Research Article

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A computational algorithm and the method of determining the temperature field along the length of the rod of variable cross section

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Abstract: Bearing elements of a number of strategic equipment are limited length rods with a variable cross-section. Most of them are exposed to certain types of heat sources. To ensure reliable operation of these equipment it is necessary to know the temperature field along the length of the rod with a variable cross section. This paper proposes a computational algorithm and method to determine the temperature field along the length of the rod with limited length and variable cross-section. They are based on fundamental laws of energy conservation. Also obtained is an approximate analytical solution of the problem.

Keywords: variable cross-section, the radius of the cross section, a heat source, the length of the rod, net rod, thermal conductivity, heat transfer, convection

1 Introduction

A complex thermo-stress-strained state appears in the bearing components of power plants, internal combustion engines and hydrogen engines. Therefore, the definition of the temperature distribution law along the length of the rod elements is relevant. In this regard, many eminent scientists are engaged in this field [1–3]. In [4, 5] based on the finite element method the law of temperature distribution along the length of the rod with limited length and constant cross section was defined. In work [5] the solution of the steady problem of determination of the temperature field along the length of the horizontal rod with insulated side surface, limited length and a constant cross-section was discusses. Thus the cross-sectional area of the

left end is under heat flux with a constant intensity and the right end is open to a convective heat transfer with the environment. The heat transfer coefficient and ambient temperature are considered constant. In [6] the effect of temperature on the deformation of the investigated element was considered. There also the law of temperature distribution along the length of the rod when the lateral surface is insulated and the left end is under the the heat flux, the right end is open to heat exchange with the environment was analytically derived. In addition, we consider the problem of determining the temperature field along the length of the horizontal rod with a constant cross section. While the left end obtains a constant temperature, and the remaining surfaces of the rod is undergoes heat transfer to the environment. The results obtained in this work show the same result as in [7–9]. In [10] based on the energy conservation law by variational method the heat transfer between the deformable shell and the surrounding fluid was determined. In [11] based on the finite element method the process of heat conduction in the rod elements of nuclear power plants was investigated. They also gave a description of a software package which was developed based on modern tools of programming to solve the considered problem. The developed program is, indeed, universal and user-friendly. In [12] the unsteady field of the temperature distribution in cylindrical rods which were under laser heat sources was investigated. The obtained results can be used in the study of nonstationary thermal processes in the rod with the laser heat source. In [13] the computational methods, algorithms and software package for the study of steady-state thermal stress - strain state of a rod with limited length and constant cross section under the influence of local heat flows, temperatures, heat exchange with consideration of local insulation were considered. For each of the considered problems corresponding regularities have been identified. For some problems the steady-state temperature fields, the components of strain and stress and the displacement field were determined. Also the expressions to calculate the elongation and axial compressive force were obtained. In addition the con-

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vergence of the method and the accuracy of the numerical results were studied. In contrast to the above mentioned works, in this paper we considered the development of methods, computational algorithms and programs based on the energy conservation law to study the steady thermal stress and strain state of a horizontal bar with a variable cross section in the form of a circle. The radius of the cross section decreases linearly with the length of the rod starting from the left end. The lateral surface of the rod is thermoinsulated. The cross-sectional area of the left end of the rod is under heat flux with a constant intensity and at the right end is open to convective heat transfer with the environment, where the heat transfer coefficient and ambient temperature are considered constant. For this problem first the temperature distribution law along the length of the rod should be defined. Further, if one end is firmly fixed and the other is free the elongation should be calculated depending on existing heat sources, physical and geometric characteristics of the rod taking into account the presence of insulation. In the case of pinching both ends of the studied rod the value of the axial compressive force should be calculated, taking into consideration the real factors. This also determines the distribution of all components of strains, stresses and the displacement field. The study revealed some patterns of the process. It should be noted that the developed Python programs proved to be effective and user-friendly.

2 Formulation of the problem

Let's consider a horizontal bar with variable cross-section. Axis Ox directs from the left to the right on the axis of the rod. Assume that the cross section of the rod is a circle. The radius of the rod's cross section varies linearly along its length, thus $r(x) = ax + b$, $0 \leq x \leq l$, where l [cm] – the length of the rod. $a, b = \text{const}$. The radius of the cross-sectional area of the left end denoted by b [cm], thus $r(x = 0) = a \cdot 0 + b = b$ [cm]. Then the radius of the cross section of the right end will be equal to $r(x = l) = a \cdot l + b$ [cm]. The cross-sectional area of the rod along its length changes according to the quadratic law, thus $F(x) = \pi r^2 = \pi(ax + b)^2$, $0 \leq x \leq l$. Further, assume that the lateral surface of the rod is insulated. The cross-sectional area of the left end is under heat flow q [$\frac{Bm}{cm^2}$] with a constant intensity, while the cross-sectional area of the right end is under convective heat transfer with the environment. The temperature of the environment is constant, thus T_{oc} [$^{\circ}C$] = const. The heat transfer coefficient between the rod's material and the environment is h [$\frac{Bm}{cm^2 \cdot ^{\circ}C}$]. The thermal properties of

the material of the rod is characterized by the coefficient of thermal conductivity k_{xx} [$\frac{Bm}{cm \cdot ^{\circ}C}$]. Under such impacts it is required to determine the temperature distribution law along the length of the rod with variable cross section. Calculation scheme of the considered problem is shown in Figure 1.

Figure 1: Calculation scheme of the task.

3 The solution to the problem using the energy conservation law

To solve this problem in accordance with the fundamental law of energy conservation it the functional of total thermal energy for the considered problem is needed to write

$$I = \int_{S(x=0)} q T ds + \int_v \frac{k_{xx}}{2} \left(\frac{\partial T}{\partial x} \right)^2 dv + \int_{S(x=l)} \frac{h}{2} (T - T_{oc})^2 ds \quad (1)$$

where $S(x = 0)$ - the cross-sectional area of the left end of the rod which is under heat flux q ; $S(x = l)$ - the cross-sectional area of the right end of rod which is open to convective heat transfer. $S(x = l)$ – the volume of the test rod. Now, considering that the studied process is established and the lateral surface of the rod is fully insulated, also $q, h, T_{oc} = \text{const}$, the field of temperature distribution along the length of the rod is approximated by the full second-order polynomial.

$$T(x) = ax^2 + bx + c = \phi_i(x) \cdot T_i + \phi_j(x) T_j + \phi_k(x) T_k \quad (2) \\ 0 \leq x \leq l$$

where $a, b, c = \text{const}$;

$$\left. \begin{aligned} \phi_i(x) &= \frac{2x^2 - 3lx + l^2}{l^2} \\ \phi_j(x) &= \frac{4lx - 4x^2}{l^2} \\ \phi_k(x) &= \frac{2x^2 - lx}{l^2} \end{aligned} \right\} \quad (3)$$

$$T_i = T(x = 0); \quad T_j = T(x = \frac{l}{2}); \quad T_k = T(x = l);$$

Using (2-3) it is possible to determine the temperature gradient.

$$\begin{aligned} \frac{\partial T}{\partial x} &= \frac{\partial \phi_i}{\partial x} T_i + \frac{\partial \phi_j}{\partial x} T_j + \frac{\partial \phi_k}{\partial x} T_k \\ &= \left(\frac{4x - 3L}{L^2} \right) T_i + \left(\frac{4L - 8x}{L^2} \right) T_j + \left(\frac{4x - L}{L^2} \right) T_k \end{aligned}$$

Besides, the function (2) must give a minimum to the functional (1) of the full thermal power of the rod. First calculate the integrals in (1) considering (2).

$$I_1 = \int_{S(x=0)} q T ds = F_0 q T_i \quad (4)$$

where $F_0 = F(x = 0) = \pi(a \cdot 0 + b)^2 = \pi b^2$ – the cross-sectional area of the left end of the rod which is under heat flux – q .

$$\begin{aligned} I_2 &= \int_V \frac{k_{xx}}{2} \left(\frac{\partial T}{\partial x} \right)^2 ds = \frac{k_{xx}}{2} \int_0^l F(x) \left(\frac{\partial T}{\partial x} \right)^2 dx \quad (5) \\ &= \frac{k_{xx}}{2} \int_0^l (ax + b) \left[\frac{4x - 3l}{l^2} T_i + \frac{4l - 8x}{l^2} T_j \right. \\ &\quad \left. + \frac{4x - l}{l^2} T_k \right]^2 dx = \frac{k_{xx}}{12l} \left[(3al + 14b) T_i^2 + 16(al + 2b) T_j^2 \right. \\ &\quad \left. + (11ab + 14b) T_k^2 - 8(al + 4b) T_i T_j \right. \\ &\quad \left. + 2(al + 2b) T_i T_k - 8(3al + 4b) T_j T_k \right] \end{aligned}$$

$$I_3 = \int_{S(x=l)} \frac{h}{2} (T - T_{oc})^2 = \frac{F_l \cdot h}{2} (T_k - T_{oc})^2 \quad (6)$$

where $F_l = a \cdot l + b$.

Substituting (4-6) into (1) we find the integral form of the functional of total thermal energy for the considered problem.

$$\begin{aligned} I &= bqT_i + \frac{k_{xx}}{12l} \left[(3al + 14b) T_i^2 + 16(al + 2b) T_j^2 \right. \\ &\quad \left. + (11al + 14b) T_k^2 - 8(al + 4b) T_i T_j \right. \\ &\quad \left. + 2(al + 2b) T_i T_k - 8(3al + 4b) T_j T_k \right] \\ &\quad + \frac{(al + b)h}{2} (T_k - T_{oc})^2; \end{aligned} \quad (7)$$

In this expression $I = I(T_i, T_j, T_k)$. Minimizing I by T_i , T_j and T_k we can get a system of equations with the natural

boundary conditions.

$$\left. \begin{aligned} \frac{\partial I}{\partial T_i} &= 0; \Rightarrow (3al + 14b) T_i - 4(al + 4b) T_j \\ &\quad + (al + 2b) T_k = -\frac{6bql}{k_{xx}} \\ \frac{\partial I}{\partial T_j} &= 0; \Rightarrow -(al + 4b) T_i + 4(al + 2b) T_j \\ &\quad - (3al + 4b) T_k = 0 \\ \frac{\partial I}{\partial T_k} &= 0; \Rightarrow (al + 2b) T_i - 4(3al + 4b) T_j \\ &\quad + [(11al + 14b) + \frac{6l(al + b)h}{k_{xx}}] T_k \\ &\quad = \frac{6l(al + b)h}{k_{xx}} T_{oc} \end{aligned} \right\} \quad (8)$$

After a little simplification, system (8) can be rewritten in the following form.

$$\left. \begin{aligned} a_1 T_i - 4a_2 T_j + a_3 T_k &= b_1 \\ -a_2 T_i + 4a_3 T_j - a_4 T_k &= 0 \\ a_3 T_i - 4a_4 T_j + a_5 T_k &= b_2 \end{aligned} \right\} \quad (9)$$

where $a_1 = (3al + 14b)$; $a_2 = (al + 4b)$; $a_3 = (al + 2b)$; $a_4 = 3al + 4b$;

$$\begin{aligned} a_5 &= (11al + 14b) + \frac{6l(al + b)h}{k_{xx}}; \\ b_1 &= \frac{6lbq}{k_{xx}}; \quad b_2 = \frac{6l(al + b)h \cdot T_{oc}}{k_{xx}}; \end{aligned} \quad (10)$$

Further, solving the system (9) define nodal temperature value T_i , T_j and T_k .

$$\begin{aligned} T_k &= \frac{C_1 b_4 - b_3 C_3}{C_1 C_4 - C_2 C_3}; \\ T_j &= -\frac{C_2}{C_1} T_k + \frac{b_3}{C_1}; \quad T_i = \frac{4a_2}{a_1} T_j - \frac{a_3}{a_1} T_k + \frac{b_1}{a_1} \end{aligned} \quad (11)$$

where

$$\begin{aligned} C_1 &= \frac{4(a_1 a_3 - a_2^2)}{a_1}; \quad C_2 = \frac{a_2 a_3 - a_1 a_4}{a_1}; \\ C_3 &= \frac{4(a_2 a_3 - a_1 a_4)}{a_1}; \quad C_4 = \frac{a_1 a_5 - a_3^2}{a_1}; \\ b_3 &= \frac{a_2 b_1}{a_1}; \quad b_4 = \frac{a_1 b_2 - b_1 a_3}{a_1}; \end{aligned} \quad (12)$$

Then the law of temperature distribution along the length of the rod can be defined as:

$$\begin{aligned} T(x, l, h, k_{xx}, T_{oc}, q, a, b) &= \frac{2x^2 - 3lx + l^2}{l^2} T_i \\ &\quad + \frac{4lx - 4x^2}{l^2} T_j + \frac{2x^2 - lx}{l^2} T_k \end{aligned} \quad (13)$$

$$0 \leq x \leq l$$

4 Analysis of the obtained results

Let's consider the solution of the problem with the following initial data:

$$\begin{aligned} l &= 90\text{cm}; a = -\frac{1}{15}; b = 12\text{cm}; q = -500 \frac{\text{Bm}}{\text{cm}^2}; \\ k_{xx} &= 100 \frac{\text{Bm}}{\text{cm} \cdot ^\circ\text{C}}; h = 10 \frac{\text{Bm}}{\text{cm}^2 \cdot ^\circ\text{C}}; T_{oc} = 20^\circ\text{C}; \\ T_k &= T(x=l) = 120^\circ\text{C}; T_j = T\left(x = \frac{l}{2}\right) = 483,46^\circ\text{C}; \\ T_i &= T(x=0) = 743,077^\circ\text{C}. \end{aligned}$$

In this case, the temperature distribution along the length of the rod according to (13) has the following form:

$$T(x, l, h, k_{xx}, T_{oc}, q, a, b) = -0,02564x^2 - 4,6154x + 743,077. \quad (14)$$

Then:

$$\begin{aligned} T(x=0) &= T_i = 743,077^\circ\text{C}; \\ T(x = \frac{l}{2}) &= T(x=45) = T_j = 483,46^\circ\text{C}; \\ T(x=l) &= T(x=90) = 120^\circ\text{C}. \end{aligned}$$

Then the graph of the temperature distribution field along the length of the rod under the assumed initial data will look like Figure 2.

Figure 2: The field of temperature distribution along the length of the rod.

The figure shows that the distribution of temperatures has a slightly parabolic character. This is due to the variability of the cross-section of the rod along its length that varies according to the square law, and the radius which varies according to the linear law.

If one of the ends is firmly fixed and the other is free, due to the heat source it becomes longer. The value of elongation can be determined on the General laws of thermophysics:

$$\begin{aligned} \Delta l_T &= \int_0^l \alpha T(x) dx = \int_0^l \alpha (-0.02564x^2 - 4.6154x \\ &\quad + 743.077) dx = 0.0000125(-0.02564x^3 - 4.6154x^2 \\ &\quad + 743.077x) \Big|_0^l = 0.524455 \end{aligned}$$

Here $\alpha = 0.0000125 \left(\frac{1}{^\circ\text{C}}\right)$ – coefficient of thermal expansion of the material of the rod.

If both ends are firmly fixed, then axial compressive force in the rod due to thermal expansion occurs. Its value can be determined by using the compatibility conditions of deformation [13]. In our case, according to [13] the value of the axial force is determined by the next formula:

$$R = -\frac{EF\Delta l_T}{l} = -\frac{E\Delta l_T}{l} \int_0^l F(x) dx = \frac{EF\Delta l_T}{l}.$$

Because $E = 2 \cdot 10^6 F_{cp} = 9$, then $R = -3062209.933$. In this case the field distribution of thermo-elastic stress in the cross-sections of the studied rod occurs, which is defined as:

$$\sigma(x) = \frac{-3062209.933}{\left(-\frac{1}{15}x + 12\right)^2}, \quad 0 \leq x \leq l = 90\text{cm},$$

The law of deformation component of thermo-elastic distribution is determined based on the laws of physics [13]:

$$\varepsilon(x) = \frac{\sigma(x)}{E}, \quad 0 \leq x \leq l = 90\text{cm},$$

Using the found temperature distribution law (14) we can calculate the distribution law of thermal component of deformation:

$$\begin{aligned} \varepsilon_T(x) &= \alpha T(x) = \alpha (0.02564x^2 + 4.6154x - 743.077), \\ 0 &\leq x \leq l = 90\text{cm}, \end{aligned}$$

Then according to the generalized Hooke's law we can determine the distribution law of thermal component of stress:

$$\begin{aligned} \sigma_T(x) &= E\varepsilon_T(x) \\ &= 2 \cdot 10^6 (0.02564x^2 + 4.6154x - 743.077), \\ 0 &\leq x \leq l = 90\text{cm}, \end{aligned}$$

Further, it's possible to determine the distribution law of elastic component of deformation:

$$\varepsilon_x(x) = \varepsilon(x) - \varepsilon_T(x), \quad 0 \leq x \leq l = 90\text{cm},$$

Further, based on generalized Hooke's law we can determine the distribution law of elastic component of stress:

$$0 \leq x \leq l = 90\text{cm},$$

Figure 3 shows the distribution of the three strain components along the length of the rod with variable cross-section, firmly fixed at both ends.

Figure 3: The distribution law of deformation: 1 - $\varepsilon(x)$; 2 - $\varepsilon_T(x)$ u 3 - $\varepsilon_x(x)$.

The graph shows that the temperature and thermo-elastic components of deformation are compressive in nature throughout the length of the rod. When the elastic component of deformation on the length of the rod $0 \leq x \leq l = 67\text{cm}$ has stretchable nature, on the length $67 \leq x \leq l = 90\text{cm}$ - compressive nature. This process caused by the presence of a large heat flow ($q = -500 \frac{\text{Bm}}{\text{cm}^2}$) on the left end of the rod, where the cross-sectional area at the left end is four times more than on the right end.

Figure 4 shows the distribution of three stress components along the length of the rod with variable cross-section area and firmly fixed at both ends.

From the figure it is evident that all stress components have the same character as the deformation.

Figure 5 shows the distribution of displacement of cross-sections of the rod.

The figure shows that all cross sections move from left to right. It is also due to the large heat flux at the left end of the rod. The largest displacement is on the cross section of the rod with coordinates $x=51\text{ cm}$, as the displacement on both ends of the rod is equal to 0.

Figure 4: The laws of stress distribution: 1 - $\sigma(x)$; 2 - $\sigma_T(x)$ u 3 - $\sigma_x(x)$.

Figure 5: The distribution law of displacement of cross-sections of the rod.

5 Conclusions

The developed computational algorithms and methods based on the fundamental energy conservation law allows us to solve a variety of applied engineering problems with high accuracy. Also, they provide an opportunity to solve many applied engineering problems of thermal conductivity taking into account the presence of various types of local heat sources.

It should be noted that the proposed algorithm and method can simultaneously determine the distribution laws of the temperature, three components of strain and stress, and also displacement. Along with that was defined the value of the elongation and the resultant axial force. The obtained results allow to carry out deep analysis of thermo-stressed rod with variable cross section and lim-

ited length. Using the developed approach it is possible to explore the emerging and complex thermo-stress-strain state of load-bearing elements of power plants, internal combustion engines, jet engines and the hydrogen, and oil heating stations used in transportation of highly waxy oil through the pipeline.

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