RESEARCH PAPER

ACCURATE RELATIONSHIPS BETWEEN FRACTALS
AND FRACTIONAL INTEGRALS:
NEW APPROACHES AND EVALUATIONS

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*Dedicated to Professor Virginia Kiryakova on the occasion
of her 65th birthday and the 20th anniversary of FCAA*

Abstract

In this paper the accurate relationships between the averaging procedure of a smooth function over 1D- fractal sets and the fractional integral of the RL-type are established. The numerical verifications are realized for confirmation of the analytical results and the physical meaning of these obtained formulas is discussed. Besides, the generalizations of the results for a combination of fractal circuits having a discrete set of fractal dimensions were obtained. We suppose that these new results help understand deeper the intimate links between fractals and fractional integrals of different types. These results can be used in different branches of the interdisciplinary physics, where the different equations describing the different physical phenomena and containing the fractional derivatives and integrals are used.

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1. Introduction

Now the acronym FDA (Fractional Derivative and its Application) received a very wide propagation. The “hot spot” is formed in the end of the 80-th of the last century when many researches working in different application fields understood that this new tool suggested by the mathematics of the fractional calculus can open new features and generalizations of the previous phenomena associated with fractal geometry studied. For beginners one can recommend some monographs [14, 1, 12, 15, 13] and reviews [6, 7] included extended old and recent historical survey, where the foundations of this “hot spot” are explained. The interest to relationship between fractals and fractional calculus is renewed again. Some original approach (but without proper physical interpretation) was outlined in papers [16, 4, 5]. One of the basic problems that did not accurately solved yet in the fractional calculus community is the finding of the justified and accurate relationships between the smoothed functions averaged over fractal objects and fractional operators. This problem was solved partly for the time-dependent functions averaged over Cantor sets in monograph [9] and paper [10], where the influence of unknown log-periodic function (leading finally to the understanding of the meaning of the fractional integral with the complex-conjugated power-law exponents) was taken into account. Possible generalizations helping to understand the role of a spatial fractional integral as a mathematical operator replacing the operation of averaging of the smoothed functions over fractal objects were considered in monograph [9] as well. However, in order to receive as a generalization the desired expressions for the gradient, divergence and curl expressed by means of the fractional operator in the limits of mesoscale (when the current scale η lies in the interval ($\eta < \lambda$) determining the limits of a possible self-similarity) it was necessary to apply the additional averaging procedure over possible places of location of the fractal object considered. This procedure provides the correct convergence of the microscopic function $f(z)$ at small ($\eta \simeq \lambda$) and large ($\eta \simeq \Lambda$) scales. Nevertheless, the basic reason that serves as a specific mathematical obstacle in accurate establishing of the desired relationship between the fractal object and the corresponding fractional integral is the absence of the 2D- and 3D-Laplace transformations. Therefore, the basic problem that is considered in this paper can be formulated as:

What accurate form of the fractional operator is generated in the results of the averaging procedure of a smoothed function over the given fractal set if we want to realize this procedure without any approximations?

We should stress also the results obtained in the recent paper [3] where new attempt to relate a fractal object (branching flow stream of a liquid passes

through porous medium) with the fractional integral is presented. However, in this model the definition of the effective velocity that is averaged over the discrete structure of the fractal considered does not allow receiving typical log-periodic oscillations that naturally appears in any fractal object with discrete structure.

In this paper, we want to discuss some new ideas that can help to understand deeper the desired relationship between the accurate averaging of the smoothed functions over fractal sets and fractional integrals in time and space. These new results we consider as a natural generalization of the previous and approximate results achieved in papers [11, 10].

2. New exact relationships connecting 1D fractals and fractional integrals

2.1. The exact relationship between temporal fractional integral and the smoothed function averaged over the Cantor set with M bars

In paper [10] the relationship between fractional integral with complex power-law exponent and Cantor set has been established. But this relationship was approximate and obtained in one-mode approximation and it would be desirable to establish the exact relationship between 1D fractals and fractional integrals in time-domain. Attentive analysis shows that one important point in the previous results leading to the desired exact relationship was missed. In order to show it let us reproduce some mathematical expressions that will be helpful for further manipulations and understanding the problem posed. As it has been shown in [10] the Laplace image of the kernel of the Cantor set is described by expression

$$\lim_{N \rightarrow \infty} K^{(N)}(z) \equiv K_\nu(z) = \frac{\pi_\nu(\ln(z))}{z^\nu}, \quad (1)$$

where $z = pT(1 - \xi)$, p defines the Laplace parameter, T is a period of location of the Cantor set, ξ is the scaling factor. The kernel $K^{(N)}(z)$ can be presented as

$$K^{(N)}(z) = \prod_{n=-(N-1)}^{N-1} g(z\xi^n) = \prod_{n=0}^{N-1} g(z\xi^n) \prod_{n=1}^{N-1} g(z\xi^{-n}). \quad (2)$$

Here $g(z)$ describes the structure of the given fractal with asymptotic given below. In particular, the Laplace image of the function $g(z)$ for Cantor set

having M bars has the form

$$g(z) = \frac{1}{M} \frac{1 - \exp\left(-\frac{zM}{M-1}\right)}{1 - \exp\left(-\frac{z}{M-1}\right)}. \quad (3)$$

In order to satisfy to the functional equation of the type (4) in the limit ($N \ll 1$) for the kernel

$$K(\xi z) = \frac{1}{\bar{g}} K(z), \quad (4)$$

the function $g(z)$ should have the following decompositions for small and large values of z :

for $Re(z) \ll 1$

$$g(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad (5)$$

for $Re(z) \gg 1$

$$g(z) = \bar{g} + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots. \quad (6)$$

It is easy to note that for the function $g(z)$ from (3) having M bars in each self-similar stage these requirements are satisfied ($\bar{g} = 1/M$).

The solution of the functional equation (4) has the form (1) with power-law exponent equaled [16, 4]

$$\nu = \frac{\ln(\bar{g})}{\ln(\xi)} = \frac{\ln(1/M)}{\ln(\xi)}, \quad 0 < \nu < 1. \quad (7)$$

The log-periodic function $\pi_\nu(\ln z \pm \ln \xi) = \pi_\nu(\ln z)$ with period $\ln(\xi)$ that figures in 1 can be decomposed to the infinite Fourier series [9, 10]

$$\pi_\nu\left(\frac{\ln(z)}{\ln(\xi)}\right) = \sum_{n=-\infty}^{\infty} C_n \exp\left(2\pi n i \frac{\ln(z)}{\ln(\xi)}\right) \equiv C_0 + \sum_{n=1}^{\infty} (C_n z^{i\Omega_n} + C_n^* z^{-i\Omega_n}). \quad (8)$$

Here $\Omega_n = 2\pi n / \ln \xi$ is a set of frequencies providing a periodicity with $\ln$ of product (4). Taking into account this decomposition one can present the limiting solution of (1) in the form

$$K_\nu(z) = C_0 z^{-\nu} + \sum_{n=1}^{\infty} (C_n z^{-\nu+i\Omega_n} + C_n^* z^{-\nu-i\Omega_n}). \quad (9)$$

Here the real exponent is defined by expression (7). The presentation of kernel (2) in the form (9) allows reproducing the previous cases (when the

sum in (9) is negligible and or it can be presented in one mode approximation [10]) and finding the desired exact relationship. Really, taking into account the well-known relationship [2]

$$(p)^{-a} =: \frac{1}{\Gamma(a)} \int_0^t t^{a-1} \exp(-pt) dt, \quad \operatorname{Re}(a) \geq 0, \quad (10)$$

it is easy to find the desired original for the kernel (9)

$$K_\nu(t) = C_0 \frac{t^{\nu-1}}{\Gamma(\nu)} + \sum_{n=1}^{\infty} \left(C_n \frac{t^{\nu-1+i\Omega_n}}{\Gamma(\nu+i\Omega_n)} + C_n^* \frac{t^{\nu-1-i\Omega_n}}{\Gamma(\nu-i\Omega_n)} \right). \quad (11)$$

Here we should take into account that temporal variable t should be dimensionless. Based on the determination of the variable $z = pT(1 - \xi)$ (see expression (1)) it is easy to restore the dimension of the constant $t \rightarrow t/T(1 - \xi)$. Based on the definition of the fractional integral in the form of the Riemann-Liouville (RL) type one can write the desired relationship

$$\int_0^t K(t - \tau) \cdot f(\tau) d\tau = C_0 J^\nu f(t) + \sum_{n=1}^{\infty} [C_n J^{\nu+i\Omega_n} f(t) + C_n^* J^{\nu-i\Omega_n} f(t)]. \quad (12)$$

Here we define the RL-integral of the complex order as

$$J_0^{\nu \pm i\Omega} f(t) = \frac{1}{\Gamma(\nu \pm i\Omega)} \int_0^t (t - \tau)^{\nu-1 \pm i\Omega} f(\tau) d\tau. \quad (13)$$

From the exact relationship (12), it follows that the averaging procedure of a smooth function over the generalized Cantor set (having M bars) is accompanied *always* by an infinite set of fractional integrals having complex-conjugated power-law exponents. Only in the partial case when the contribution of log-periodic function becomes negligible relationship (13) restores the RL fractional integral with real power-law exponent [9, 10]. When the total sum in (11) can be replaced approximately by one term

$$\sum_{n=1}^{\infty} \left(C_n \frac{t^{\nu-1+i\Omega_n}}{\Gamma(\nu+i\Omega_n)} + (C_n)^* \frac{t^{\nu-1-i\Omega_n}}{\Gamma(\nu-i\Omega_n)} \right) \cong C \frac{t^{\nu-1+i\langle\Omega\rangle}}{\Gamma(\nu+i\langle\Omega\rangle)} + C^* \frac{t^{\nu-1-i\langle\Omega\rangle}}{\Gamma(\nu-i\langle\Omega\rangle)}, \quad (14)$$

then we restore the basic result of paper [10] in the so-called one-mode approximation. In relationship (14) $\langle\Omega\rangle$ defines the leading mode, which replaces approximately other modes that figure in the left-hand side of expression (14).

2.2. Numerical verification

For verification purposes one can take time-domain expression for the kernel defined by (11) and the corresponding one-mode approximation expression from (14).

Numerical calculations were realized with the help of the following procedure:

1. Calculation of the right side of expression (14) using one-mode approximation constants for some given M (number of the Cantor columns) and the scaling parameter ξ from [10] (the first 3 rows of Table 1).
2. Calculation of the fitting constants C_n ($n = 1, \dots, N_m$), where N_m defines the finite number of modes in the corresponding sum (11) by the linear least-square method (LLSM).

The values of fitting parameters for the given values of M and ξ are collected in last 3 rows of Table 1. One can see that the fitting parameters C_0 , A_1 , Ω_1 , are very close to the corresponding initial ones obtained in one-mode approximation.

M	ξ	ν	C_0	A	$\langle\Omega\rangle$	$\langle n \rangle$
2	0.125	0.3333	0.63	0.0082	3.01161	0.9967
5	0.05	0.5372	0.6117	0.0217	2.09144	0.9972
15	0.0167	0.6614	0.606	0.0353	1.5331	0.9991
M	ξ	ν	C_0	C_1	Ω_1	N_m
2	0.125	0.3333	0.62328	0.00805	3.02157	20
5	0.05	0.5372	0.61139	0.02157	2.09738	20
15	0.0167	0.6614	0.60640	0.03523	1.5346	20

TABLE 1. The basic initial (the first 3 rows) and the fitting parameters (rest rows) obtained in the result of numerical verification of expression (11).

The results of further numerical calculations are collected in Figures 1 and 2. There one can see that contribution of each next term in the sum (11) decreases drastically in comparison with the previous one. This observation serves as a good proof of the validity of one-mode approximation approach proposed in [10].

Analyzing these results obtained in this section we found that higher terms $n = 1, 2, \dots, N_m$ give relatively insignificant contribution in the total result and the value of the fitting error has a plateau with increasing of N_m . One can notice that fitting error mostly depends on $\langle n \rangle$ parameter, characterizing the contribution of the main harmonics.

FIGURE 1. The fitted C_n coefficients presented in log-scale (y -axis) correspond to the successive terms of the sum in (11) (x -axis) for $M = 2$ (open circles), $M = 5$ (open squares) and $M = 15$ (open triangles).

FIGURE 2. Relative fitting error in % (y -axis) versus number of modes (x -axis) for (a) $M = 2$, (b) $M = 5$ and (c) $M = 15$.

2.3. Physical interpretation of this result and possible generalizations

To understand deeper the results from physical point of view it makes sense to start from the simplest physical examples which can clarify the procedure considered above. For this purpose we consider at first the following mechanical problem. How to calculate the total and averaged path $\langle L(t) \rangle$ for some interval of the time t for a set of material points (particles) if every particle is moving in one direction with the constant velocity V coinciding with the points of Cantor set and is idle outside the set? The expression for the path $L_N(t)$ on the N -th stage of Cantor set construction has the form:

$$L_N(t) = V \int_0^t K_{T,\nu}^{(N)}(\tau) d\tau. \quad (15)$$

Here the value $K_{T,\nu}^{(N)}$ determines the Cantor set located on the temporal interval T on the N -th stage of its self-similarity having M bars and dimension defined by expression (7). The value of the normalization interval T for the given case can be found from the condition that during the interval T the body passes the distance L^* . Hence, $T = L^*/V$. Then, integrating expression (11) we obtain

$$L_\nu(t) = \frac{L^*}{\Gamma(1+\nu)} \left(\frac{t}{T} \right)^\nu \left[1 + \sum_{n=1}^{\infty} \left(c_n \frac{\Gamma(1+\nu)}{\Gamma(\nu+1+i\Omega_n)} \left(\frac{t}{T} \right)^{i\Omega_n} + c_n^* \frac{\Gamma(1+\nu)}{\Gamma(\nu+1-i\Omega_n)} \left(\frac{t}{T} \right)^{-i\Omega_n} \right) \right]. \quad (16)$$

The sum entering in (16) determines the log-periodic corrections appearing in the result of discretization of the Cantor set. Then, realizing the averaging procedure for the total assembly of particles described in a book [9], we obtain

$$\langle L(t) \rangle = L^* \frac{[B(\nu)]}{\Gamma(1+\nu)} \left(\frac{Vt}{L^*} \right)^\nu. \quad (17)$$

Here $B(\nu)$ determines the averaged value of the sum located in the square brackets (16). For the binary Cantor set it can be evaluated analytically that gives

$$B(\nu) = \langle \pi_{T,\nu}(z) \rangle = \int_{-1/2}^{1/2} \pi_{T,\nu}(z + x \ln \xi) dx = \frac{2^{-(1+\nu/2)}}{\ln 2}. \quad (18)$$

Actually, the formula (17) expresses the dependence of the averaged path as the function of time for a set of particles if every particle is moving only

inside the Cantor set with the fractal dimension and we are interested only by the averaged path of the total assembly. The fractal dimension shows what part of states of the assembly are involved in this movement. We also want to notice that exact intervals where one type of movement is transformed into the state of idleness are averaged and after the performing of the averaging procedure we have the information only about a part of all the states involved into the distribution over Cantor sets. The averaged procedure of a smoothed function over Cantor set described above represents the Cantor's "filter" which enables to filtrate one type of movement and delete (because of the normalization of the states) another one. The averaged log-periodic states are described by a constant (18). It represents the result of this "filtration" procedure. To stress the filtration properties of the Cantor set we consider more complicated example. Let us assume that it in the intervals between Cantor stripes an assembly of bodies is moved with various and random velocities defined as $\{U_i\}$. The index $i = 1, 2, \dots, N, \dots$ refers to the stages of Cantor construction. Two stages of the modified Cantor set are shown on Figure 3.

FIGURE 3. Here two successive stages of the modified binary set are shown. The set of velocities outside of the Cantor set are random. The widths of Cantor bars are denoted as $\{\Delta_i \ (i = 0, 1, 2)\}$, the set of velocities $\{V_i\}$ defines the movement inside the set.

It is interesting to set up the following question: *Is there a condition for the distribution of velocities $\{U_i\}$ when the influence of movement outside the set becomes negligible?*

In this case the recurrence relationship for the density is expressed by formula:

$$K_{\Delta_N}^{(N)}(t) = V_N \left[K_{\Delta_N}^{(N-1)}(t) + K_{\Delta_N}^{(N-1)}(t - (\Delta_{N-1} - \Delta_N)) \right] + U_N \left[K_{\Delta_N}^{(N-1)}(t - (\Delta_{N-1} - \Delta_N)) - K_{\Delta_N}^{(N-1)}(t - \Delta_N) \right]. \quad (19)$$

For Laplace-image with the help of retardation theorem

$$F(t - a) =: \exp(-pa) F(p), \quad (20)$$

one can obtain the following expression:

$$K_{\Delta_N}^{(N)}(p) = V_N \frac{1 - \exp(-p\Delta_N)}{p} \prod_{n=0}^{N-1} (1 + \exp(-p(\Delta_n - \Delta_{n+1}))) + \sum_{k=0}^{N-1} U_{k+1} \times \frac{\exp(-p\Delta_{k+1}) - \exp(-p(\Delta_k - \Delta_{k+1}))}{p} \prod_{n=0}^{k-1} (1 + \exp(-p(\Delta_n - \Delta_{n+1}))). \quad (21)$$

The total area on the N -th stage is given by the following expression:

$$2^N V_N \Delta_N + \sum_{k=0}^{N-1} 2^k U_{k+1} (\Delta_k - 2\Delta_{k+1}) = S_N. \quad (22)$$

We require that the total area on every stage remains the same (it is equivalent to the condition of conservation of the total number of states):

$$S_N = S_{N-1} = \dots = S_0. \quad (23)$$

Let us assume that $U_i = \bar{U} + \delta U_i$, where

$$\bar{U} = \frac{1}{N} \sum_{i=1}^N U_i \quad (24)$$

is the arithmetic mean of random velocity. The set U_i ($i = 1, 2, \dots, N, \dots$) defines the random deviations of U_i . We assume that these deviations $\{U_i\}$ satisfy the condition:

$$\sum_{k=0}^{N-1} 2^k \delta U_{k+1} (\Delta_k - 2\Delta_{k+1}) = 0. \quad (25)$$

Based on this condition we can rewrite (21) as

$$K_{\Delta_N}^{(N)}(p) = \frac{(S_0 - \bar{U}T)}{(2\xi)^N T} \left[\frac{1 - \exp(-pT\xi^N)}{p} \right] \prod_{n=0}^{N-1} (1 + \exp(-pT(1 - \xi)\xi^n)) + \bar{U} \frac{1 - \exp(-pT)}{p}. \quad (26)$$

If we propagate the binary Cantor set on the whole temporal interval (in and out of the given T) we obtain from

$$K_{\Delta_N}^{(N>>1)}(z) \cong \frac{S_0 - \bar{U}T}{T} \prod_{n=-(N-1)}^{N-1} \left[\frac{1 + \exp(-z\xi^n)}{2} \right] + \bar{U} \cdot T \frac{1 - \exp(-pT)}{pT}. \quad (27)$$

The first part of (27) satisfies to the scaling equation (4) with $\bar{g} = 2$ and the second part is proportional to t . If we, again, apply the averaging procedure then we obtain

$$\langle L(t) \rangle = L^* \left(1 - \frac{\bar{U}}{V} \right) \frac{B(\nu)}{\Gamma(1+\nu)} \left(\frac{Vt}{L^*} \right)^\nu + \bar{U}t. \quad (28)$$

The last expression confirms again the filtration properties of the Cantor set. It divides all motion on two parts: (a) the first motion of the particles located inside the Cantor set, (b) the second part describes the motion that takes place outside the fractal set.

2.4. New relationships connecting a fractal process in time with the smoothed function by means of the fractional integral

It is natural to consider another self-similar fractal process, which can be presented in the form of an additive summation [9, 10]. These sums are appeared naturally when the self-similar RLC elements connected in parallel or in-series are considered [9, 10]

$$S(z) = s_0 \sum_{n=-N}^N b^n f(z\xi^n). \quad (29)$$

Here z is dimensionless Laplace parameter, b and ξ are some constant scaling factors. Each term in (29) can be associated, for example, with the scaled resistor ($R_n = R_0 b^n$), capacitance ($C_n = R/z\xi^n$, $z = j\omega RC$) or inductance ($L_n = Rz\xi^n$, $z = j\omega L_0/R$) that can form a self-similar fractal circuit [9] or with additive contribution of a bar belonging to some Cantor set. In this case, an additive contribution of each bar is expressed by the function

$$f(z\xi^n) = \frac{1 - \exp(-z\xi^n)}{z\xi^n}. \quad (30)$$

It is easy to note that sum (29) satisfies to the following equation

$$\begin{aligned} S(z\xi) &= s_0 \sum_{n=-N}^N b^n f(z\xi^{n+1}) = s_0 \sum_{n=-N+1}^{N+1} b^{n-1} f(z\xi^n) \\ &= \frac{1}{b} S(z) + b^N f(z\xi^{N+1}) - b^{-N-1} f(z\xi^{-N}). \end{aligned} \quad (31)$$

We suppose that the contributions of the last two terms on the ends of the finite interval are negligible

$$b^N f(z\xi^{N+1}) \stackrel{N \gg 1}{\cong} 0, \quad b^{-N-1} f(z\xi^{-N}) \stackrel{N \gg 1}{\cong} 0. \quad (32)$$

Below we will discuss these suppositions in detail. Equation (31) at conditions (32) satisfies the following functional equation that formally coincides with equation (4) considered above

$$S(z\xi) = \frac{1}{b} S(z), \quad (33)$$

So, based on the results obtained in the previous section one can write

$$S(z) = z^{-\nu} \text{Pr}(\ln z), \quad \nu = \ln(b)/\ln(\xi), \quad \text{Pr}(\ln z \pm \ln \xi) = \text{Pr}(\ln z). \quad (34)$$

In spite of the formal coincidence of equation (33) with (4), in order to satisfy to conditions (32) the function $f(z)$ in (31) should have another asymptotic decompositions

for $\text{Re}(z) \ll 1$

$$f(z) = c_1 z + c_2 z^2 + \dots, \quad (35)$$

for $\text{Re}(z) \gg 1$

$$f(z) = \frac{A_1}{z} + \frac{A_2}{z^2} + \dots, \quad (36)$$

and condition

$$1 < b < \xi \quad (37)$$

As it follows from (34) the last condition provides in addition the obvious inequality

$$0 < \nu = \frac{\ln(b)}{\ln \xi} < 1. \quad (38)$$

So, the results obtained for the sum (31) coincides with the results obtained earlier for the product (2) and the exact relationship in time-domain (11) with subsequent convolution (12) of the kernel (11) with a smooth function remains invariant. Is it possible to consider more complicated self-similar process that is presented by an additive combination of two sums of the type?

$$S(z) = s_1 \sum_{n=-N}^N b_1^n f_1(z\xi^n) + s_2 \sum_{n=-N}^N b_2^n f_2(z\xi^n) \equiv S_1(z) + S_2(z). \quad (39)$$

If we assume, by analogy with requirement (32) that the contribution of each sum on the end of the intervals are negligible

$$b_{1,2}^N f_{1,2}(z\xi^{N+1}) \stackrel{N \gg 1}{\cong} 0, \quad b_{1,2}^{-N-1} f_{1,2}(z\xi^{-N}) \stackrel{N \gg 1}{\cong} 0, \quad (40)$$

(here the number of elements N and the scaling factor for both sums are kept the same values) then from (39) we have a system of equations

$$\begin{aligned} S(z\xi) &= \frac{1}{b_1}S_1(z) + \frac{1}{b_2}S_2(z), \\ S(z\xi^2) &= \frac{1}{b_1^2}S_1(z) + \frac{1}{b_2^2}S_2(z). \end{aligned} \quad (41)$$

Excluding from (39) and the first row of (41) the unknown sums $S_{1,2}(z)$ (defined above by expressions (39)) and inserting them to the second row of (41) we obtain the generalized scaling equation of the second order

$$S(z\xi^2) = \left(\frac{1}{b_1} + \frac{1}{b_2} \right) S(z\xi) - \frac{1}{b_1 b_2} S(z). \quad (42)$$

Solutions of the scaling equations of the second order was considered in paper [11] and, therefore, we can write

$$S(z) = z^{\nu_1} \text{Pr}_1(\ln z) + z^{\nu_2} \text{Pr}_2(\ln z), \quad (43)$$

where the power-law exponents are found from the equation

$$\xi^{2\nu} - \left(\frac{1}{b_1} + \frac{1}{b_2} \right) \xi^\nu + \frac{1}{b_1 b_2} = 0, \quad \nu_{1,2} = \frac{\ln(1/b_{1,2})}{\ln \xi}. \quad (44)$$

In solution (43) we have now two log-periodic functions $\text{Pr}_{1,2}(\ln z \pm \ln \xi) = \text{Pr}_{1,2}(\ln z)$ that can be decomposed to the Fourier series relatively variable $\ln(z)$. Therefore, one can present the solution (43) in the form

$$\begin{aligned} S(z) &= C_0^{(1)} z^{-\nu_1} + \sum_{n=1}^{\infty} \left[C_n^{(1)} z^{-\nu_1 + i\Omega_n} + \left(C_n^{(1)} \right)^* z^{-\nu_1 - i\Omega_n} \right] \\ &+ C_0^{(2)} z^{-\nu_2} + \sum_{n=1}^{\infty} \left[C_n^{(2)} z^{-\nu_2 + i\Omega_n} + \left(C_n^{(2)} \right)^* z^{-\nu_2 - i\Omega_n} \right], \end{aligned} \quad (45)$$

where we introduced new definitions of the power-law exponents and discrete frequencies reflecting the discrete structure of the fractal process considered.

Moreover, these results can be further generalized. Let us suppose that we have three additive combination of self-similar processes

$$\begin{aligned} S(z) &= s_1 \sum_{n=-N}^N b_1^n f_1(z\xi^n) + s_2 \sum_{n=-N}^N b_2^n f_2(z\xi^n) + s_3 \sum_{n=-N}^N b_3^n f_3(z\xi^n) \\ &\equiv S_1(z) + S_2(z) + S_3(z). \end{aligned} \quad (46)$$

The functions $f_k(z)$ ($k = 1, 2, 3$) are chosen in a such way that their contributions on the ends of the interval remain negligible

$$b_k^N f_k(z\xi^{N+1}) \stackrel{N \gg 1}{\cong} 0, \quad b_k^{-N-1} f_k(z\xi^{-N}) \stackrel{N \gg 1}{\cong} 0, \quad k = 1, 2, 3. \quad (47)$$

We *suppose* again that for each function $f_k(z)$ ($k = 1, 2, 3$) the asymptotic behavior similar to expressions (35) and (36) is conserved. Therefore, the suppositions (47) are satisfied if the constants b_k and ξ follow the requirements

$$1 < b_k < \xi, \quad k = 1, 2, 3. \quad (48)$$

Therefore, we obtain the following system of equations for the finding of unknown $S_k(z)$:

$$\begin{aligned} S(z) &= S_1(z) + S_2(z) + S_3(z), \\ S(z\xi) &= \frac{1}{b_1}S_1(z) + \frac{1}{b_2}S_2(z) + \frac{1}{b_3}S_3(z), \\ S(z\xi^2) &= \frac{1}{b_1^2}S_1(z) + \frac{1}{b_2^2}S_2(z) + \frac{1}{b_3^2}S_3(z). \end{aligned} \quad (49)$$

Excluding from the system (49) the unknown sums $S_k(z)$ and inserting them into the relationship

$$S(z\xi^3) = \frac{1}{b_1^3}S_1(z) + \frac{1}{b_2^3}S_2(z) + \frac{1}{b_3^3}S_3(z), \quad (50)$$

after simple algebraic manipulations we obtain the following functional equation with respect to the function $S(z)$

$$\begin{aligned} S(z\xi^3) &= a_2S(z\xi^2) + a_1S(z\xi) + a_0S(z) \\ a_2 &= \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}, \quad a_1 = -\left(\frac{1}{b_1b_2} + \frac{1}{b_1b_3} + \frac{1}{b_3b_2}\right), \quad a_0 = \frac{1}{b_1b_2b_3}. \end{aligned} \quad (51)$$

The solution of the functional equation (51) can be written as

$$S(z) = z^{\nu_1}\text{Pr}_1(\ln z) + z^{\nu_2}\text{Pr}_2(\ln z) + z^{\nu_3}\text{Pr}_3(\ln z). \quad (52)$$

The power-law exponents entering into (52) are found from the cubic equation

$$\xi^{3\nu} - a_2\xi^{2\nu} - a_1\xi^\nu - a_0 = 0, \quad (53)$$

and can be expressed as

$$\nu_k = \frac{\ln(1/b_k)}{\ln \xi} = -\frac{\ln(b_k)}{\ln \xi}, \quad 0 < |\nu_k| < 1, \quad k = 1, 2, 3. \quad (54)$$

Therefore, expression (45) conserves its form and for this case the solution (52) can be rewritten as

$$S(z) = \sum_{k=1}^3 C_0^{(k)} z^{-\nu_k} + \sum_{k=1}^3 \sum_{n=1}^{\infty} \left[C_n^{(k)} z^{-|\nu_k|+i\Omega_n} + \left(C_n^{(k)}\right)^* z^{-|\nu_k|-i\Omega_n} \right]. \quad (55)$$

It is quite obvious that this result admits the further generalization for the case of $k = 1, 2, \dots, K$ sums

$$S(z) = \sum_{k=1}^K s_k \sum_{n=-N}^N b_k^n f_k(z\xi^n) \equiv \sum_{k=1}^K S_k(z). \quad (56)$$

Using the mathematical induction method one can write the final result for this case

$$\begin{aligned} S(z) &= \sum_{k=1}^K z^{\nu_k} \text{Pr}_k(\ln z) \\ &= \sum_{k=1}^K C_0^{(k)} z^{-\nu_k} + \sum_{k=1}^K \sum_{n=1}^{\infty} \left[C_n^{(k)} z^{-|\nu_k|+i\Omega_n} + \left(C_n^{(k)} \right)^* z^{-|\nu_k|-i\Omega_n} \right]. \end{aligned} \quad (57)$$

The desired roots are found from the relationships

$$|\nu_k| = \frac{\ln(b_k)}{\ln \xi}, \quad 1 < b_k < \xi, \quad 0 < |\nu_k| < 1. \quad (58)$$

So, the kernel connecting the given fractal process with a smooth function in time-domain accept the following general form

$$K_\nu(t) = \sum_{k=1}^K C_0^{(k)} \frac{t^{\nu_k-1}}{\Gamma(\nu_k)} + \sum_{k=1}^K \sum_{n=1}^{\infty} \left(C_n^{(k)} \frac{t^{\nu_k-1+i\Omega_n}}{\Gamma(\nu_k+i\Omega_n)} + \left(C_n^{(k)} \right)^* \frac{t^{\nu_k-1-i\Omega_n}}{\Gamma(\nu_k-i\Omega_n)} \right). \quad (59)$$

To the best of our knowledge, this is the most general result that can be obtained for a set of additive self-similar processes in time. As one notice from the calculations, all dynamical functions describe a self-similar process with the *same* scale ξ . If the scales are different and the discrete boundaries of the processes are different also $N_k \neq N$ then these results are becoming questionable and require the further research. But nevertheless, one can show the situation taking place for different scales $\xi_i \neq \xi$ when the result can be reduced to the case considered above. Let us suppose that we have a product combining a random combination of different scales

$$P_n = \prod_{s=1}^n \xi_s, \quad (60)$$

but the deviations from some averaged scale (considered as the dominant scale) are small. We have for this case

$$P_n = \prod_{s=1}^n \xi_s = \prod_{s=1}^n (\langle \xi \rangle + \delta_s) = \langle \xi \rangle^n \prod_{s=1}^n \left(1 + \frac{\delta_s}{\langle \xi \rangle} \right)$$

$$= \langle \xi \rangle^n \exp \left(\sum_{s=1}^n \ln \left(1 + \frac{\delta_s}{\langle \xi \rangle} \right) \right). \quad (61)$$

For the condition $\delta_s / \langle \xi \rangle \ll 1$ we have approximately

$$P_n \cong \left(\langle \xi \rangle \exp \left[\frac{\langle \delta \rangle}{\langle \xi \rangle} - \frac{\langle \delta^2 \rangle}{2\langle \xi \rangle^2} + \frac{\langle \delta^3 \rangle}{3\langle \xi \rangle^3} \right] \right)^n, \quad (62)$$

$$\langle \delta^p \rangle = \frac{1}{n} \sum_{s=1}^n \delta_s^p, \quad p = 1, 2, \dots$$

Therefore, this result allows to consider different scales if they are not strongly deviated from each other. All the previous results are valid if we simply replace

$$\xi \rightarrow \langle \xi \rangle \exp \left(\frac{\langle \delta \rangle}{\langle \xi \rangle} \right) \exp \left(-\frac{\langle \delta^2 \rangle}{2\langle \xi \rangle^2} \right) \dots, \quad \text{with } \langle \xi \rangle = \frac{1}{n} \sum_{s=1}^n \xi_s. \quad (63)$$

3. Results and discussion

The accurate relationships between fractals and fractional integrals remain a “hot” spot for many researches working in the fractional calculus and fractal geometry field. From the results obtained in this paper one can notice that the complex-conjugated part figuring in the fractional power-law exponent plays a crucial role. It is necessary to say that the power-law exponent with complex additive are appeared in some papers but the physical/geometrical origin of this additive was not clear. From the results obtained above one can say that the complex-conjugated part is tightly associated with the discrete structure of the fractal process and should be taken into account in fractional and kinetic equations that pretend on description of self-similar processes in time-domain, at least. As for the spatial fractional integral the finding of the accurate relationship for the given fractal in space remains an open problem. This problem can be divided at least on two parts:

- (a) The finding of the proper fractional integral based on the given fractal structure
- (b) To find a proper fractal for the fractional integral that is chosen for description of the self-similar process in space.

From our point of view, the general solution of this complex problem is absent because each fractal in space can generate a specific fractional integral [11] but any efforts of researches actively working in this interesting field are very welcome [3, 8].

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