RESEARCH PAPER

EXISTENCE AND UNIQUENESS FOR A CLASS OF
STOCHASTIC TIME FRACTIONAL SPACE
PSEUDO-DIFFERENTIAL EQUATIONS

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Abstract

Some classes of stochastic fractional differential equations with respect to time and a pseudo-differential equation with respect to space are investigated. Using estimates for the Mittag-Leffler functions and a fixed point theorem, existence and uniqueness of mild solutions of the equations under consideration are established.

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1. Introduction

The theory of the fractional calculus has been developed for more than 300 years, we highlight the contributions of Leibniz, Liouville, Riemann and Weyl, just to name a few. There exists an extensive literature in this area, we here only mention Awawdeh, Rawashdeh and Jaradat [1], Gorenflo and Mainardi [3], Hilfer [5], Mainardi [13], Metzler and Klafter [15] and Podlubny [16] among others. Moreover, many mathematicians (see [11], [13] and [16]) have paid attention to the Riemann-Liouville fractional derivative and fractional differential equations composed by Riemann-Liouville fractional derivatives, i.e. equations of a form such as

$${}_0\mathcal{D}_t^\alpha u(x, t) + \lambda u(x, t) = g(x, t), \quad \text{for } 0 < \alpha < 1, \quad (1.1)$$

for example under the initial data

$${}_0\mathcal{D}_t^{\alpha-1}u(x, 0) = 0. \quad (1.2)$$

The main tool for solving such equations is to use the Laplace transform.

On the other hand, it is well-known that many phenomena in various branches of science and industry can be modelled by stochastic differential equations (SDEs). Recently, attempts were made to combine fractional derivatives and SDEs. For example, El-Borai [2] has investigated the existence and uniqueness of stochastic fractional integro-differential equations. In this paper, motivated by the previous references, we will study the existence and uniqueness for a class of stochastic fractional differential equations. Our results are different from [2], since we use the Riemann-Liouville fractional derivatives.

In Section 2 we deal with auxiliary results, and in Section 3 we prove the existence and uniqueness for a class of stochastic fractional differential equations.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, i.e. it is right continuous and $\mathcal{F}_0$ contains all P -null sets. $B(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. We consider a stochastic fractional differential equation of the form

$${}_0\mathcal{D}_t^\alpha u(x, t) + \psi(D_x)u(x, t) + p(x, D_x)u(x, t) - \int_0^t f(u(x, \theta))dB(\theta) = g(x, t), \quad (2.1)$$

with the initial data

$${}_0\mathcal{D}_t^{\alpha-1}u(x, t)|_{t=0} = 0, \quad (2.2)$$

for $x \in \mathbb{R}$, $0 < t \leq T$, $T < \infty$. Here, $0 < \alpha < 1$, $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous negative definite function of class C^2 , satisfying the growth condition (2.11), and

$$p(x, D_x)v(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} p(x, \xi) \hat{v}(\xi) d\xi \quad (2.3)$$

is a pseudo-differential operator with symbol $p : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. For the details on Itô's stochastic integral, we refer the reader to [12]. We will discuss later on certain properties which we have to impose on the symbol $p(x, \xi)$. In this paragraph, however, the only property needed is that $p(x, D_x)$ satisfies the estimate

$$\|p(x, D_x)u(\cdot, t)\|_{L^2} \leq \mu \|u(\cdot, t)\|_{L^2}, \quad u(\cdot, t) \in L^2(\mathbb{R}), \quad (2.4)$$

where μ is independent of t , and

$$\|u(\cdot, t)\|_{L^2}^2 = \int_{\mathbb{R}} |u(x, t)|^2 dx. \quad (2.5)$$

We refer to [7, Remark 3.2], where suitable conditions for (2.4) to hold were discussed.

Let $\omega \mapsto u(\omega, x, t)$ be a random variable, which we often write just as $u(x, t)$. The space $L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))$ defined by

$$L^2(\Omega \times \mathbb{R}; L^\infty([0, T])) := \left\{ u : \Omega \times \mathbb{R} \rightarrow L^\infty([0, T]) \mid \left(\mathbb{E} \int_{\mathbb{R}} \sup_{0 \leq t \leq T} |u(x, t)|^2 dx \right)^{1/2} < \infty \right\} \quad (2.6)$$

is a Banach space equipped with norm

$$\|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} = \left(\mathbb{E} \int_{\mathbb{R}} \sup_{0 \leq t \leq T} |u(x, t)|^2 dx \right)^{1/2}. \quad (2.7)$$

Comparing [16, Example 4.3, p.140], (2.1) can be rewritten as:

$$\begin{aligned} u(x, t) &= (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} g(x - y, \tau) (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} \left(\int_0^\tau f(u(x - y, \theta)) dB(\theta) \right) \\ &\quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} (p(x, D_x)u(x - y, \tau)) \\ &\quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy, \end{aligned} \quad (2.8)$$

and the equality holds P -almost everywhere for which we will write P -a.e. Here by $E_{\alpha, \beta}(z)$ we denote the (2-indices) Mittag-Leffler function defined by

$$E_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \quad (2.9)$$

This is an entire function and we refer to [11], [13] or [16] where basic properties of $E_{\alpha, \beta}(z)$ are discussed. We now estimate the Mittag-Leffler function $E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)$. For this we need the following theorem which can be found in [16, Section 1.2, p.16]. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous negative definite function. In fact we assume that ψ has the representation

$$\psi(\xi) = \int_{\mathbb{R} \setminus \{0\}} (1 - \cos y\xi) \nu(dy) \quad (2.10)$$

with a Lévy measure ν satisfying $\int_{\mathbb{R} \setminus \{0\}} 1 \wedge |y|^2 \nu(dy) < \infty$. We assume further that ψ is of class C^2 , note that $\psi(\xi) \geq 0$, then for $l \leq 2$,

$$\left| \frac{d^l \psi(\xi)}{d\xi^l} \right| \leq \begin{cases} \psi(\xi), & l = 0; \\ C_1 \psi^{1/2}(\xi), & l = 1; \\ C_2, & l = 2. \end{cases}$$

Moreover, we require a lower bound for ψ , i.e.

$$\psi(\xi) \geq C_\psi (1 + |\xi|^2)^{\gamma/2} \quad (2.11)$$

for some $\gamma \in (1, 2)$ and all $\xi \in \mathbb{R}$. We refer to [8]-[10] where examples of continuous negative definite functions are given. Using the table of Bernstein functions provided in [18], we can easily construct many examples of type $f(|\xi|^2)$ where f is suitable Bernstein function.

3. Existence and Uniqueness

We start with a definition. A random function $u \in L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))$ is called a mild solution of the problem (2.1) and (2.2), if

$$\begin{aligned} u(x, t) &= (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} g(x - y, \tau) (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} \left(\int_0^\tau f(u(x - y, \theta)) dB(\theta) \right) \\ &\quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} (p(x, D_x)u(x - y, \tau)) \\ &\quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy, \quad \text{P-a.e.} \end{aligned} \quad (3.1)$$

Let us introduce the operator

$$\begin{aligned} L_t u(x, t) &= (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} g(x - y, \tau) (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} \left(\int_0^\tau f(u(x - y, \theta)) dB(\theta) \right) \\ &\quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t - \tau)^{\alpha-1} (p(x, D_x)u(x - y, \tau)) \\ &\quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(y) d\tau dy, \quad \text{P-a.e.} \end{aligned} \quad (3.2)$$

It follows that $u \in L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))$ is a mild solution to (2.1) and (2.2) if and only if u is a fixed point of L_t , i.e.

$$L_t u = u. \quad (3.3)$$

THEOREM 3.1. *Let $u, v \in L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))$ and let f satisfy the Lipschitz conditions*

$$\|f(u) - f(v)\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq K_1 \|u - v\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}, \quad (3.4)$$

and the linear growth condition

$$\|f(u)\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq K_2(1 + \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}), \quad (3.5)$$

where K_1 and K_2 are two positive constants. For $R > 0$ consider

$$B(g, R) := \{v \in L^2(\Omega \times \mathbb{R}; L^\infty([0, T])) \mid \|v - g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq R\} \quad (3.6)$$

and assume

$$\|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq \frac{R}{4}. \quad (3.7)$$

Then there exists $T > 0$ such that L_t is a contraction on $L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))$ and L_t leaves $B(g, R)$ invariant, i.e. $u \in B(g, R)$ implies $L_t u \in B(g, R)$. Consequently, L_t has a unique fixed point which is a mild solution to the problem (2.1) and (2.2).

P r o o f. For $u, v \in L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))$ we have

$$\begin{aligned} & \|L_t u - L_t v\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\ &= \left\| (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (f(u(x-y, \theta)) - f(v(x-y, \theta))) dB(\theta) \right. \\ & \quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \\ & \quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} (p(x, D_x)(u(x-y, \tau) - v(x-y, \tau))) \\ & \quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \Big\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\ &\leq (2\pi)^{-1/2} \left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau (f(u(x-y, \theta)) - f(v(x-y, \theta))) dB(\theta) \right. \\ & \quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \Big\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\ & \quad + (2\pi)^{-1/2} \left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} (p(x, D_x)(u(x-y, \tau) - v(x-y, \tau))) \right. \\ & \quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \Big\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \end{aligned}$$

$$\begin{aligned}
& \leq (2\pi)^{-1/2} \left\| \int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right) \right. \\
& \quad \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& + (2\pi)^{-1/2} \left\| \int_0^t (t-\tau)^{\alpha-1} (p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))) \right. \\
[4pt] & \quad \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}, \tag{3.8}
\end{aligned}$$

where the $*$ means convolution. In (3.8), we estimate the first term

$$\begin{aligned}
& \left\| \int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right) \right. \\
& \quad \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau dy \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
[4pt] & = \left\{ \mathbb{E} \int_{\mathbb{R}} \sup_{0 \leq t \leq T} \left(\int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right) \right. \right. \\
& \quad \left. \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right)^2 dx \right\}^{1/2} \\
& \leq \left\{ \mathbb{E} \int_{\mathbb{R}} \left(\sup_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right) \right. \right. \\
& \quad \left. \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right)^2 dx \right\}^{1/2} \\
& = \left\{ \mathbb{E} \left[\left(\int_{\mathbb{R}} \left(\sup_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right) \right. \right. \right. \right. \\
& \quad \left. \left. \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right)^{1/2} \right)^2 \right] \right\}^{1/2} \\
& \leq \left\{ \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1} \left\| \int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right\|_{L^2} \right. \right. \\
& \quad \left. \left. \times \left\| (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right\|_{L^1} \right)^2 \right\}^{1/2}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1} h(t, \tau) \right. \right. \\
&\quad \left. \left. \times \left\| \int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right\|_{L^2} d\tau \right)^2 \right\}^{1/2}, \tag{3.9}
\end{aligned}$$

Here we have for $\alpha > 0$

$$h(t, \tau) := \|F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha))\|_{L^1}.$$

By Lemma 2.2 and Corollary 2.2 in [7], compare also with [6], we have already proven the integrability properties of $h(t, \tau)$ and $t \mapsto \tilde{h}(t, \tau) := t^{\alpha-1}h(t, \tau)$ on every finite interval $[0, T]$. For $d > 0$ we can find T_0 such that

$$d \sup_{0 \leq t \leq T_0} \int_0^t (t-\tau)^{\alpha-1} h(t, \tau) d\tau \leq \kappa < 1.$$

Now, by using Burkholder-Davis-Gundy inequality, compare [12], we arrive at

$$\begin{aligned}
&\left\| \int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right) \right. \\
&\quad \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau dy \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
&\leq \left\{ \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1} h(t, \tau) \right. \right. \\
&\quad \left. \left. \times \left\| \sup_{0 \leq \tau \leq T} \int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right\|_{L^2} d\tau \right)^2 \right\}^{1/2} \\
&\leq \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \\
&\quad \times \left\{ \mathbb{E} \left(\left\| \sup_{0 \leq \tau \leq T} \int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right\|_{L^2} \right)^2 \right\}^{1/2} \\
&= \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \\
&\quad \times \left\{ \mathbb{E} \int_{\mathbb{R}} \sup_{0 \leq \tau \leq T} \left(\int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta))) dB(\theta) \right)^2 dx \right\}^{1/2} \\
&\leq 4 \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \\
&\quad \times \left\{ \int_{\mathbb{R}} \mathbb{E} \int_0^\tau (f(u(\cdot, \theta)) - f(v(\cdot, \theta)))^2 d\theta dx \right\}^{1/2}
\end{aligned}$$

$$\begin{aligned}
 &\leq 4T^{1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \\
 &\times \left\{ \mathbb{E} \int_{\mathbb{R}} \sup_{0 \leq t \leq T} (f(u(\cdot, \cdot)) - f(v(\cdot, \cdot)))^2 dx \right\}^{1/2} \\
 &\leq 4T^{1/2} K_1 \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|u - v\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}.
 \end{aligned} \tag{3.10}$$

For the second term in (3.8), we have

$$\begin{aligned}
 &\left\| \int_0^t (t - \tau)^{\alpha-1} (p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))) \right. \\
 &\quad \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(x) d\tau \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
 &= \left\{ \mathbb{E} \int_{\mathbb{R}} \sup_{0 \leq t \leq T} \left(\int_0^t (t - \tau)^{\alpha-1} (p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))) \right. \right. \\
 &\quad \left. \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(x) d\tau \right)^2 dx \right\}^{1/2} \\
 &\leq \left\{ \mathbb{E} \int_{\mathbb{R}} \left(\sup_{0 \leq t \leq T} \int_0^t (t - \tau)^{\alpha-1} (p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))) \right. \right. \\
 &\quad \left. \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha)))(x) d\tau \right)^2 dx \right\}^{1/2} \\
 &\leq \left\{ \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_0^t (t - \tau)^{\alpha-1} h(t, \tau) \|p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))\|_{L^2} d\tau \right)^2 \right\}^{1/2},
 \end{aligned} \tag{3.11}$$

where

$$h(t, \tau) := \|F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t - \tau)^\alpha))\|_{L^1}.$$

Using (2.4), we find

$$\|p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))\|_{L^2} \leq \mu \|u(\cdot, \tau) - v(\cdot, \tau)\|_{L^2}$$

which implies that

$$\begin{aligned}
& \left\| \int_0^t (t-\tau)^{\alpha-1} (p(x, D_x)(u(\cdot, \tau) - v(\cdot, \tau))) \right. \\
& \quad \left. * (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \leq \left\{ \mathbb{E} \left(\sup_{0 \leq t \leq T} \int_0^t (t-\tau)^{\alpha-1} h(t, \tau) \right. \right. \\
& \quad \left. \left. \times \left\| \sup_{0 \leq t \leq T} p(x, D_x)(u(\cdot, t) - v(\cdot, t)) \right\|_{L^2} d\tau \right)^2 \right\}^{1/2} \\
& \leq \mu \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|u - v\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}.
\end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12), we have

$$\begin{aligned}
& \|L_t u - L_t v\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq (2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \\
& \quad \times (4K_1 T^{1/2} + \mu) \|u - v\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}.
\end{aligned} \tag{3.13}$$

In (3.13), let T be sufficiently small such that

$$(2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) (4K_1 T^{1/2} + \mu)$$

is less than or equal to 1. Under this condition, L_t is contractive.

Now, we note

$$\begin{aligned}
& \|L_t u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& = \left\| (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} g(x-y, \tau) \right. \\
& \quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \\
& \quad + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau f(u(x-y, \theta)) dB(\theta) \\
& \quad \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \\
& \quad \left. + (2\pi)^{-1/2} \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} p(x, D_x) u(x-y, \tau) \right. \\
& \quad \left. \times (F^{-1}(E_{\alpha, \alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}
\end{aligned}$$

$$\begin{aligned}
 &\leq (2\pi)^{-1/2} \left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} g(x-y, \tau) \right. \\
 &\quad \times (F^{-1}(E_{\alpha,\alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \left. \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &+ (2\pi)^{-1/2} \left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} \int_0^\tau f(u(x-y, \theta)) dB(\theta) \right. \\
 &\quad \times (F^{-1}(E_{\alpha,\alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \left. \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &+ (2\pi)^{-1/2} \left\| \int_{\mathbb{R}} \int_0^t (t-\tau)^{\alpha-1} p(x, D_x) u(x-y, \tau) \right. \\
 &\quad \times (F^{-1}(E_{\alpha,\alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(y) d\tau dy \left. \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))}, \tag{3.14}
 \end{aligned}$$

and further,

$$\begin{aligned}
 &\|L_t u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &= (2\pi)^{-1/2} \left\| \int_0^t (t-\tau)^{\alpha-1} g(\cdot, \tau) \right. \\
 &\quad * (F^{-1}(E_{\alpha,\alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \left. \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &+ (2\pi)^{-1/2} \left\| \int_0^t (t-\tau)^{\alpha-1} \left(\int_0^\tau f(u(\cdot, \theta)) dB(\theta) \right) \right. \\
 &\quad * (F^{-1}(E_{\alpha,\alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \left. \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &+ (2\pi)^{-1/2} \left\| \int_0^t (t-\tau)^{\alpha-1} p(x, D_x) u(\cdot, \tau) \right. \\
 &\quad * (F^{-1}(E_{\alpha,\alpha}(-\psi(\cdot)(t-\tau)^\alpha)))(x) d\tau \left. \right\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &\leq 4(2\pi)^{-1/2} T^{1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|f(u(\cdot, t))\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &+ \mu(2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))} \\
 &+ (2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0,T]))}. \tag{3.15}
 \end{aligned}$$

By using the linear growth condition (3.5), we have

$$\begin{aligned}
& \|L_t u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \leq 4(2\pi)^{-1/2} T^{1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) K_2 (1 + \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}) \\
& \quad + \mu(2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \quad + (2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& = C_1 + C_1 \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} + C_2 \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \quad + C_3 \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}, \tag{3.16}
\end{aligned}$$

where

$$C_1 := 4(2\pi)^{-1/2} T^{1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right) K_2, \tag{3.17}$$

$$C_2 := \mu(2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right), \tag{3.18}$$

and

$$C_3 := (2\pi)^{-1/2} \left(\frac{\kappa_1}{\alpha(\gamma-1)} T^{\alpha(\gamma-1)/\gamma} + \frac{\kappa_2}{\alpha} T^\alpha \right). \tag{3.19}$$

Thus we obtain

$$\begin{aligned}
& \|L_t u - g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \leq \|L_t u\|_{L^2(\Omega \times \mathbb{R}^n; L^\infty([0, T]))} + \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \leq C_1 + C_1 \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} + C_2 \|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \\
& \quad + C_3 \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} + \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))}. \tag{3.20}
\end{aligned}$$

In addition for $u \in B(g, R)$ it follows

$$\|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} - \|g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq \|u - g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq R, \tag{3.21}$$

or

$$\|u\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq \frac{5R}{4}. \tag{3.22}$$

Eventually, we arrive at

$$\|L_t u - g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq C_1 + C_1 \frac{5R}{4} + C_2 \frac{5R}{4} + C_3 \frac{R}{4} + \frac{R}{4}. \tag{3.23}$$

We can make $C_1 + C_1 \frac{5R}{4} + C_2 \frac{5R}{4} + C_3 \frac{R}{4} + \frac{R}{4} \leq R$, i.e. $\frac{4C_1}{3-5C_1-5C_2-C_3} \leq R$ and we find $\|L_t u - g\|_{L^2(\Omega \times \mathbb{R}; L^\infty([0, T]))} \leq R$. $\square$

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