Research Article

Open Access

Giovanni Catino*

Rigidity of positively curved shrinking Ricci solitons in dimension four

<https://doi.org/10.1515/geofl-2019-0001>

Received August 13, 2018; accepted September 24, 2018

Abstract: We classify four-dimensional shrinking Ricci solitons satisfying $\text{Sec} \geq \frac{1}{24}R$, where Sec and R denote the sectional and the scalar curvature, respectively. They are isometric to either $\mathbb{R}^4$ (and quotients), $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics.

Keywords: Ricci solitons, Einstein metrics, positive sectional curvature

MSC: 53C24, 53C25

1 Introduction

In this paper we investigate gradient shrinking Ricci solitons with positive sectional curvature. We recall that a Riemannian manifold (M^n, g) of dimension $n \geq 3$ is a gradient Ricci soliton if there exists a smooth function f on M^n such that

$$\text{Ric} + \nabla^2 f = \lambda g$$

for some constant λ . If ∇f is parallel, then (M^n, g) is Einstein. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Ricci solitons generate self-similar solutions of the Ricci flow, play a fundamental role in the formation of singularities and have been studied by many authors (see H.-D. Cao [5] for an overview).

It is well known that (compact) Einstein manifolds can be classified, if they are enough positively curved. Sufficient conditions are non-negative curvature operator (S. Tachibana [18]), non-negative isotropic curvature (M. J. Micallef and Y. Wang [13] in dimension four and S. Brendle [3] in every dimension) and weakly $\frac{1}{4}$ -pinched sectional curvature [1] (if Sec and R denote the sectional and the scalar curvature, respectively, this condition in dimension four is implied by $\text{Sec} \geq \frac{1}{24}R$). Moreover, in dimension four, it is proved by D. Yang [19] that four-dimensional Einstein manifolds satisfying $\text{Sec} \geq \varepsilon R$ are isometric to either $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics, if $\varepsilon = \frac{\sqrt{1249-23}}{480}$. The lower bound has been improved to $\varepsilon = \frac{2-\sqrt{2}}{24}$ by E. Costa [8] and, more recently, to $\varepsilon = \frac{1}{48}$ by E. Ribeiro [16] (see also X. Cao and P. Wu [6]). It is conjectured in [19] that the result should be true assuming positive sectional curvature.

In dimension $n \leq 3$, complete shrinking Ricci solitons are classified. In the last years there have been a lot of interesting results concerning the classification of shrinking Ricci solitons which are positively curved. For instance, it follows by the work of C. Böhm and B. Wilking [2] that the only compact shrinking Ricci solitons with positive (two-positive) curvature operator are quotients of $\mathbb{S}^n$. In dimension four, A. Naber [14] classified complete shrinkers with non-negative curvature operator. Four dimensional shrinkers with non-negative isotropic curvature were classified by X. Li, L. Ni and K. Wang [12].

Recently, O. Munteanu and J.P. Wang [17] showed that every complete shrinking Ricci solitons with positive sectional curvature are compact. It is natural to ask the following question: *given $\varepsilon > 0$, are there four dimensional non-Einstein shrinking Ricci solitons satisfying $\text{Sec} \geq \varepsilon R$?*

*Corresponding Author: Giovanni Catino: Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy, E-mail: giovanni.catino@polimi.it

In this paper we give an answer to this question proving the following

Theorem 1.1. *Let (M^4, g) be a four-dimensional complete gradient shrinking Ricci soliton with $\text{Sec} \geq \frac{1}{24}R$. Then (M^4, g) is necessarily Einstein, thus isometric to either $\mathbb{R}^4$ (and quotients), $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics.*

Note that, by the work of S. Brendle and R. Schoen [4], using the Ricci flow, one can show that compact Ricci shrinkers with weakly $\frac{1}{4}$ -pinched sectional curvature are isometric to $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics. The condition $\text{Sec} \geq \frac{1}{24}R$ is a little stronger, but the proof of Theorem 1.1 that we present is completely “elliptic”.

2 Estimates on manifolds with positive sectional curvature

To fix the notation we recall that the Riemann curvature operator of a Riemannian manifold (M^n, g) is defined as in [10] by

$$Rm(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z.$$

In a local coordinate system the components of the $(3, 1)$ -Riemann curvature tensor are given by $R_{ijk}^l \frac{\partial}{\partial x^l} = Rm(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k}$ and we denote by $R_{ijkl} = g_{lp} R_{ijk}^p$ its $(4, 0)$ -version. Throughout the paper the Einstein convention of summing over the repeated indices will be adopted. The Ricci tensor Ric is obtained by the contraction $(Ric)_{ik} = R_{ik} = g^{jl} R_{ijkl}$, $R = g^{ik} R_{ik}$ will denote the scalar curvature and $(\mathring{Ric})_{ik} = R_{ik} - \frac{1}{n} R g_{ik}$ the traceless Ricci tensor. The Riemannian metric induces norms on all the tensor bundles, in coordinates this norm is given, for a tensor $T = T_{i_1 \dots i_k}^{j_1 \dots j_l}$, by

$$|T|_g^2 = g^{i_1 m_1} \dots g^{i_k m_k} g_{j_1 n_1} \dots g_{j_l n_l} T_{i_1 \dots i_k}^{j_1 \dots j_l} T_{m_1 \dots m_k}^{n_1 \dots n_l}.$$

The first key observation are the following pointwise estimates which are satisfied by every metric with $\text{Sec} \geq \varepsilon R$ for some $\varepsilon \in \mathbb{R}$.

Proposition 2.1. *Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. If the sectional curvature satisfies $\text{Sec} \geq \varepsilon R$ for some $\varepsilon \in \mathbb{R}$, then the following two estimates hold*

$$R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} \leq \frac{1 - n^2 \varepsilon}{n} R |\mathring{Ric}|^2 + \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk}$$

and

$$R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} \leq \frac{n^2 - 4n + 2 - n^2(n-2)(n-3)\varepsilon}{2n} R |\mathring{Ric}|^2 - (n-1) \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk}.$$

In particular, in dimension four

$$R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} \leq \frac{1 - 16\varepsilon}{4} R |\mathring{Ric}|^2 + \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk},$$

$$R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} \leq \frac{1 - 16\varepsilon}{4} R |\mathring{Ric}|^2 - 3 \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk}.$$

Proof. Let $\{e_i\}$, $i = 1, \dots, n$, be the eigenvectors of $\mathring{Ric}$ and let λ_i be the corresponding eigenvalues. Moreover, let σ_{ij} be the sectional curvature defined by the two-plane spanned by e_i and e_j . Since the sectional curvature satisfy $\text{Sec} \geq \varepsilon R$, it is natural to define the tensor

$$\overline{Rm} = Rm - \frac{\varepsilon}{2} R g \otimes g.$$

In particular

$$\overline{Ric} = Ric - (n-1)\varepsilon Rg, \quad \overline{R} = (1 - n(n-1)\varepsilon)R \quad \text{and} \quad \overline{\sigma}_{ij} = \sigma_{ij} - \varepsilon R \geq 0.$$

Moreover, if μ_k and $\bar{\mu}_k$ are the eigenvalues with eigenvector e_k of Ric and $\overline{Ric}$, respectively, one has

$$\mu_k = \sum_{i \neq k} \sigma_{ik} \quad \text{and} \quad \bar{\mu}_k = \sum_{i \neq k} \bar{\sigma}_{ik}.$$

Denoting by $\bar{R}_{ijkl}$ the components of $\overline{Rm}$, we get

$$\begin{aligned} \bar{R}_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} - \bar{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk} &= \sum_{i,j=1}^n \lambda_i \lambda_j \bar{\sigma}_{ij} - \sum_{k=1}^n \mu_k \lambda_k^2 \\ &= 2 \sum_{i < j} \lambda_i \lambda_j \bar{\sigma}_{ij} - \sum_{i < j} (\lambda_i^2 + \lambda_j^2) \bar{\sigma}_{ij} \\ &= - \sum_{i < j} (\lambda_i - \lambda_j)^2 \bar{\sigma}_{ij} \leq 0. \end{aligned} \quad (2.1)$$

Using the definition of $\overline{Rm}$ and $\overline{Ric}$, we obtain

$$R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} + \varepsilon R |\overset{\circ}{Ric}|^2 \leq R_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk} - (n-1)\varepsilon R |\overset{\circ}{Ric}|^2 = \bar{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk} + \frac{1 - n(n-1)\varepsilon}{n} R |\overset{\circ}{Ric}|^2$$

and this proves the first inequality of this proposition.

In order to show the second one, we will follow the proof of [7, Proposition 3.1]. We observe that

$$\bar{R}_{ijkl} \overset{\circ}{R}_{ij} \overset{\circ}{R}_{kl} - \frac{n-2}{2n} \bar{R} |\overset{\circ}{Ric}|^2 = \sum_{i,j=1}^n \lambda_i \lambda_j \bar{\sigma}_{ij} - \frac{n-2}{2n} \bar{R} \sum_{k=1}^n \lambda_k^2.$$

Since the modified scalar curvature $\bar{R}$ can be written as

$$\bar{R} = g^{ij} g^{kl} \bar{R}_{ikjl} = \sum_{i,j=1}^n \bar{\sigma}_{ij} = 2 \sum_{i < j} \bar{\sigma}_{ij},$$

one has the following

$$\begin{aligned} \sum_{i,j=1}^n \lambda_i \lambda_j \bar{\sigma}_{ij} - \frac{n-2}{2n} \bar{R} \sum_{k=1}^n \lambda_k^2 &= 2 \sum_{i < j} \lambda_i \lambda_j \bar{\sigma}_{ij} - \frac{n-2}{n} \sum_{i < j} \bar{\sigma}_{ij} \sum_{k=1}^n \lambda_k^2 \\ &= \sum_{i < j} \left(2\lambda_i \lambda_j - \frac{n-2}{n} \sum_{k=1}^n \lambda_k^2 \right) \bar{\sigma}_{ij}. \end{aligned}$$

On the other hand, one has

$$\sum_{k=1}^n \lambda_k^2 = \lambda_i^2 + \lambda_j^2 + \sum_{k \neq i,j} \lambda_k^2.$$

Moreover, using the Cauchy-Schwarz inequality and the fact that $\sum_{k=1}^n \lambda_k = 0$, we obtain

$$\sum_{k \neq i,j} \lambda_k^2 \geq \frac{1}{n-2} \left(\sum_{k \neq i,j} \lambda_k \right)^2 = \frac{1}{n-2} (\lambda_i + \lambda_j)^2$$

with equality if and only if $\lambda_k = \lambda_{k'}$ for every $k, k' \neq i, j$. Hence, the following estimate holds

$$\sum_{k=1}^n \lambda_k^2 \geq \frac{n-1}{n-2} (\lambda_i^2 + \lambda_j^2) + \frac{2}{n-2} \lambda_i \lambda_j.$$

Using this, since $\bar{\sigma}_{ij} \geq 0$, it follows that

$$\begin{aligned} \sum_{i,j=1}^n \lambda_i \lambda_j \bar{\sigma}_{ij} - \frac{n-2}{2n} \bar{R} \sum_{k=1}^n \lambda_k^2 &\leq \frac{n-1}{n} \sum_{i < j} (2\lambda_i \lambda_j - (\lambda_i^2 + \lambda_j^2)) \bar{\sigma}_{ij} \\ &= -\frac{n-1}{n} \sum_{i < j} (\lambda_i - \lambda_j)^2 \bar{\sigma}_{ij} \\ &= \frac{n-1}{n} \left(\bar{R}_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} - \bar{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk} \right), \end{aligned}$$

where in the last equality we have used equation (2.1). Hence, we proved

$$\bar{R}_{ijkl} \overset{\circ}{R}_{ij} \overset{\circ}{R}_{kl} - \frac{n-2}{2n} \bar{R} |\overset{\circ}{Ric}|^2 \leq \frac{n-1}{n} \left(\bar{R}_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} - \bar{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk} \right),$$

i.e.

$$\bar{R}_{ijkl} \overset{\circ}{R}_{ij} \overset{\circ}{R}_{kl} \leq \frac{n-2}{2} \bar{R} |\overset{\circ}{Ric}|^2 - (n-1) \bar{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk}.$$

Finally, substituting $\bar{R}m$, $\bar{Ric}$ and $\bar{R}$ we obtain the the second inequality of this proposition. $\square$

Taking the convex combination of the two previous estimates we obtain the following.

Corollary 2.2. *Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. If the sectional curvature satisfies $\text{Sec} \geq \varepsilon R$ for some $\varepsilon \in \mathbb{R}$, then, for every $s \in [0, 1]$, one has*

$$R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \leq \left(\frac{n^2 - 4n + 2 - n^2(n-2)(n-3)\varepsilon}{2n} - \frac{n-4}{2} (1 - n(n-1)\varepsilon)s \right) R |\overset{\circ}{Ric}|^2 - (n-1 - ns) \overset{\circ}{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk}.$$

In particular, in dimension four, for every $s \in [0, 1]$, one has

$$R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \leq \frac{1-16\varepsilon}{4} R |\overset{\circ}{Ric}|^2 - (3-4s) \overset{\circ}{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk},$$

Remark 2.3. Taking $\varepsilon = 0$ and $s = \frac{n-1}{n}$, we recover the estimate on manifolds with non-negative sectional curvature which was proved in [7].

3 Some formulas for Ricci solitons

Let (M^n, g) be a n -dimensional complete gradient shrinking Ricci solitons

$$\text{Ric} + \nabla^2 f = \lambda g$$

for some smooth function f and some positive constant $\lambda > 0$. First of all we recall the following well known formulas (for the proof see [9])

Lemma 3.1. *Let (M^n, g) be a gradient Ricci soliton. Then the following formulas hold*

$$\Delta f = n\lambda - R$$

$$\Delta_f R = 2\lambda R - 2|\overset{\circ}{Ric}|^2$$

$$\Delta_f R_{ik} = 2\lambda R_{ik} - 2R_{ijkl} R_{jl}$$

where the Δ_f denotes the f -Laplacian, $\Delta_f = \Delta - \nabla \nabla f$.

In particular, defining $\overset{\circ}{R}_{ij} = R_{ij} - \frac{1}{n}Rg_{ij}$, a simple computation shows the following equation for the f -Laplacian of the squared norm of the trace-less Ricci tensor $\overset{\circ}{Ric}$

Lemma 3.2. *Let (M^n, g) be a gradient Ricci soliton. Then the following formula holds*

$$\frac{1}{2}\Delta_f |\overset{\circ}{Ric}|^2 = |\nabla \overset{\circ}{Ric}|^2 + 2\lambda |\overset{\circ}{Ric}|^2 - 2R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \frac{2}{n}R|\overset{\circ}{Ric}|^2.$$

Moreover we have the following scalar curvature estimate [15].

Lemma 3.3. *Let (M^n, g) be a complete gradient shrinking Ricci soliton. Then either g is flat or its scalar curvature is positive $R > 0$.*

Finally, we show this simple identity.

Lemma 3.4. *Let (M^n, g) be a compact gradient Ricci soliton. Then the following formula holds*

$$\int |\nabla R|^2 dV = \frac{n-4}{2}\lambda \int R^2 dV - \frac{n-4}{2n} \int R^3 dV + 2 \int R|\overset{\circ}{Ric}|^2 dV.$$

In particular, in dimension four

$$\int |\nabla R|^2 dV = 2 \int R|\overset{\circ}{Ric}|^2 dV.$$

Proof. Integrating by parts and using Lemma 3.1 we obtain

$$\begin{aligned} \int |\nabla R|^2 dV &= - \int R \Delta R dV \\ &= -\frac{1}{2} \int \langle \nabla R^2, \nabla f \rangle dV - 2\lambda \int R^2 dV + 2 \int R|\overset{\circ}{Ric}|^2 dV + \frac{2}{n} \int R^3 dV \\ &= \frac{1}{2} \int R^2 \Delta f dV - 2\lambda \int R^2 dV + 2 \int R|\overset{\circ}{Ric}|^2 dV + \frac{2}{n} \int R^3 dV \\ &= \frac{n-4}{2}\lambda \int R^2 dV - \frac{n-4}{2n} \int R^3 dV + 2 \int R|\overset{\circ}{Ric}|^2 dV. \end{aligned}$$

□

4 Proof of Theorem 1.1

Let (M^4, g) be a complete gradient shrinking Ricci soliton of dimension four and assume that $\text{Sec} \geq \varepsilon R$ on M^4 for some $\varepsilon > 0$. By Lemma 3.3 either g is flat or $R > 0$. In this second case, by the result in [17] we know that M^4 must be compact. From now on we can assume that (M^4, g) is compact with $\text{Sec} \geq \varepsilon R > 0$. Lemma 3.2 gives

$$\frac{1}{2}\Delta_f |\overset{\circ}{Ric}|^2 = |\nabla \overset{\circ}{Ric}|^2 + 2\lambda |\overset{\circ}{Ric}|^2 - 2R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \frac{1}{2}R|\overset{\circ}{Ric}|^2.$$

Integrate over M^4 and using equation (3.1) we obtain

$$\begin{aligned} 0 &= \frac{1}{2} \int \langle \nabla |\overset{\circ}{Ric}|^2, \nabla f \rangle dV + \int |\nabla \overset{\circ}{Ric}|^2 dV + 2 \int \lambda |\overset{\circ}{Ric}|^2 dV - 2 \int R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} dV - \frac{1}{2} \int R|\overset{\circ}{Ric}|^2 dV \\ &= \int |\nabla \overset{\circ}{Ric}|^2 dV - 2 \int R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} dV. \end{aligned} \quad (4.1)$$

On the other hand, given $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$ we define the three tensor

$$F_{ijk} := \nabla_k \mathring{R}_{ij} + a_1 \nabla_j \mathring{R}_{ik} + a_2 \nabla_i \mathring{R}_{jk} + b_1 \nabla_k R g_{ij} + b_2 \nabla_j R g_{ik} + b_3 \nabla_i R g_{jk}.$$

Using the Bianchi identity $\nabla_i \mathring{R}_{ij} = \frac{1}{4} \nabla_j R$, a computation gives

$$\begin{aligned} |F|^2 &= (1 + a_1^2 + a_2^2) |\nabla \mathring{Ric}|^2 + 2(a_1 + a_2 + a_1 a_2) \nabla_k \mathring{R}_{ij} \nabla_j \mathring{R}_{ik} \\ &\quad + \frac{1}{2} \left(a_1(b_1 + b_3) + a_2(b_1 + b_2) + b_2 + b_3 + 8(b_1^2 + b_2^2 + b_3^2) + 4(b_1 b_2 + b_1 b_3 + b_2 b_3) \right) |\nabla R|^2. \end{aligned}$$

In particular,

$$\begin{aligned} \int |\nabla \mathring{Ric}|^2 dV &= \frac{1}{1 + a_1^2 + a_2^2} \int |F|^2 dV - \frac{2(a_1 + a_2 + a_1 a_2)}{1 + a_1^2 + a_2^2} \int \nabla_k \mathring{R}_{ij} \nabla_j \mathring{R}_{ik} dV \\ &\quad - \frac{a_1(b_1 + b_3) + a_2(b_1 + b_2) + b_2 + b_3 + 8(b_1^2 + b_2^2 + b_3^2) + 4(b_1 b_2 + b_1 b_3 + b_2 b_3)}{2(1 + a_1^2 + a_2^2)} \int |\nabla R|^2 dV \\ &= \frac{1}{1 + a_1^2 + a_2^2} \int |F|^2 dV - \frac{2(a_1 + a_2 + a_1 a_2)}{1 + a_1^2 + a_2^2} \int \nabla_k \mathring{R}_{ij} \nabla_j \mathring{R}_{ik} dV \\ &\quad - \frac{a_1(b_1 + b_3) + a_2(b_1 + b_2) + b_2 + b_3 + 8(b_1^2 + b_2^2 + b_3^2) + 4(b_1 b_2 + b_1 b_3 + b_2 b_3)}{1 + a_1^2 + a_2^2} \int R |\mathring{Ric}|^2 dV, \end{aligned} \quad (4.2)$$

where, in the last equality we have used Lemma 3.4. On the other hand, integrating by parts and commuting the covariant derivatives, one has

$$\begin{aligned} \int \nabla_k \mathring{R}_{ij} \nabla_j \mathring{R}_{ik} dV &= - \int \mathring{R}_{ij} \nabla_k \nabla_j \mathring{R}_{ik} dV \\ &= - \int \left(\mathring{R}_{ij} \nabla_j \nabla_k \mathring{R}_{ik} + R_{kji} \mathring{R}_{ij} \mathring{R}_{kl} + R_{ij} \mathring{R}_{ik} \mathring{R}_{jl} \right) dV \\ &= - \int \left(\frac{1}{4} \mathring{R}_{ij} \nabla_i \nabla_j R - R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} + \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jl} + \frac{1}{4} R |\mathring{Ric}|^2 \right) dV \\ &= \int \left(\frac{1}{16} |\nabla R|^2 + R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} - \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jl} - \frac{1}{4} R |\mathring{Ric}|^2 \right) dV \\ &= \int \left(R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} - \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jl} - \frac{1}{8} R |\mathring{Ric}|^2 \right) dV. \end{aligned} \quad (4.3)$$

From equation (4.2), we obtain

$$\int |\nabla \mathring{Ric}|^2 dV = \frac{1}{1 + a_1^2 + a_2^2} \int |F|^2 dV - \frac{2(a_1 + a_2 + a_1 a_2)}{1 + a_1^2 + a_2^2} \int \left(R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} - \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jl} \right) dV + Q_1 \int R |\mathring{Ric}|^2 dV$$

with

$$Q_1 := \frac{a_1 + a_2 + a_1 a_2}{4(1 + a_1^2 + a_2^2)} - \frac{a_1(b_1 + b_3) + a_2(b_1 + b_2) + b_2 + b_3 + 8(b_1^2 + b_2^2 + b_3^2) + 4(b_1 b_2 + b_1 b_3 + b_2 b_3)}{1 + a_1^2 + a_2^2}.$$

Using this inequality in (4.1), we obtain that

$$\begin{aligned} 0 &= \frac{1}{1 + a_1^2 + a_2^2} \int |F|^2 dV - \frac{2(1 + a_1^2 + a_2^2 + a_1 + a_2 + a_1 a_2)}{1 + a_1^2 + a_2^2} \int R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} dV \\ &\quad + \frac{2(a_1 + a_2 + a_1 a_2)}{1 + a_1^2 + a_2^2} \int \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jl} dV + Q_1 \int R |\mathring{Ric}|^2 dV. \end{aligned} \quad (4.4)$$

From Corollary 2.2 we have

$$R_{ijkl} \mathring{R}_{ik} \mathring{R}_{jl} \leq \frac{1 - 16\varepsilon}{4} R |\mathring{Ric}|^2 - (3 - 4s) \mathring{R}_{ij} \mathring{R}_{ik} \mathring{R}_{jk} \quad (4.5)$$

for every $s \in [0, 1]$. Thus, if $a_1 + a_2 + a_1 a_2 \geq 0$, for every $s \in [0, 1]$, estimate (4.4) gives

$$0 \geq \frac{1}{1 + a_1^2 + a_2^2} \int |F|^2 dV + \frac{2 \left((3 - 4s)(1 + a_1^2 + a_2^2) + 4(1 - s)(a_1 + a_2 + a_1 a_2) \right)}{1 + a_1^2 + a_2^2} \int \overset{\circ}{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} dV \\ + Q_2 \int R |\overset{\circ}{Ric}|^2 dV \quad (4.6)$$

with

$$Q_2 := Q_1 - \frac{(1 - 16\varepsilon)(1 + a_1^2 + a_2^2 + a_1 + a_2 + a_1 a_2)}{2(1 + a_1^2 + a_2^2)} \\ = \frac{a_1 + a_2 + a_1 a_2}{4(1 + a_1^2 + a_2^2)} - \frac{(1 - 16\varepsilon)(1 + a_1^2 + a_2^2 + a_1 + a_2 + a_1 a_2)}{2(1 + a_1^2 + a_2^2)} \\ - \frac{a_1(b_1 + b_3) + a_2(b_1 + b_2) + b_2 + b_3 + 8(b_1^2 + b_2^2 + b_3^2) + 4(b_1 b_2 + b_1 b_3 + b_2 b_3)}{1 + a_1^2 + a_2^2}.$$

Now, choose $a_1 = a_2 = 1$ and $b_1 = b_2 = b_3 =: b$. Then

$$Q_2 = -12b^2 - 2b + 16\varepsilon - \frac{3}{4}.$$

In particular, the maximum is attained at $b = -1/12$ and is given by

$$Q_2 = \frac{48\varepsilon - 2}{3}. \quad (4.7)$$

Actually a (long) computation gives that the maximum of the function Q_2 defined for general variables $(a_1, a_2, b_1, b_2, b_3)$ is attained at the point

$$(a_1, a_2, b_1, b_2, b_3) = \left(1, 1, -\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}\right) \quad (4.8)$$

and is given by the value (4.7). Moreover, under the choice (4.8), one has

$$\frac{2 \left((3 - 4s)(1 + a_1^2 + a_2^2) + 4(1 - s)(a_1 + a_2 + a_1 a_2) \right)}{1 + a_1^2 + a_2^2} = 2(7 - 8s).$$

In particular, choosing

$$s = \frac{7}{8},$$

from (4.6) we obtain

$$0 \geq \frac{1}{3} \int |F|^2 dV + \frac{48\varepsilon - 2}{3} \int R |\overset{\circ}{Ric}|^2 dV.$$

Thus, if $\varepsilon > 1/24$, then $\overset{\circ}{Ric} \equiv 0$, i.e. (M^4, g) is Einstein. By Berger classification result [1] we conclude the proof of Theorem 1.1 in this case.

If $\varepsilon = 1/24$, then $Q_1 = 1/3$, $Q_2 = 0$ and all previous inequalities become equalities. In particular, $F \equiv 0$. Moreover, from (4.5), we get

$$R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \equiv \frac{1}{12} R |\overset{\circ}{Ric}|^2 \quad \text{and} \quad \overset{\circ}{R}_{ij} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jk} \equiv 0. \quad (4.9)$$

From equation (4.3) and Lemma 3.4 we get

$$\int \nabla_k \overset{\circ}{R}_{ij} \nabla_j \overset{\circ}{R}_{ik} dV = -\frac{1}{24} \int R |\overset{\circ}{Ric}|^2 dV = -\frac{1}{48} \int |\nabla R|^2 dV.$$

Thus, equation (4.2) gives

$$\int |\nabla \overset{\circ}{Ric}|^2 dV = \frac{1}{12} \int |\nabla R|^2 dV. \quad (4.10)$$

Now, to conclude, we have to use the fact that $F \equiv 0$, i.e.

$$0 = \nabla_k \overset{\circ}{R}_{ij} + \nabla_j \overset{\circ}{R}_{ik} + \nabla_i \overset{\circ}{R}_{jk} - \frac{1}{12} (\nabla_k R g_{ij} + \nabla_j R g_{ik} + \nabla_i R g_{jk}) .$$

Taking the diverge in k and contracting with $\overset{\circ}{R}_{ij}$, we obtain

$$\begin{aligned} 0 &= \overset{\circ}{R}_{ij} \left[\Delta \overset{\circ}{R}_{ij} + \nabla_k \nabla_j \overset{\circ}{R}_{ik} + \nabla_k \nabla_i \overset{\circ}{R}_{jk} - \frac{1}{12} (\Delta R g_{ij} + 2 \nabla_i \nabla_j R) \right] \\ &= \frac{1}{2} \Delta |\overset{\circ}{Ric}|^2 - |\nabla \overset{\circ}{Ric}|^2 + \overset{\circ}{R}_{ij} \left[\nabla_j \nabla_k \overset{\circ}{R}_{ik} + \nabla_i \nabla_k \overset{\circ}{R}_{jk} - \frac{1}{6} \nabla_i \nabla_j R \right] - 2 R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \\ &= \frac{1}{2} \Delta |\overset{\circ}{Ric}|^2 - |\nabla \overset{\circ}{Ric}|^2 + \frac{1}{3} \overset{\circ}{R}_{ij} \nabla_i \nabla_j R - 2 R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \\ &= \frac{1}{2} \Delta |\overset{\circ}{Ric}|^2 - |\nabla \overset{\circ}{Ric}|^2 + \frac{1}{3} \overset{\circ}{R}_{ij} \nabla_i \nabla_j R - \frac{1}{6} R |\overset{\circ}{Ric}|^2 , \end{aligned}$$

where we used (4.9). Integrating by parts over M , using (4.10), we obtain

$$0 = -\frac{1}{6} \int |\nabla R|^2 dV - \frac{1}{6} \int R |\overset{\circ}{Ric}|^2 dV$$

which implies $\overset{\circ}{Ric} \equiv 0$, i.e. (M^4, g) is Einstein and the thesis follows again by Berger result. This concludes the proof of Theorem 1.1.

Acknowledgments. *The author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).*

References

- [1] M. Berger, *Sur quelques varietes d'Einstein compactes*, Ann. Mat. Pura Appl. (4) **53** (1961), 89-95.
- [2] C. Böhm and B. Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) **167** (2008), no. 3, 1079–1097.
- [3] S. Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. J. **151** (2010), no.1, 1–21.
- [4] S. Brendle and R. Schoen, *Classification of manifolds with weakly $\frac{1}{4}$ -pinched curvatures*, Acta Math. **200** (2008), 1–13.
- [5] H.-D. Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 1–38.
- [6] X. Cao and P. Wu, *Einstein four-manifolds of three-nonnegative curvature operator*, Unpublished, 2014.
- [7] G. Catino, *Some rigidity results on critical metrics for quadratic functionals*, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2921–2937.
- [8] E. Costa *On Einstein four-manifolds*. J. Geom. Phys. **51** (2004), no. 2, 244–255.
- [9] M. Eminenti, G. La Nave, and C. Mantegazza, *Ricci solitons: the equation point of view*, Manuscripta Math. **127** (2008), no. 3, 345–367.
- [10] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, 1990.
- [11] R. S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.
- [12] X. Li, L. Ni and K. Wang, *Four-dimensional gradient shrinking solitons with positive isotropic curvature*, Int. Math. Res. Not. **2018** (2018), no.3, 949–959.
- [13] M. J. Micallef and Y. Wang, *Metrics with nonnegative isotropic curvature*, Duke Math. J. **72** (1993), 649–672.
- [14] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153.
- [15] S. Pigola, M. Rimoldi and A. Setti, *Remarks on non-compact gradient Ricci solitons*, Math. Z. **268** (2011), no. 3-4, 777–790.
- [16] E. Ribeiro, *Rigidity of four-dimensional compact manifolds with harmonic Weyl tensor*, Ann. Mat. Pura Appl. **195** (2016), no. 6, 2171–2181.
- [17] O. Munteanu and J. Wang, *Positively curved shrinking Ricci solitons are compact*, J. Differential Geom. **106** (2017), no. 3, 499–505.
- [18] S. Tachibana, *A theorem of Riemannian manifolds of positive curvature operator*, Proc. Japan Acad. **50** (1974), 301–302.
- [19] D. Yang, *Rigidity of Einstein 4-manifolds with positive curvature*, Invent. Math. **142** (2000), no. 2, 435–450.