltoid and rectangular trapezoid. Furthermore, they were required to sketch each of the prescribed quadrilaterals.

The test results were subsequently qualitatively evaluated in terms of the exclusive definition of a trapezoid (Figure 1) and in terms of the incorporation of a trapezoid and rectangular trapezoid in the classification of quadrilaterals. In addition, we took note of whether they sketched a trapezoid and a rectangular trapezoid correctly and whether they chose a prototype shape and position (horizontal parallel sides). Finally, we took into consideration the relationship between the right/wrong definition and the right/wrong classification. For an example of a student’s classification see Figure 4, section 4.3.

4 Results

In this section we will present the results of the three parts of our research.

4.1 Textbook analysis results

Prior to the actual testing of school and university students, we analysed five mathematics textbooks for lower secondary school level and four textbooks for upper secondary school level used in the Czech Republic and that dealt with trapezoids. We summarized the results of our analysis in Tables 1 and 2.

From the perspective of the understanding of the concept of trapezoid, it is important that all the textbooks

Table 1: Results of the lower secondary schools textbooks analysis. Symbols: || and X (one pair of parallel sides and one pair of non-parallel sides), only || (only one pair of parallel sides).

	(Binterová et al., 2008)	(Herman et al., 2006)	(Molnár et al., 1999)	(Odvárko & Kadleček, 2012)	(Šarounová et al., 1998)
trapezoid definition	and X	only	only	and X	only
not only prototypes	yes	yes	yes	yes	yes
not only horizontal bases	yes	no	yes	yes	yes
obtuse trapezium mentioned	no	no	yes	yes	yes
occurrence of non-models	no	no	yes	yes	no
classification of quadrilaterals	verbally described, but the layout is misleading: kite excluded from trapeziums as separate group	verbally described, incomplete (trapezium is missing)	scheme – dichotomous	missing	scheme
types of trapezoids mentioned	explicit classification: general, rectangular, isosceles	rectangular, isosceles	general, rectangular, isosceles	rectangular, isosceles	isosceles

for lower secondary school level contain not only prototypes of trapezoid, but, except for one, models that does not have horizontal bases. However, the non-models occur less frequently. The textbooks differ more in the quadrilateral classifications listed. This is sometimes described only verbally, in other textbooks a scheme is attached. In (Molnár et al., 1999) we find the dichotomous classification of quadrilaterals which are divided into two groups: with *at least one pair* of parallel sides and without parallel sides. However, this classification is not inclusive (in the first group there are two disjoint subgroups: parallelograms and trapezoids).

The textbook analysis for upper secondary schools showed that the concept of trapezoid has been adequately developed by textbooks for lower secondary schools in which it is introduced for the first time.

The trapezoid is no longer given a separate chapter, it is mentioned together with other quadrilaterals. It is therefore rather a brief repetition, in which only a few images appear, see e.g. (Molnár, 2011), where only 3 trapezoids are shown.

4.2 Test results – lower secondary schools

The results of the test conducted among the students of lower secondary schools and grammar schools (i.e.

lower secondary schools with entrance examination) were evaluated in several ways. There were significant differences between these groups.

Firstly, for each quadrilateral in the test task, we counted the number of students who considered it to be a trapezoid. The absolute and relative frequencies are presented in Tables 3 and 4; for the quadrilaterals *a–f* see Figure 3.

Next, we investigated the dependence of selected responses (*b–c*, *d–e*, *b–d*, *a–f*). For the relevant data see Table 5.

We focused on whether there is a dependence between responses *b* and *c*; figure *c* shows a prototypical trapezoid and *b* is the closest to the prototype (bases are in horizontal position but the upper one is longer). We compiled contingency table 2×2 for these two responses only. We subsequently calculated the test criterion value *K* ($K = 46.948$) for χ^2 independent characters with one degree of freedom, and then compared the value to the critical value of 6.635 at the significance level of 0.01. The test criterion significantly exceeds the critical value. As a result, the null hypothesis that these two characters do not depend on each other can be rejected. We can therefore conclude that there is a significant relationship between the choice of variants *b* and *c*.

Similarly, the relationship between the choice of both rotated trapezoids (non-prototypical) *d* and *e* ($K = 80.991$)

Table 2: Results of the upper secondary schools textbooks analysis. Symbols: || and X (one pair of parallel sides and one pair of non-parallel sides), only || (only one pair of parallel sides).

	(Molnár, 2011)	(Pomykalová, 1993)	(Pomykalová et al., 2012)	(Vondra, 2013a, 2013b)
trapezoid definition	missing	and X	and X	both types of formulation: and X, only
not only prototypes	yes	no	no	yes
not only horizontal bases	yes (3 trapezoids only)	no	no	yes
obtuse trapezium mentioned	no	no	no	yes
occurrence of non-models	no	no	no	no
classification of quadrilaterals	scheme	verbally described	verbally described	scheme (restricted to convex quadrilaterals)
types of trapezoids mentioned	explicit classification: general, rectangular, isosceles	rectangular, isosceles	rectangular, isosceles	rectangular, isosceles

Table 3: Absolute frequencies – number of students who marked the quadrilaterals a, b, c, d, e, f . The quadrilateral a and the rhomboid f are not trapezoids. LSS – lower secondary school, GrS – grammar school (LSS with entrance examination).

	a	b	c	d	e	f
LSS	44	142	160	137	94	35
GrS	35	217	245	237	193	29
Total	79	359	405	374	287	64

Table 4: Relative frequencies – number of students who marked the quadrilaterals a, b, c, d, e, f ; 180 students from LSS and 257 students from GrS; it was possible to choose more than one answer – the sums of the percentages in lines are therefore greater than 100% (combination bcd represents the correct solution).

	a	b	c	d	e	f
LSS	24.44%	78.89%	88.89%	76.11%	52.22%	19.44%
GrS	13.62%	84.44%	95.33%	92.22%	75.10%	11.28%
Total	18.08%	82.15%	92.68%	85.58%	65.68%	14.65%

was shown. Compared to variants b and c , both of these trapezoids were simultaneously chosen by fewer students, but the number of students who did not choose either of them increased.

On the other hand, we did not find any relationship at the significance level 0.01 between the choice of variants b and d . The test criterion value $K = 5.772$ is less than the critical value of 6.635. As a result, the null hypothesis that the choice of variants b and d is not related cannot be rejected at the given significance level.

For the quadrilaterals a and f , the test criterion value K is 7.083, which is higher than the critical value. In this case, we can talk about the dependence of these two responses.

Finally, we were interested in the frequency of the choices of the combinations of quadrilaterals that the students considered trapezoids. The results are summarised in Tables 6 and 7.

After separation of the results of lower secondary school and grammar school students, it became clear that

Table 5: Contingency tables – dependencies between particular choices.

$c \backslash b$	1	0	n_i
1	347	12	359
0	58	20	78
n_j	405	32	437

$d \backslash e$	1	0	n_i
1	277	97	374
0	10	53	63
n_j	287	150	437

$b \backslash d$	1	0	n_i
1	314	45	359
0	60	18	78
n_j	374	63	437

$a \backslash f$	1	0	n_i
1	4	75	79
0	60	298	358
n_j	64	373	437

Table 6: Most frequent answers – absolute frequencies (*OK* means *bcde*).

	<i>OK</i>	<i>abcde</i>	<i>acde</i>	<i>bc</i>	<i>bcd</i>	<i>bcdef</i>	<i>cde</i>	<i>OA</i>
LSS	51	12	8	10	29	8	5	57
GrS	141	10	12	8	28	16	10	32
Total	192	22	20	18	57	24	15	89

Table 7: Most frequent answers – relative frequencies (*OK* means *bcde*).

	<i>OK</i>	<i>abcde</i>	<i>acde</i>	<i>bc</i>	<i>bcd</i>	<i>bcdef</i>	<i>cde</i>	<i>OA</i>
LSS	28.33%	6.67%	4.44%	5.56%	16.11%	4.44%	2.78%	31.67%
GrS	54.86%	3.89%	4.67%	3.11%	10.89%	6.23%	3.89%	12.45%
Total	43.94%	5.03%	4.58%	4.12%	13.04%	5.49%	3.43%	20.37%

frequency with which the correct answer was forthcoming was greater among grammar school students.

4.3 Test results – pre-service mathematics teachers

The short test conducted among 28 pre-service mathematics teachers consisted of two tasks: 1) to create a conceptual map with a classification of the prescribed quadrilaterals and to make a sketch of each of them, see Figure 4; 2) to write a definition of a trapezoid. We evaluated the test results according to the exclusive definition (Figure 1) used in the Czech Republic and from several points of view.

With regards to the first task, we evaluated the:

- A) sketch of the trapezoid (whether the students sketched a prototype or trapezoid in a general position);
- B) inclusion of trapezoids in the classification of quadrilaterals.

With regards to the second task, we evaluated the:

- C) accuracy of the definition of a trapezoid.

We subsequently evaluated the:

- D) relationship between the correctness of the inclusion of trapezoids in the classification of quadrilaterals and the accuracy of the definition.

A) Sketch of the trapezoid

Among the trapezoidal sketches, the prototype dominated (horizontal bases, lower base longer, mostly isosceles trapezoid). This type was sketched by 25 of the 28 students. This is understandable for sketches within the classification. A single student sketched an obtuse trapezoid and one student sketched 4 different trapezoids (both students sketched all the figures with horizontal bases). One student did not sketch anything.

With regards to the rectangular trapezoid, the prototype also dominated (horizontal bases, lower base longer). This type was sketched by 27 of the 28 students, of which 25 students sketched the left shoulder perpendicular to the bases, 1 student the right shoulder,

Figure 4: Example of a student's classification of quadrilaterals

and 1 student both variants. A single student drew a rectangular trapezoid with vertical bases.

B) Inclusion of trapezoids in the classification of quadrilaterals

Of the 28 students, only 13 correctly divided the quadrilaterals into parallelograms, trapezoids and trapeziums. In total, 13 students divided the quadrilaterals into two basic groups (dichotomous classification): parallelograms and trapeziums, 12 of them classified a trapezoid as a special form of trapezium³ (see Figure 4), and one student included a trapezoid as a special form of parallelogram. These dichotomous classifications do not correspond to either exclusive or inclusive definition. For the two remaining respondents, the classification was unclear.

The rectangular trapezoid, as the only prescribed figure subordinated to a trapezoid, was correctly classified by 24 students. For three of the remaining responses the classification was unclear, and in one case, the rectangular trapezoid was classified as a special form of trapezoid together with a deltoid.

C) Accuracy of the definition of a trapezoid

Of the 28 students, 18 formulated a correct definition of a trapezoid. Of the remaining 10 responses, 4 students gave inaccurate definitions (trapezoid as a quadrilateral with one pair of parallel sides, without specification "only one pair"), 5 students incorrect definitions, and 1 student did not write anything at all.

From the 18 correct answers:

- 12 students defined a trapezoid as having one pair of parallel and one pair of non-parallel sides;
- 5 students defined a trapezoid as having just one pair of parallel sides;
- 1 student defined a trapezoid as having a pair of parallel sides that have different lengths (thereby eliminating the parallelism of the other pair of sides).

The evaluations of the students' definitions of a trapezoid are summarised in Table 8.

D) Relationship between the correctness of the inclusion of trapezoids in the classification of quadrilaterals and the accuracy of the definition

The correct classification here is the one in which three disjoint groups (parallelogram – trapezoid – trapezium) are subordinated to the quadrilateral. Since the test was given to a small number of students, statistical methods (the χ^2 test) cannot be used for the evaluation. However, it

³ These 13 students understood the trapezium as a quadrilateral with at least one pair of non-parallel sides.

Table 8: Evaluation of the accuracy of the definition of a trapezoid. Symbols: || (a pair of parallel sides), X (a pair of non-parallel opposite sides).

	correct definitions			incorrect definitions			total
absolute values	18			10			28
relative values	64%			36%			100%
	and X	just 2	Other	2	erroneous	missing	total
absolute values	12	5	1	4	5	1	28
relative values	43%	18%	4%	14%	18%	4%	≈ 100%

Table 9: Evaluation of the accuracy of the definition of a trapezoid (absolute values).

		Definition		
		OK	Incorrect	Total
Classification	OK	9	4	13
	Incorrect	9	6	15
	Total	18	10	28

is clear that those who correctly included trapezoids in the classification on the whole also gave a correct definition (see Table 9). This cannot be said in the reverse, i.e. the correct formulation of the definition of a trapezoid did not mean that the student correctly placed trapezoids in the classification of quadrilaterals. On the other hand, even if a student misplaced a trapezoid in the conceptual map, it did not mean that the student did not have a good idea of how a trapezoid looks.

5 Discussion

The results of the test clearly show that 9th grade students (15 years of age) prefer typical shapes of trapezoids (*c*, *d*, *b*; see Figure 3) over less common models. It was also confirmed that they preferred the prototypical position, where the parallel sides of the trapezoid are horizontal and the lower base is longer than the upper one. Similar results were obtained by Okazaki and Fujita (2007) and by Türnüklü (2014). Students chose the rotated trapezoid more often than the trapezoid with horizontal parallel sides, whose lower side is shorter than the upper one and therefore does not closely resemble the prototypical form.

The χ^2 test showed that the *b*–*c* and *d*–*e* choices are dependent. If the students recognised the trapezoid with horizontal bases, it was not so important, which of the bases was longer (whether the lower or the upper one). Similarly, the dependence of *d*–*e* suggests that rotated

trapezoids are difficult for students or that they dealt with them and correctly identified both as trapezoids.

On the other hand, *b*–*d* are not dependent, so the rotation of the bases is important. However, it is not possible to say whether students always find it easier to recognise a trapezoid when its parallel sides are horizontal. Trapezoid *b* is isosceles, it has horizontal parallel bases, but its lower base is shorter, making it unusual for its shape. On the other hand, trapezoid *d* (rotated and not isosceles) was chosen by the students more often. The problems associated with both types may be related to the used textbooks, as some of them do not present a rotated trapezoid.

Although the *a*–*f* choice shows dependency, it is possible that students designated both figures as trapezoids for various reasons. Some of them may have had such serious problems with the concept of trapezoids that their choices may not have been based on rational consideration. Figure *a* remotely resembles a “rotated” trapezoid, and so some students may have confused it with a trapezoid. With regards to figure *f*, the vague notion that “something is parallel” may persist, which could have resulted in the figure being marked as a trapezoid.

The difference between lower secondary school students and grammar school students was significant, especially with regards to the correct responses (choices *b*, *c*, *d* and *e*), whereas the difference was not so significant in the case of incorrect responses (choices *a* and *f*). However, grammar school students marked non-models (*a* and *f*) as trapezoids less frequently than lower secondary school students. The better results of the grammar school students are possibly influenced by the fact that the students passed an entrance exam.

The students’ results of our test could be influenced by the ways of teaching mathematics. Some research studies point out the influence of the teacher. Martínková, Goldhaber & Erosheva (2018) stated that the qualification and quality of teachers can influence the achievement of their students. Žilková et al. (2018) identified factors

that may create barriers in the younger students' geometric thinking, e.g. teacher's communication, incorrect terminology, choice of models and non-models; concerning the textbooks, the factors can be the models used or inaccuracies in definitions. Moreover, the textbook can influence the teacher and his/her teaching.

Where it concerns the definition of a trapezoid, the success of Czech pre-service mathematics teachers in our research is significantly better than in the above findings (Fujita & Jones, 2007; Törnklö, 2014). The preferred option is a descriptive approach to the definition, whereby the students indicate the relationship between both two pairs of opposite sides. This definition is consistent with that listed in the textbooks by Odvárko and Kadleček (2012) and Binterová et al. (2008) and in all textbooks for upper secondary schools (Pomykalová, 1993; Pomykalová et al., 2012; Vondra, 2013a) with one exception (Molnár, 2011) where the definition is omitted. The second most common approach, which is consistent with the textbooks by Molnár et al. (1999), Šarounová et al. (1998) and Herman et al. (2006) – a trapezoid has just one pair of parallel sides – is problematic. A similar conclusion was reached by Törnklö (2014). In this definition, the word “just” or “only” must be included, otherwise it could refer to a parallelogram and the inclusive definition (Figure 2). Four of the tested students made this mistake, although based on their sketches, we can assume that they are aware that one pair of opposite sides is not parallel.

Based on the analysis of the conceptual maps, it can be said that university students are well aware of the properties of trapezoids, although they classify them differently in the classification of quadrilaterals. Two approaches prevail: the division of quadrilaterals into three disjoint groups and the dichotomous approach. The division into three groups corresponds to the school interpretation in the Czech Republic (Figure 1) and it is included in all the Czech textbooks for upper secondary schools.

We believe that students used dichotomous classification because, in our opinion, it is easier to think in counterparts (both pairs of sides are parallel, and the other figures) than to distinguish whether one, two or no pairs of sides are parallel. In all cases, however, it is still an exclusive definition, only the structure is formed differently. No Czech students classify a trapezoid as being superior to a parallelogram. The idea held by students that a trapezoid has just one pair of parallel sides is also illustrated by their sketches. Students clearly preferred dichotomous classifications, even though they had never been taught it. Therefore it seems that if a dichotomous

classification were taught instead, students might make fewer mistakes.

We have found the pre-service teachers drew trapezoids only with horizontal bases (with one exception for a rectangular trapezoid). They prefer a prototype, which may indicate an insufficiently generalised notion of a trapezoid. Concerning the textbooks, our analysis has shown that they mostly contain prototypes of trapezoid. Only a few of them mention non-prototypical models. This may negatively interfere with a student's conceptual understanding of a trapezoid (Hejný, 2012), as well as result in their concept image not corresponding to their concept definition (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981); eventually resulting in their personal concept definition not corresponding to a mathematical definition (Rösken & Rolka, 2007). Textbook authors are generally aware that the classification of quadrilaterals is a challenging topic for lower secondary school. In some textbooks there is a scheme containing a classification of quadrilaterals, but without further explanation. In other textbooks, the authors restrict the classification only to verbal description, which can be less understandable for students.

Students' success in defining a trapezoid was better than in the classification task. This may be because the students' personal figural concepts influenced the quadrilateral classification (Fujita & Jones, 2006; Fujita & Jones, 2007). Students therefore have formal knowledge of the concept, but do not have sufficient structural knowledge thereof (Okazaki & Fujita, 2007; Fujita & Jones, 2006; Fujita & Jones, 2007). On the contrary, Fujita and Jones (2006) have found that pre-service primary school teachers have difficulty with formulating definition of trapezoid. However, our sample consisted of pre-service mathematics teachers of lower and upper secondary school level. Our results concerning correctness of trapezoid definition is better (64% of the 28 students gave the correct definition) than the result presented by Erdogan and Dur (2014) (where 46% of the 57 pre-service mathematics teachers tested formulated a correct definition). Why? The reason may be that university students of mathematics are used to formulating definitions, whereby the correct wording may be based on formal knowledge without a well-developed conceptual map. In accordance with Tall and Vinner (1981), we observed that the concept images and concept definitions of these students did not correspond.

6 Conclusion

In our test given to secondary school students of approximately 15 years of age, they preferred a prototype of a trapezoid. Nevertheless, they quite often recognised a trapezoid which did not have horizontal parallel sides. On the other hand, almost a fifth of the students marked the trapezium as a trapezoid and almost one-sixth of them considered the parallelogram to be a trapezoid. Furthermore, it turned out that grammar school students had almost double the number of correct answers.

The problems with the concept of trapezoids detected among the students, led us to supplement our research with a short test for pre-service teachers. It turned out that the university students preferred the prototype in their sketches. The wording of the definition of a trapezoid as formulated by some pre-service teachers was inaccurate, sometimes even nonsensical. The risk is that this lack of clarity can be transferred to students. A useful tool for preparing pre-service mathematics teachers is the use of conceptual maps during the pedagogical maths courses as a diagnostic tool to determine a student's concept image and their personal concept definition.

It follows that the declarative and structural knowledge of secondary school students should be linked to textbooks; it is important to take into account the structural grasp of the concept and to support this through the examination of the properties of individual figures. It is not only necessary to highlight models of the concept, but also non-models. Trapezoids should therefore appear in textbooks not only in prototypical form, but also in different positions (Hromadová et al., 2017), so that students can acquire a better understanding of the concept. The definition of a trapezoid, which we consider to be more comprehensible to students, is the one that is formulated with two parallel and two non-parallel opposite sides.

The analysis of the Czech textbooks showed that the definition of a trapezoid is sometimes given in a form that is not sufficiently comprehensible to secondary school students (*only one pair of parallel sides*). Most Czech textbooks for lower secondary schools contain models in various positions, but in most textbooks the non-models are missing. The classification of quadrilaterals at lower secondary school level is sometimes not sufficiently illustrative. In some textbooks it is only mentioned in passing, whereas in others it is inconsistent. As a result, the structural knowledge of concepts is therefore not sufficiently developed. Thus, we think it is important to add more non-standard tasks and non-models into textbooks and improve the pre-service teachers' preparation

concerning the classification of quadrilaterals. Moreover, in upper secondary school, quadrilaterals are no longer repeated thoroughly, only basic facts are revised, so the students do not develop their conceptual understanding of the trapezoid. Testing of students at the end of lower secondary school level has shown that the students did not recognize all of the trapezoids presented. Upper secondary school textbooks authors therefore have unrealistic expectations and, as the analysis of the upper secondary school textbooks revealed in connection with the results of our testing of students at the end of the lower secondary school level, the textbooks devote very little space to classifying quadrilaterals, thus more than half of the tested students did not correctly identify all offered trapezoids. Therefore, more attention should be paid to this topic in upper secondary school textbooks – especially to make a clear classification of quadrilaterals (this is higher level of conceptual understanding, so it is too early to do so at lower secondary school), and to present enough non-models and non-standard tasks leading to deeper conceptual understanding. The results of our survey can be used by the authors/publishers of the math textbooks and modify their tasks on the basis of our survey.

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