Research Article

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Semigroups of pathological sets

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Abstract: In this article, we demonstrate how the Vitali and Bernstein constructions together with a simple theory on semigroups and ideals of sets can be used for producing different semigroups of sets without the Baire property and/or non-measurable in the Lebesgue sense.

Keywords: Sets with the Baire property, measurable sets in the Lebesgue sense, Vitali selectors, Bernstein sets, semigroups of sets

MSC 2010: 54A10

Dedicated to Professor Alexander Kharazishvili on the occasion of his 70th birthday

1 Introduction

The family $\mathcal{B}_p(\mathbb{R})$ of subsets of the real line $\mathbb{R}$ with the Baire property and the family $\mathcal{L}(\mathbb{R})$ of subsets of $\mathbb{R}$ measurable in the Lebesgue sense are well known in topology and analysis (cf. [5, 7, 8, 12], etc). They have a number of similarities. Thus the families $\mathcal{B}_p(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ are σ -algebras of sets on $\mathbb{R}$, i.e., they are closed under all basic set theoretic operations. Moreover, $\mathcal{B}_p(\mathbb{R})$ is the minimal σ -algebra of sets which contains the family $\mathcal{B}(\mathbb{R})$ of all Borel subsets of $\mathbb{R}$ and all meager sets on $\mathbb{R}$, while $\mathcal{L}(\mathbb{R})$ is the minimal σ -algebra of sets which contains the family $\mathcal{B}(\mathbb{R})$ and all subsets of $\mathbb{R}$ having the Lebesgue measure equal to zero (*null sets*). None of the families contains Vitali sets as well as Bernstein sets of the real line. Furthermore, the families $\mathcal{B}_p(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ are invariant under action of the group $\mathcal{T}(\mathbb{R})$ of all translations of $\mathbb{R}$, i.e., if $A \in \mathcal{B}_p(\mathbb{R})$ (resp. $A \in \mathcal{L}(\mathbb{R})$) and $h \in \mathcal{T}(\mathbb{R})$, then $h(A) \in \mathcal{B}_p(\mathbb{R})$ (resp. $h(A) \in \mathcal{L}(\mathbb{R})$).

Despite these similarities, the families $\mathcal{B}_p(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ are incomparable by inclusion. Indeed, recall (cf. [12]) that each subset of $\mathbb{R}$ can be decomposed into a disjoint union of a first category set and a null set. So if we decompose a Vitali subset (or a Bernstein subset) of $\mathbb{R}$ into a disjoint union of a meager set and a null set, then one of the sets of this decomposition must be Lebesgue non-measurable but meager, and the other must be a null set but without the Baire property. Thus none of the families $\mathcal{B}_p(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ contains the other (the same is evidently valid for their complements $\mathcal{B}_p^c(\mathbb{R})$ and $\mathcal{L}^c(\mathbb{R})$ in the family $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$). Let us mention a difference between $\mathcal{B}_p(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ (resp. $\mathcal{B}_p^c(\mathbb{R})$ and $\mathcal{L}^c(\mathbb{R})$). The first family is evidently invariant under self-homeomorphisms of the real line, but the second one is not.

Now we will name a similarity of the families $\mathcal{B}_p^c(\mathbb{R})$ and $\mathcal{L}^c(\mathbb{R})$ which is fundamental for us. Unlike the pair $\mathcal{B}_p(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$, the families $\mathcal{B}_p^c(\mathbb{R})$ and $\mathcal{L}^c(\mathbb{R})$ are not closed under such basic set theoretic operations as the union and intersection of sets. So, in the study of $\mathcal{B}_p^c(\mathbb{R})$ and $\mathcal{L}^c(\mathbb{R})$, various authors pay more attention to their elements with very curious properties (cf. [5]) than to the families themselves. Differently, we are

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interested in the properties of the family $\mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{L}^c(\mathbb{R})$). Thus, in [4] (see also [11]), the following project was started.

Look for sufficiently rich subfamilies of $\mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{L}^c(\mathbb{R})$ or $\mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$) which have some algebraic structures and which are invariant under action of infinite subgroups of the group $\mathcal{H}(\mathbb{R})$ of all homeomorphisms of $\mathbb{R}$.

Example 1.1. Consider a subgroup $\mathcal{G}$ of the additive group $(\mathbb{R}, +)$ of reals which is a Bernstein set (hence $\mathcal{G}$ is an element of $\mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$) and for which cardinality of the factor group $(\mathbb{R}, +)/\mathcal{G}$ is continuum $\mathfrak{c}$ (cf. [5]). Then let $\mathcal{F}$ be the family of all cosets of $\mathcal{G}$ in $\mathbb{R}$. So $\mathcal{F}$ is invariant under translations of $\mathbb{R}$. Since elements of $\mathcal{F}$ are Bernstein sets (as translations of $\mathcal{G}$), $\mathcal{F}$ is a subfamily of $\mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$. Moreover, $\mathcal{F}$ is an abelian group with the group operation “ $*$ ” defined as follows. For $F_1, F_2 \in \mathcal{F}$ such that $F_1 = x_1 + \mathcal{G}$ and $F_2 = x_2 + \mathcal{G}$ for some $x_1, x_2 \in \mathbb{R}$, we have $F_1 * F_2 = (x_1 + x_2) + \mathcal{G}$. Let us note that $F_1 * F_2 = F_1 + F_2 = \{y_1 + y_2 : y_1 \in F_1, y_2 \in F_2\}$. In particular, if $x \notin \mathcal{G}$, then $\mathcal{G} * (x + \mathcal{G}) = x + \mathcal{G}$ and the set $x + \mathcal{G}$ is equal to neither $\mathcal{G} \cup (x + \mathcal{G}) = \mathcal{G} \Delta (x + \mathcal{G})$ nor $\mathcal{G} \cap (x + \mathcal{G}) = \emptyset$. So the operation “ $*$ ” is not elementary set theoretic.

It is natural to ask if we can find subfamilies of $\mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{L}^c(\mathbb{R})$ or $\mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$) endowed with some elementary set theoretic operation(s).

Proposition 1.2. Let $\mathcal{H}_1$ be a non-trivial subgroup of $\mathcal{H}(\mathbb{R})$, and let $\mathcal{A}$ be a subfamily of $\mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{L}^c(\mathbb{R})$) invariant under action of $\mathcal{H}_1$. Assume that, for each $n \geq 2$ and each $A_1, \dots, A_n \in \mathcal{A}$, we have $\bigcup_{i=1}^n A_n \in \mathcal{B}_p^c(\mathbb{R})$ (resp. $\bigcup_{i=1}^n A_n \in \mathcal{L}^c(\mathbb{R})$). Then the family $\mathcal{S}_{\mathcal{A}}$ consisting of all unions of finitely many elements of $\mathcal{A}$ is an abelian semigroup of sets with respect to the operation union of sets. Moreover, $\mathcal{S}_{\mathcal{A}}$ is invariant under action of $\mathcal{H}_1$, and $\mathcal{S}_{\mathcal{A}} \subseteq \mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{S}_{\mathcal{A}} \subseteq \mathcal{L}^c(\mathbb{R})$).

In addition, the dual family $\mathcal{S}_{\mathcal{A}}^* = \{\mathbb{R} \setminus A : A \in \mathcal{S}_{\mathcal{A}}\}$ is also an abelian semigroup of sets with respect to the operation intersection of sets. Moreover, $\mathcal{S}_{\mathcal{A}}^*$ is invariant under action of $\mathcal{H}_1$, and $\mathcal{S}_{\mathcal{A}}^* \subseteq \mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{S}_{\mathcal{A}}^* \subseteq \mathcal{L}^c(\mathbb{R})$).

Let us note that the Bernstein sets as well as the Vitali sets provide families satisfying Proposition 1.2 (see below).

Example 1.3. The family $\mathcal{F} \subseteq \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$ considered in Example 1.1 is invariant under translations of the real line, and for each $n \geq 2$ and each $F_1, \dots, F_n \in \mathcal{F}$ we have $\bigcup_{i=1}^n F_n \in \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$ (the union is evidently a Bernstein set). So the family $\mathcal{S}_{\mathcal{F}}$ consisting of all unions of finitely many elements of $\mathcal{F}$ is an abelian semigroup of sets with respect to the operation union of sets. Moreover, $\mathcal{S}_{\mathcal{F}}$ is invariant under translations of $\mathbb{R}$ and $\mathcal{S}_{\mathcal{F}} \subseteq \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$.

Similarly, the dual family $\mathcal{S}_{\mathcal{F}}^* = \{\mathbb{R} \setminus F : F \in \mathcal{S}_{\mathcal{F}}\}$ is also an abelian semigroup of sets with respect to the operation intersection of sets which is invariant under translations of $\mathbb{R}$ and for which $\mathcal{S}_{\mathcal{F}}^* \subseteq \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$.

Example 1.4. The family $\mathcal{V} \subseteq \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$ of all Vitali sets of the real line is invariant under translations of the real line, and for each $n \geq 2$ and each $V_1, \dots, V_n \in \mathcal{V}$, we have $\bigcup_{i=1}^n V_n \in \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$ (see [3] and [6], respectively). So the family $\mathcal{S}_{\mathcal{V}}$ consisting of all unions of finitely many elements of $\mathcal{V}$ is an abelian semigroup of sets with respect to the operation union of sets. Moreover, $\mathcal{S}_{\mathcal{V}}$ is invariant under translations of $\mathbb{R}$ and $\mathcal{S}_{\mathcal{V}} \subseteq \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$.

Similarly, the dual family $\mathcal{S}_{\mathcal{V}}^* = \{\mathbb{R} \setminus V : V \in \mathcal{S}_{\mathcal{V}}\}$ is also an abelian semigroup of sets with respect to the operation intersection of sets which is invariant under translations of $\mathbb{R}$ and for which $\mathcal{S}_{\mathcal{V}}^* \subseteq \mathcal{B}_p^c(\mathbb{R}) \cap \mathcal{L}^c(\mathbb{R})$.

Let us point out some evident differences between the semigroups $\mathcal{S}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{V}}$ from the examples above. As we already observed, each element of the family $\mathcal{S}_{\mathcal{F}}$ is a Bernstein set, but elements of $\mathcal{S}_{\mathcal{V}}$ are neither Bernstein sets nor Vitali sets as a rule. Furthermore, each element of $\mathcal{S}_{\mathcal{F}}$ as a Bernstein set must be everywhere dense in $\mathbb{R}$; however, each element of $\mathcal{S}_{\mathcal{V}}$ does not need to be everywhere dense in $\mathbb{R}$.

In this article, we will demonstrate how the ideas of Proposition 1.2, Examples 1.3, 1.4 and some simple theory on semigroups and ideals of sets can be used for producing various rich semigroups of sets without the Baire property (see also [1, 4, 11]) and/or non-measurable in the Lebesgue sense. The mutual relationship between the semigroups of sets are also under our consideration.

2 Elementary algebraic objects in set theory

By an algebraic property on a family of sets we mean any elementary set theoretic operation which is closed on the family. We continue with several known types of families of sets having algebraic properties.

2.1 Rings and algebras

Let X be a non-empty set and let $\mathcal{P}(X)$ be the family of all subsets of X . A non-empty family $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a *ring of sets* on X if $A \Delta B \in \mathcal{R}$ and $A \cap B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$. Since $A \cup B = (A \Delta B) \Delta (A \cap B)$, $A \setminus B = A \Delta (A \cap B)$, we have also $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$ for any $A, B \in \mathcal{R}$. Since $A \setminus A = \emptyset$ for any $A \in \mathcal{R}$, each ring of sets contains the empty set $\emptyset$. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a ring of sets on X . The family $\mathcal{A}$ is called an *algebra of sets* on X if $X \in \mathcal{A}$. In particular, for each element $A \in \mathcal{A}$, its complement $A^c = X \setminus A$ in X is also an element of $\mathcal{A}$. A ring $\mathcal{R}$ of sets on X is called a σ -*ring of sets* on X if $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ whenever $A_i \in \mathcal{R}$ for each $i \geq 1$. A σ -ring of sets $\mathcal{A}$ on X is called a σ -*algebra of sets* on X if $X \in \mathcal{A}$. It is easy to see that each σ -algebra of sets on X is also closed under countable intersections of sets. Let $\mathcal{A} \subseteq \mathcal{P}(X)$. The smallest σ -ring (resp. σ -algebra) of sets on X containing $\mathcal{A}$ is called the σ -*ring* (resp. the σ -*algebra*) of sets on X generated by the family $\mathcal{A}$.

2.2 Ideals

A non-empty family $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal of sets* on X if $\mathcal{I}$ satisfies the following conditions:

- (a) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;
- (b) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

Let us note that each ideal of sets on X contains the empty set $\emptyset$ as an element. An ideal of sets $\mathcal{I}$ on X is called a σ -*ideal of sets* on X if it is closed under countable unions.

Example 2.1. Consider some ideals of sets.

- (i) The family of all finite subsets of a set X is an ideal of sets on X . We will denote the ideal by $\mathcal{I}_f$.
- (ii) The family of all bounded subsets of the reals $\mathbb{R}$ is an ideal of sets on $\mathbb{R}$. We will denote the ideal by $\mathcal{I}_b$.
- (iii) The family of all countable subsets of a set X is a σ -ideal of sets on X . We will denote the ideal by $\mathcal{I}_c$.
- (iv) Let $A \subseteq X$. Then the family $\mathcal{I}_A = \{B : B \subseteq A\}$ is a σ -ideal of sets on X .

Let $\mathcal{I}$ be an ideal of sets on a set X and $A, B \subseteq X$. We say (cf. [8]) that $A = B$ *modulo* $\mathcal{I}$ (in brief, $A = B \bmod \mathcal{I}$) if $A \setminus B$ and $B \setminus A$ are elements of $\mathcal{I}$ (or, equivalently, $A \Delta B \in \mathcal{I}$).

Proposition 2.2 ([8]). *For subsets A, B of X , the following statements are equivalent.*

- (i) $A = B \bmod \mathcal{I}$.
- (ii) *There are two elements M, N of $\mathcal{I}$ such that $A = (B \setminus M) \cup N$.*
- (iii) *There is an element P of $\mathcal{I}$ such that $A = B \Delta P$.*

Proposition 2.3 ([8]). *If $A_i = B_i \bmod \mathcal{I}$, $i = 1, 2$, then $A_1 \cup A_2 = B_1 \cup B_2 \bmod \mathcal{I}$. Moreover, if $\mathcal{I}$ is a σ -ideal of sets on X and $A_i = B_i \bmod \mathcal{I}$ for each integer $i \geq 1$, then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \bmod \mathcal{I}$.*

The following statement shows how to produce from one set outside an algebra a family of sets outside the algebra.

Proposition 2.4. *Let $\mathcal{I}$ be an ideal of sets on X . Let $\mathcal{A}$ be an algebra of sets on X containing $\mathcal{I}$, and let $Y \subseteq X$ be such that $Y \notin \mathcal{A}$. Then, for every $I \in \mathcal{I}$, we have $Y \setminus I, Y \cup I, Y \Delta I \notin \mathcal{A}$.*

2.3 Semigroups of sets

Rings of sets, algebras of sets or ideals of sets are used in topology and analysis (cf. [7]). In this section, we consider semigroups of sets with respect to a set theoretic operation.

Let us recall that a non-empty set S is called a *semigroup* if there is a binary operation $*$: $S \times S \rightarrow S$ for which the equality $(x * y) * z = x * (y * z)$ holds for all $x, y, z \in S$. The semigroup S is *abelian* if $x * y = y * x$ for all $x, y \in S$.

Since the operation union (resp. intersection) of sets is associative, we have the following natural definition. A non-empty family $S \subseteq \mathcal{P}(X)$ of sets is called a *semigroup of sets with respect to the operation union (resp. intersection) of sets* if, for each pair of elements $A, B \in S$, we have $A \cup B \in S$ (resp. $A \cap B \in S$). Let us note that the semigroups of sets are abelian.

A semigroup of sets with respect to the operation union (resp. intersection) of sets on X is called a *σ -semigroup of sets with respect to the operation union (resp. intersection) of sets* if S is closed under unions (resp. intersections) of countably many sets.

Note that if a proper subfamily S of $\mathcal{P}(X)$ is a (σ) -semigroup of sets with respect to the operation union of sets, then the dual family $S^* = \{X \setminus S : S \in S\}$ is also a (σ) -semigroup of sets with respect to the operation intersection of sets, and vice versa.

In this article, we will discuss mostly (σ) -semigroups of sets with respect to the operation union of sets (shortly, (σ) -semigroups of sets).

Example 2.5. Let $\mathcal{A} \subseteq \mathcal{P}(X)$.

(i) Put $S_{\mathcal{A}} = \{\bigcup_{i=1}^n A_i : A_i \in \mathcal{A}, n \in \mathbb{N}\}$ (resp. $S_{\mathcal{A}}^{\sigma} = \{\bigcup_{i=1}^{\infty} A_i : A_i \in \mathcal{A}\}$), and note that the family $S_{\mathcal{A}}^{(\sigma)}$ is a (σ) -semigroup of sets on X . We will call $S_{\mathcal{A}}^{(\sigma)}$ a *(σ) -semigroup of sets on X generated by the family $\mathcal{A}$* .

(ii) Put

$$\mathcal{I}_{\mathcal{A}} = \{B \in \mathcal{P}(X) : \text{there is } A \in S_{\mathcal{A}} \text{ such that } B \subseteq A\}$$

(resp. $\mathcal{I}_{\mathcal{A}}^{\sigma} = \{B \in \mathcal{P}(X) : \text{there is } A \in S_{\mathcal{A}}^{\sigma} \text{ such that } B \subseteq A\}$) and note that the family $\mathcal{I}_{\mathcal{A}}^{(\sigma)}$ is an (σ) -ideal of sets on X . If $\mathcal{A} = \{A\}$, where $A \subseteq X$, we denote $\mathcal{I}_{\mathcal{A}}$ by $\mathcal{I}_A$. It is evident that $\mathcal{I}_A$ is a σ -ideal of sets on X . We will call $\mathcal{I}_A^{(\sigma)}$ a *(σ) -ideal of sets on X generated by the family $\mathcal{A}$* .

2.4 New semigroups of sets from old by ideals of sets

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. Put

(i) $\mathcal{A} \cup \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$,

(ii) $\mathcal{A} \Delta \mathcal{B} = \{A \Delta B : A \in \mathcal{A}, B \in \mathcal{B}\}$,

(iii) $\mathcal{A} * \mathcal{B} = \{(A \setminus B_1) \cup B_2 : A \in \mathcal{A}; B_1, B_2 \in \mathcal{B}\}$.

Propositions 2.2 and 2.3 imply the following statement.

Proposition 2.6 ([1]). *Let S be a (σ) -semigroup of sets, and let $\mathcal{I}$ be an (σ) -ideal of sets. Then the families $S * \mathcal{I}$, $\mathcal{I} * S$ are (σ) -semigroups of sets such that the following holds:*

(i) $S * \mathcal{I} = S \Delta \mathcal{I}$ and $S \cup \mathcal{I} = \mathcal{I} * S$;

(ii) $S * \mathcal{I} \supseteq \mathcal{I} * S \supseteq S$; moreover, if $\mathcal{I}$ contains an element of S , then $S * \mathcal{I} \supseteq \mathcal{I}$; furthermore, $\mathcal{I} * S \supseteq \mathcal{I}$ if and only if S contains the empty set as an element;

(iii) $(S * \mathcal{I}) * \mathcal{I} = S * \mathcal{I}$ and $\mathcal{I} * (\mathcal{I} * S) = \mathcal{I} * S$;

(iv) $(\mathcal{I} * S) * \mathcal{I} = \mathcal{I} * (S * \mathcal{I}) = S * \mathcal{I}$.

Remark 2.7. If $\mathcal{I}_1, \mathcal{I}_2$ are (σ) -ideals of sets, then the family $\mathcal{I}_1 * \mathcal{I}_2$ is an (σ) -ideal of sets. Moreover, we have $\mathcal{I}_1 * \mathcal{I}_2 = \mathcal{I}_2 * \mathcal{I}_1 = \mathcal{I}_1 \Delta \mathcal{I}_2 = \mathcal{I}_1 \cup \mathcal{I}_2$.

Simple conditions allowing to distinguish the semigroups $S \Delta \mathcal{I}$ and $S \cup \mathcal{I}$ can be found in the following statement.

Proposition 2.8. *Let S be a semigroup of sets and $\mathcal{I}$ be an ideal of sets.*

(i) *If there exist an $S \in S$ and a point $x \in S$ such that $S \setminus \{x\} \notin S \cup \mathcal{I}$ and $\{x\} \in \mathcal{I}$, then $S \Delta \mathcal{I} \neq S \cup \mathcal{I}$.*

(ii) *If $\mathcal{I}$ contains an element of S and for each $S \in S$ we have $S \neq \emptyset$, then $S \Delta \mathcal{I} \neq S \cup \mathcal{I}$.*

For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, define $\mathcal{A} \cap \mathcal{B} = \{Y : Y \in \mathcal{A} \text{ and } Y \in \mathcal{B}\}$.

The following statement can be used in the constructions of various semigroups of sets without the Baire property.

Proposition 2.9 ([1]). *Let $\mathcal{I}$ be an ideal of sets, and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ be such that*

- (i) $\mathcal{A} \cap \mathcal{I} = \emptyset$,
- (ii) *for each element $U \in \mathcal{S}_{\mathcal{A}}$ and each non-empty element $B \in \mathcal{B}$, there exists an element $A \in \mathcal{A}$ such that $A \subseteq B \setminus U$.*

Then $(\mathcal{S}_{\mathcal{A}} * \mathcal{I}) \cap (\mathcal{S}_{\mathcal{B}} * \mathcal{I}) = \emptyset$.

3 Facts from measure theory and category

3.1 Lebesgue measurable sets on the real line

Recall (cf. [12]) that the *outer measure* m^* of a subset A of the real line $\mathbb{R}$ is defined by the formula $m^*(A) = \inf\{\sum |I_i| : A \subseteq \cup I_i\}$, where the infimum is taken over all countable coverings of A by open intervals, and $|I| = b - a$ whenever $I = (a, b)$.

A subset A of $\mathbb{R}$ is called a *null set* if $m^*(A) = 0$. It is easy to see that the family $\mathcal{N}(\mathbb{R})$ of all null sets of $\mathbb{R}$ is a σ -ideal of sets on $\mathbb{R}$. Moreover, $\mathcal{N}(\mathbb{R})$ is invariant under translations of $\mathbb{R}$.

Let us recall that the outer measure m^* possesses the following evident properties:

- (a) if $A \subseteq B$, then $m^*(A) \leq m^*(B)$ (the monotonicity);
- (b) if $A = \bigcup_{i=1}^{\infty} A_i$, then $m^*(A) \leq \sum_{i=1}^{\infty} m^*(A_i)$ (the countable subadditivity);
- (c) $m^*(A) = m^*(x + A)$ for each $x \in \mathbb{R}$ (the invariance under translations of $\mathbb{R}$);
- (d) $m^*(I) = |I|$ for each interval I of $\mathbb{R}$.

A subset A of the real line $\mathbb{R}$ is said to be *measurable in the sense of Lebesgue*, or simply *measurable*, if, for each $\epsilon > 0$, there exists a closed set F of $\mathbb{R}$ and an open set G of $\mathbb{R}$ such that $F \subseteq A \subseteq G$ and $m^*(G - F) < \epsilon$.

Proposition 3.1 (cf. [12]). *A subset A of $\mathbb{R}$ is measurable if and only if A can be represented as the disjoint union $F \cup M$ of an F_{σ} -set F and a null set M (or as the difference $G \setminus N$ of a G_{δ} -set G and a null set N).*

Since each open (resp. closed) subset of $\mathbb{R}$ and each null subset of $\mathbb{R}$ are elements of $\mathcal{L}(\mathbb{R})$, it follows from Proposition 3.1 that $\mathcal{L}(\mathbb{R})$ is the smallest σ -algebra of sets in $\mathbb{R}$ containing all open (resp. closed) and all null subsets of $\mathbb{R}$. Recall that the smallest σ -algebra of sets containing all open subsets of $\mathbb{R}$ is the algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$. So $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$. The restriction m of m^* on the family $\mathcal{L}(\mathbb{R})$ is called the *Lebesgue measure* on $\mathbb{R}$.

Let us note that, besides the mentioned earlier properties (a)–(d) of m^* , for the function m , we have additionally the following: if $A_1, A_2, \dots$ are disjoint measurable sets and $A = \bigcup_{i=1}^{\infty} A_i$, then $m(A) = \sum_{i=1}^{\infty} m(A_i)$ (the countable additivity).

3.2 Meager sets in topological spaces

Let X be a topological space and $A \subseteq X$.

A subset A of X is called *discrete* if, for each point $x \in A$, there exists an open neighborhood U of x such that $U \cap A = \{x\}$. Note that the family of all closed and discrete subsets of X forms an ideal of sets. We will denote the ideal by $\mathcal{I}_{cd}$.

A subset A of X is said to be *nowhere dense* in X if $\text{Int}_X(\text{Cl}_X(A)) = \emptyset$. Note that the family of nowhere dense sets in a given topological space is an ideal of sets. We will denote the ideal by $\mathcal{I}_n$.

A subset A of X is said to be *dense* in X if $\text{Cl}_X(A) = X$. It is easy to see that if a set A is a nowhere dense subset of X , then the set $X \setminus A$ is dense in X , and if a set B is open and dense in X , then $X \setminus B$ is nowhere dense in X .

A subset A of X is said to be *meager* in X if A is the union of countably many nowhere dense subsets of X . Let us note that the family of all meager sets in the space X is a σ -ideal of sets on X . We will denote the ideal by $\mathcal{M}(X)$.

Remark 3.2. For the real line $\mathbb{R}$, we have the following:

$$\mathcal{I}_f \not\subseteq \mathcal{I}_{cd} \not\subseteq \mathcal{I}_c \not\subseteq \mathcal{M}(\mathbb{R}) \not\subseteq \mathcal{P}(\mathbb{R}) \quad \text{and} \quad \mathcal{I}_{cd} \not\subseteq \mathcal{I}_n \not\subseteq \mathcal{M}(\mathbb{R}).$$

Since the standard Cantor set $\mathbb{C}$ on the interval $[0, 1]$ is an uncountable nowhere dense subset of $\mathbb{R}$, while the set $\mathbb{Q}$ of rational numbers is a countable dense subset of $\mathbb{R}$, the ideals $\mathcal{I}_c$ and $\mathcal{I}_n$ are not comparable in the sense of inclusion.

3.3 Sets with Baire property in topological spaces

Let X be a topological space with the topology τ_X .

A subset A of X is said to *have the Baire property* in X if $A = O \Delta M$ for some $O \in \tau_X$ and some $M \in \mathcal{M}(X)$. Denote the family of all sets having the Baire property in a topological space X by $\mathcal{B}_p(X)$. Proposition 2.2 easily implies the following.

Proposition 3.3. *For a subset A of X , the following statements are equivalent.*

- (i) A has the Baire property.
- (ii) $A = V \bmod \mathcal{M}(X)$ for some $V \in \tau_X$.
- (iii) $A = (W \setminus P) \cup Q$ for some $W \in \tau_X$ and $P, Q \in \mathcal{M}(X)$.

Similarly, one can introduce and describe the sets with the Baire property on a topological space via closed sets of the space.

Proposition 3.4 (cf. [12]). *A subset A of X has the Baire property if and only if A can be represented as the disjoint union $G \cup R$ of a G_δ -set G of X and $R \in \mathcal{M}(X)$ (or as the difference $F \setminus S$ of an F_σ -set F of X and $S \in \mathcal{M}(X)$).*

Since each open (resp. closed) subset of X and each meager subset of X are elements of $\mathcal{B}_p(X)$, it follows from Proposition 3.4 that $\mathcal{B}_p(X)$ is the smallest σ -algebra of sets in X containing all open (resp. closed) and all meager subsets of X . In particular, we have $\mathcal{B}(X) \subseteq \mathcal{B}_p(X)$.

Remark 3.5. The families $\mathcal{B}(X)$, $\mathcal{B}_p(X)$ are invariant under action of the group $\mathcal{H}(X)$ of all self-homeomorphisms of the space X .

Proposition 3.6. *Let $A \subseteq \mathbb{R}$, and let M, N be subsets of $\mathbb{R}$ such that $M \in \mathcal{M}(\mathbb{R})$, $N \in \mathcal{N}(\mathbb{R})$ and $M \cup N = \mathbb{R}$ (see [12]). Then the following holds: if $A \supseteq M$, then $A \in \mathcal{L}(\mathbb{R})$, and if $A \supseteq N$, then $A \in \mathcal{B}_p(\mathbb{R})$.*

4 Pathological sets of the real line

4.1 Vitali selectors

Below Q is any countable dense subgroup of the additive group $(\mathbb{R}, +)$.

Following [5] (see also [11]), let us consider the following equivalence relation $E(Q)$ on $\mathbb{R}$: for $x, y \in \mathbb{R}$, let $x E(Q) y$ if and only if $x - y \in Q$. Denote by $E_\alpha(Q)$, $\alpha \in I$, the equivalence classes of the relation. Observe that $|I| = \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of continuum, and for each $\alpha \in I$ and each $x \in E_\alpha(Q)$, we have $E_\alpha(Q) = Q + x$ (so $E_\alpha(Q)$ is dense in $\mathbb{R}$).

A *Vitali Q -selector* (shortly, a *Vitali selector*) of $\mathbb{R}$ is any subset V of $\mathbb{R}$ such that $|V \cap E_\alpha(Q)| = 1$ for each $\alpha \in I$. If Q is the group $\mathbb{Q}$ of rational numbers, then the Vitali Q -selectors of $\mathbb{R}$ are Vitali sets [13]. It is easy to see that, for each point $p \in \mathbb{R}$, there exists a Vitali Q -selector which contains the point p , and for each Vitali Q -selector V we have $|V| = \mathfrak{c}$.

Proposition 4.1 (cf. [5]). *Let V be a Vitali Q -selector of $\mathbb{R}$.*

- (i) *Any proper subset of V is no Vitali Q -selector of $\mathbb{R}$.*
- (ii) *If $q_1, q_2 \in Q$ and $q_1 \neq q_2$, then $(V + q_1) \cap (V + q_2) = \emptyset$.*
- (iii) $\mathbb{R} = \bigcup_{q \in Q} (V + q)$.
- (iv) *V does not have the Baire property, and it is non-measurable in the sense of Lebesgue; in particular, $m^*(V) \neq 0$.*

Since the equivalence classes E_α , $\alpha \in I$, are dense in $\mathbb{R}$, $(\mathbb{R} \setminus V) \cap O \neq \emptyset$ for every non-empty open subset O of $\mathbb{R}$. In particular, $\dim V = 0$, where $\dim$ is the Lebesgue covering dimension (cf. [2]).

Proposition 4.2 (cf. [3]). *For any open non-empty subset O of $\mathbb{R}$, there is a Vitali selector V which is a (dense) subset of O .*

Corollary 4.3. *A closed subset A of $\mathbb{R}$ contains a Vitali selector if and only if $\text{Int}_{\mathbb{R}} A \neq \emptyset$.*

Proof. The necessity: If A contains a Vitali selector, then A is not meager. Since A is closed, $\text{Int}_{\mathbb{R}} A \neq \emptyset$.

The sufficiency: If $O = \text{Int}_{\mathbb{R}} A \neq \emptyset$, then, by Proposition 4.2, there exists a Vitali selector V such that $V \subseteq O \subseteq A$. $\square$

Proposition 4.4. *Any subset A of $\mathbb{R}$ such that $\text{Int}_{\mathbb{R}} A \neq \emptyset$ is a disjoint union of countably many Vitali Q -selectors.*

Proof. Let us note that $|E_\alpha(Q) \cap A| = \aleph_0$ for each $\alpha \in I$. Denote $E_\alpha(Q) \cap A = \{x_{\alpha,i}\}_{i=1}^\infty$. For each $i = 1, 2, \dots$, put $V_i = \{x_{\alpha,i} : \alpha \in I\}$. Observe that each V_i is a Vitali Q -selector, $\bigcup_{i=1}^\infty V_i = A$ and all V_i are disjoint. $\square$

Remark 4.5. Let us note that the family $\mathcal{A}$ of all bounded subsets of $\mathbb{R}$, which are measurable in the Lebesgue sense, is a ring of sets. Moreover, the Lebesgue measure m restricted to the ring $\mathcal{A}$ is finitely additive, translation invariant, and $m([0, 1]) = 1$. Observe that the family $\mathcal{B}$ of all bounded subsets of $\mathbb{R}$ is also a ring having the family $\mathcal{A}$ as a subring. The outer measure m^* on $\mathcal{B}$ can be considered as an extension of m on $\mathcal{A}$ which is translation invariant, and $m^*([0, 1]) = 1$, but not finitely additive, i.e., there are disjoint bounded subsets A and B of $\mathbb{R}$ such that $m^*(A \cup B) < m^*(A) + m^*(B)$.

Theorem 4.6 (Banach extension theorem, cf. [6, 14]). *Let $\mathcal{B}$ be the family of all bounded subsets of $\mathbb{R}$ and $\mathcal{A}$ a translation invariant ring of subsets of $\mathbb{R}$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $[0, 1] \in \mathcal{A}$. Let $v: \mathcal{A} \rightarrow [0, \infty)$ be a finitely additive translation invariant function such that $v([0, 1]) = 1$. Then there exists a finitely additive translation invariant extension $v': \mathcal{B} \rightarrow [0, \infty)$ of v .*

4.2 Bernstein sets

Recall that a subset B of the real line $\mathbb{R}$ is called a *Bernstein set* of $\mathbb{R}$ if both B and $B^c = \mathbb{R} \setminus B$ meet every uncountable closed subset of $\mathbb{R}$. It is easy to see that if B is a Bernstein set of $\mathbb{R}$, then B^c is also a Bernstein set of $\mathbb{R}$.

Proposition 4.7 (cf. [12]). *Bernstein sets of $\mathbb{R}$ exist.*

Remark 4.8. Let us recall that one can replace in the definition of the Bernstein sets the closed uncountable sets by the uncountable G_δ -sets or by the topological copies of the Cantor set $\mathbb{C}$.

Since $\mathbb{C}$ is homeomorphic to $\mathbb{C} \times \mathbb{C}$, each Bernstein set has cardinality continuum. It is easy to see that the Bernstein sets on the real line are also invariant under self-homeomorphisms of $\mathbb{R}$. Moreover, they are dense in $\mathbb{R}$ and zero-dimensional.

Propositions 3.1 and 3.4 easily imply the following statements.

Proposition 4.9 (cf. [12]). *Let B be a Bernstein set of the real line. Then the following holds.*

- (i) *Each Lebesgue measurable subset A of B is a null set of $\mathbb{R}$.*
- (ii) *Each subset U of B with the Baire property is a meager set of $\mathbb{R}$.*

Corollary 4.10 (cf. [12]). *Let B be a Bernstein set B . Then B is neither Lebesgue measurable nor with the Baire property.*

The following statement is important for Example 1.1.

Proposition 4.11 (cf. [5]). *There exist two subgroups G_1 and G_2 of the additive group $(\mathbb{R}, +)$ such that*

- (i) $G_1 \cap G_2 = \{0\}$,
- (ii) G_1, G_2 are Bernstein sets in the real line $\mathbb{R}$,
- (iii) $|(\mathbb{R}, +)/G_i| = \mathfrak{c}$ for each $i \in \{1, 2\}$.

Remark 4.12. The family of all cosets of the subgroup $\mathcal{G}_1$ (or $\mathcal{G}_2$) from Proposition 4.11 forms a decomposition of the reals into continuum many Bernstein sets (as translations of $\mathcal{G}_i$). Let us note that there exist decompositions of the real line $\mathbb{R}$ into countably (resp. finitely) many Bernstein sets with different arithmetic properties (see [10]).

Let us note that there are Vitali sets which are no Bernstein sets, and Bernstein sets which are no Vitali sets.

Proposition 4.13 (cf. [5]). *There is a subset A of $\mathbb{R}$ such that A is both a Vitali set and a Bernstein set.*

5 Vitali construction and semigroups of sets on the real line

5.1 Vitali Q -selectors of the reals and related semigroups of sets

Let V be a Vitali Q -selector of $\mathbb{R}$. Put $V_Q = \{V + q : q \in Q\}$, and denote by $\mathcal{T}_Q(\mathbb{R}) = \{T_q : q \in Q\}$ the countable subgroup of the group $\mathcal{T}(\mathbb{R})$ of all translations of the reals. (Here $T_x(y) = y + x$ for any $x, y \in \mathbb{R}$.)

Proposition 5.1. *The following statements hold.*

- (i) *The family $\mathcal{S}_{V_Q}$ of all non-empty unions of finitely many elements of V_Q is a countable semigroup of sets invariant under the action of $\mathcal{T}_Q(\mathbb{R})$.*
- (ii) *The family $\mathcal{S}_{V_Q}^\sigma$ of all non-empty unions of countably many elements of V_Q is a σ -semigroup of sets invariant under the action of $\mathcal{T}_Q(\mathbb{R})$. Moreover, $|\mathcal{S}_{V_Q}^\sigma| = \mathfrak{c}$, $\mathcal{S}_{V_Q} \subseteq \mathcal{S}_{V_Q}^\sigma$ and $\mathbb{R} \in \mathcal{S}_{V_Q}^\sigma \setminus \mathcal{S}_{V_Q}$.*

Remark 5.2. Let us note that all elements of $\mathcal{S}_{V_Q}^\sigma$ besides the set $\mathbb{R}$ are without the Baire property [3].

Lemma 5.3. *For each Vitali Q -selector V and each element $x \in \mathbb{R}$, the set $V + x$ is also a Vitali Q -selector.*

Let $\mathcal{V}(Q)$ be the family of all Vitali Q -selectors, and let $\mathcal{S}_{\mathcal{V}(Q)}$ (resp. $\mathcal{S}_{\mathcal{V}(Q)}^\sigma$) be the semigroup (resp. the σ -semigroup) of sets generated by $\mathcal{V}(Q)$.

It is easy to see that, for each Vitali Q -selector V , we have $V_Q \subseteq \mathcal{V}(Q)$, $\mathcal{S}_{V_Q} \subseteq \mathcal{S}_{\mathcal{V}(Q)}$ and $\mathcal{S}_{V_Q}^\sigma \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma$. Since $|\mathcal{V}(Q)| = 2^{\mathfrak{c}}$, we have $V_Q \neq \mathcal{V}(Q)$, $\mathcal{S}_{V_Q} \neq \mathcal{S}_{\mathcal{V}(Q)}$ and $\mathcal{S}_{V_Q}^\sigma \neq \mathcal{S}_{\mathcal{V}(Q)}^\sigma$.

Moreover, it is obvious that $\mathcal{S}_{\mathcal{V}(Q)} \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma$. Since $1 \leq |U \cap E_\alpha(Q)| < \aleph_0$ for each $U \in \mathcal{S}_{\mathcal{V}(Q)}$ and each $\alpha \in I$, we have additionally that $\mathcal{S}_{\mathcal{V}(Q)} \neq \mathcal{S}_{\mathcal{V}(Q)}^\sigma$.

Proposition 5.4. *The families $\mathcal{V}(Q)$, $\mathcal{S}_{\mathcal{V}(Q)}$ and $\mathcal{S}_{\mathcal{V}(Q)}^\sigma$ are invariant under translations of $\mathbb{R}$.*

Proposition 5.5. *Let $A \subseteq \mathbb{R}$. Then $A \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma$ if and only if $|A \cap (x + Q)| \neq \emptyset$ for each $x \in \mathbb{R}$, i.e., the set A must contain a Vitali Q -selector.*

Moreover, if $A = \bigcup_{i=1}^\infty V_i$, where each V_i is a Vitali Q -selector, then the Vitali Q -selectors $\{V_i\}_{i=1}^\infty$ can be chosen disjoint if and only if $|A \cap (x + Q)| = \aleph_0$ for each $x \in \mathbb{R}$.

Proof. We will show the sufficiency of the first statement. Indeed, for each $\alpha \in I$, let $A \cap E_\alpha(Q) = \{x_1^\alpha, x_2^\alpha, \dots\}$ (some elements in the sequence can be equal). Set $V_i = \{x_i^\alpha : \alpha \in I\}$ for each positive integer i . Note that V_i is a Vitali Q -selector of $\mathbb{R}$ and $\bigcup_{i=1}^\infty V_i = A$. $\square$

Corollary 5.6. *The following statements hold.*

- (i) *If $A \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma$ and $A \subseteq B \subseteq \mathbb{R}$, then $B \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma$.*
- (ii) *Every subset A of $\mathbb{R}$ with $\dim A = 1$ is the union of a countable disjoint family of Vitali Q -selectors, i.e., $\mathcal{S}_{\mathcal{V}(Q)}^\sigma$ contains all one-dimensional subsets of $\mathbb{R}$, in particular, all open non-empty sets of $\mathbb{R}$.*

Proof. (ii) Note that $\text{Int}_{\mathbb{R}} A \neq \emptyset$ (cf. [2, Brouwer dimension theorem for $\mathbb{R}$]). Hence $|E_\alpha(Q) \cap A| = \aleph_0$ for each $\alpha \in I$. Apply now Proposition 5.5. $\square$

Example 5.7. Let V be a Vitali Q -selector of $\mathbb{R}$ such that $0 \in V$ and $(-1, 1) \cap (V \setminus \{0\}) = \emptyset$. Consider any $p \in (0, 1) \setminus Q$, and define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \leq -1, \\ (p+1)x + p & \text{if } -1 \leq x \leq 0, \\ (1-p)x + p & \text{if } 0 \leq x \leq 1, \\ x & \text{if } x \geq 1. \end{cases}$$

Note that f is a self-homeomorphism of the real line and $f(V) \cap Q = \emptyset$. This implies that $f(V) \notin \mathcal{S}_{\mathcal{V}(Q)}^\sigma$, and hence the families $\mathcal{V}_Q$, $\mathcal{S}_{\mathcal{V}_Q}$, $\mathcal{V}(Q)$, $\mathcal{S}_{\mathcal{V}(Q)}$ and $\mathcal{S}_{\mathcal{V}(Q)}^\sigma$ are not invariant under self-homeomorphisms of the real line.

Proposition 5.8. *Let $\mathcal{I}$ be an ideal of sets on $\mathbb{R}$ such that $\mathcal{I} \neq \{\emptyset\}$. Then we have the following.*

- (i) *The families $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{I}$ are semigroups of sets, and*
 - (1) $\mathcal{S}_{\mathcal{V}(Q)} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{I}$,
 - (2) $\mathcal{S}_{\mathcal{V}(Q)}^\sigma = \mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{I} = \{A \setminus I : A \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma, I \in \mathcal{I}\}$,
 - (3) $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I} \setminus (\mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}) \neq \emptyset$.
- Moreover, if $\mathcal{I}$ contains some Vitali Q -selector of $\mathbb{R}$ as an element, then $\mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$. Furthermore, if $\mathcal{I}$ contains as an element an infinite subset A of some class $E_\alpha(Q)$, then $\mathcal{S}_{\mathcal{V}(Q)} \neq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$.*
- (ii) *If the ideal $\mathcal{I}$ is invariant under translations of $\mathbb{R}$, then the semigroups $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}$ and $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{I}$ are also invariant under translations of $\mathbb{R}$.*

Proof. (i) The families $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}$ are semigroups of sets by Propositions 2.6. The inclusions (1) follow also Proposition 2.6, and the equalities (2) follow Corollary 5.6 (i). Let us show (3). Since $\mathcal{I} \neq \{\emptyset\}$, there exists a point x for which $\{x\} \in \mathcal{I}$. Consider a Vitali Q -selector V of $\mathbb{R}$ containing the point x . Note that $V \setminus \{x\} = (V \setminus \{x\}) \cup \emptyset \in \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$. Since no proper subset of any Vitali Q -selector of $\mathbb{R}$ is an element of $\mathcal{V}(Q) \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma$, we have $V \setminus \{x\} \notin \mathcal{S}_{\mathcal{V}(Q)}^\sigma \cup \mathcal{I}$. In order to get the inclusion $\mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$, apply Proposition 2.6. Note that, for any set $U \in \mathcal{S}_{\mathcal{V}(Q)}$, we have $|U \cap E_\alpha(Q)| < \aleph_0$, but for $U \cup A \in \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$, we have $|(U \cup A) \cap E_\alpha(Q)| = \aleph_0$.

(ii) The invariance of the families $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$ and $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$ under translations of $\mathbb{R}$ follows from Proposition 5.4 and the assumption made on $\mathcal{I}$. $\square$

Lemma 5.9 ([3]). *For each $U \in \mathcal{S}_{\mathcal{V}(Q)}$ and each non-empty open set O of $\mathbb{R}$, there is an element $V \in \mathcal{V}(Q)$ such that $V \subseteq O \setminus U$. In particular, the set U cannot contain $O \setminus M$, where M is any meager set of the real line.*

Let $\tau_{\mathbb{R}}$ be the family of all open subsets of the real line $\mathbb{R}$. Note that $\tau_{\mathbb{R}}$ is a semigroup of sets and $\mathcal{S}_{\tau_{\mathbb{R}}} = \tau_{\mathbb{R}}$.

Proposition 5.10 (cf. [11]). *Let $\mathcal{I}$ be an ideal of sets on $\mathbb{R}$ and $\mathcal{V}(Q) \cap \mathcal{I} = \emptyset$. Then $(\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}) \cap (\tau_{\mathbb{R}} \Delta \mathcal{I}) = \emptyset$. In particular, $\mathcal{S}_{\mathcal{V}(Q)} \cap (\tau_{\mathbb{R}} \Delta \mathcal{I}) = \emptyset$. Moreover, each element of $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$ (as well as each element of $\mathcal{S}_{\mathcal{V}(Q)}$) is zero-dimensional.*

Proof. In order to get the equality $(\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}) \cap (\tau_{\mathbb{R}} \Delta \mathcal{I}) = \emptyset$, apply Proposition 2.9 together with Lemma 5.9. Namely, consider the families $\mathcal{V}(Q)$ and $\tau_{\mathbb{R}}$ as the families $\mathcal{A}$ and $\mathcal{B}$ of Proposition 2.9, respectively. Lemma 2.9 plays the same role as condition (ii) of Proposition 2.9.

Let us prove the second part of the statement. Let $A \in \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{I}$. It follows from Proposition 2.6 that $A = (U \setminus F) \cup E$ for some $U \in \mathcal{S}_{\mathcal{V}(Q)}$ and $E, F \in \mathcal{I}$. Since $\mathcal{V}(Q) \cap \mathcal{I} = \emptyset$, we have $U \setminus F \neq \emptyset$. Hence $A \neq \emptyset$. Moreover, $0 \leq \dim A \leq 1$.

Assume that $\dim A = 1$. By the Brouwer dimension theorem, there must exist a non-empty open set O of the real line $\mathbb{R}$ such that $O \subseteq A$. Note that $A = (U \setminus F) \cup E \subseteq U \cup E$. By Lemma 5.9, there exists $V \in \mathcal{V}(Q)$ such that $V \subseteq O \setminus U$. So $V \subseteq A \setminus U \subseteq (U \cup E) \setminus U \subseteq E$, which implies that $V \in \mathcal{I}$. This is a contradiction. Hence $\dim A = 0$. $\square$

Let us note that $\mathcal{V}(Q) \cap \mathcal{M}(\mathbb{R}) = \emptyset$, $\mathcal{B}_p(\mathbb{R}) = \tau_{\mathbb{R}}\Delta\mathcal{M}(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$ is invariant under translations of $\mathbb{R}$.

Corollary 5.11 ([4]). *The families $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})$, $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R})$ are semigroups of sets, and*

$$\mathcal{S}_{\mathcal{V}(Q)} \not\subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R}) \not\subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R}).$$

These semigroups are invariant under translations of $\mathbb{R}$ and consist of zero-dimensional sets without the Baire property.

Proof. From Proposition 5.10, we get $(\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R})) \cap (\tau_{\mathbb{R}}\Delta\mathcal{M}(\mathbb{R})) = \emptyset$, i.e., $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R}) \subseteq \mathcal{B}_p^c(\mathbb{R})$, and that each element of $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R})$ is zero-dimensional. The rest follows Proposition 5.8. $\square$

Remark 5.12. It follows from Proposition 3.6 that the semigroup of sets $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})$ contains measurable, in the sense of Lebesgue, sets.

Note that the ideals $\mathcal{I}_f$, $\mathcal{I}_c$, $\mathcal{I}_{cd}$, $\mathcal{I}_n$ are invariant under translations of $\mathbb{R}$.

Corollary 5.13 (cf. [11]). *Let $\mathcal{I}$ be $\mathcal{I}_f$, $\mathcal{I}_c$, $\mathcal{I}_{cd}$ or $\mathcal{I}_n$. Then the families $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}$ are semigroups of sets, $\mathcal{S}_{\mathcal{V}(Q)} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}$ and $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I} \neq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}$. Moreover, $\mathcal{S}_{\mathcal{V}(Q)} \neq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}$ whenever $\mathcal{I}$ is $\mathcal{I}_{cd}$ (and hence also $\mathcal{I}_c$ or $\mathcal{I}_n$). The considered semigroups are invariant under translations of $\mathbb{R}$ and consist of zero-dimensional sets without the Baire property.*

Proof. The results follows from Propositions 5.8 and Corollary 5.11. $\square$

5.2 Relationship between some semigroups of sets defined by Vitali Q -selectors and ideals of sets

Let Q be any countable dense subgroup of $(\mathbb{R}, +)$.

Lemma 5.14. *Let $U_1, U_2 \in \mathcal{S}_{\mathcal{V}(Q)}$ and $\emptyset \neq A \subseteq U_2$. Then $U_1 \cup A \in \mathcal{S}_{\mathcal{V}(Q)}$.*

Proof. Since $\mathcal{S}_{\mathcal{V}(Q)}$ is a semigroup of sets, it is enough to prove the statement for Vitali Q -selectors of $\mathbb{R}$. Consider $V_1, V_2 \in \mathcal{V}(Q)$ and $A \subseteq V_2$.

Put $J = \{\alpha \in I : E_{\alpha}(Q) \cap A \neq \emptyset\}$. Choose the Vitali Q -selector V_3 of $\mathbb{R}$ such that $V_3 \cap E_{\alpha}(Q) = V_1 \cap E_{\alpha}(Q)$ for each $\alpha \in I \setminus J$ and $V_3 \cap E_{\alpha}(Q) = V_2 \cap E_{\alpha}(Q)$ for each $\alpha \in J$. It is easy to see that $V_1 \cup A = V_1 \cup V_3 \in \mathcal{S}_{\mathcal{V}(Q)}$. $\square$

Proposition 5.15 (cf. [11]). $\mathcal{S}_{\mathcal{V}(Q)} = \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_f$.

Proof. Since, for each point of $\mathbb{R}$, there exists a Vitali Q -selector containing the point, the statement follows Lemma 5.14. $\square$

Remark 5.16. It is easy to see that $\mathcal{S}_{\mathcal{V}(Q)} = \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_A$ whenever $A \subseteq U$ for some $U \in \mathcal{S}_{\mathcal{V}(Q)}$.

Proposition 5.17 (cf. [11]). *The following statements hold.*

- (i) $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_f \subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_{cd} \subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_c \subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R})$ and $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_{cd} \subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_n \subseteq \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R})$.
- (ii) $\mathcal{S}_{\mathcal{V}(Q)} = \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_f \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_{cd} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_c \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})$ and $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_{cd} \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_n \subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})$.

Proof. The statement easily follows from the inclusions $\mathcal{I}_f \subseteq \mathcal{I}_{cd} \subseteq \mathcal{I}_c \subseteq \mathcal{M}(\mathbb{R})$ and $\mathcal{I}_{cd} \subseteq \mathcal{I}_n \subseteq \mathcal{M}(\mathbb{R})$. $\square$

Proposition 5.18. *Let $\mathcal{I}_1, \mathcal{I}_2$ be ideals of sets and $\mathcal{I}_1 \neq \{\emptyset\}$. Then $(\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_1) \setminus (\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_2) \neq \emptyset$.*

Proof. Let $x \in \mathbb{R}$ such that $\{x\} \in \mathcal{I}_1$. Consider any Vitali Q -selector of $\mathbb{R}$ containing the point x . Note that $V \setminus \{x\} \in (\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{I}_1) \setminus (\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{I}_2)$. $\square$

Lemma 5.19. *Let $\mathcal{J}$ be an ideal of sets such that $\mathcal{J}_f \subseteq \mathcal{J}$. Then, for each element $A \in \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}$, we have $A \cap Q \in \mathcal{J}$.*

Proof. Consider $A \in \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}$. It follows from Proposition 2.6 that $A = (U \setminus I_1) \cup I_2$, where $U \in \mathcal{S}_{\mathcal{V}(Q)}$ and $I_1, I_2 \in \mathcal{J}$. Recall that $U = \bigcup_{i=1}^n V_i$, where $V_i \in \mathcal{V}(Q)$. Thus,

$$A \cap Q = ((U \setminus I_1) \cup I_2) \cap Q \subseteq (U \cup I_2) \cap Q = (U \cap Q) \cup (I_2 \cap Q).$$

Since $U \cap Q \in \mathcal{J}_f \subseteq \mathcal{J}$ and $I_2 \cap Q \in \mathcal{J}$, we have $A \cap Q \in \mathcal{J}$. $\square$

Lemma 5.20. *There exists a meager set $A \subseteq \mathbb{R}$ such that $|\{\alpha \in I : |A \cap E_\alpha(Q)| = \aleph_0\}| = \mathfrak{c}$.*

Proof. Consider a Vitali Q -selector V of $\mathbb{R}$ and the standard Cantor set $\mathbb{C} \subseteq [0, 1] \subseteq \mathbb{R}$. Since $\mathbb{R} = \bigcup_{q \in Q} (V + q)$, there exists $q_0 \in Q$ such that $|(V + q_0) \cap \mathbb{C}| = \mathfrak{c}$ (apply König's theorem, cf. [9]). Let

$$J = \{\alpha \in I : ((V + q_0) \cap \mathbb{C}) \cap E_\alpha(Q) \neq \emptyset\}.$$

Note that the set $A = \bigcup_{q \in Q} (((V + q_0) \cap \mathbb{C}) + q)$ is a meager set of the real line $\mathbb{R}$, and for each $\alpha \in J$, we have $A \cap E_\alpha(Q) = E_\alpha(Q)$. Hence $|\{\alpha \in I : |A \cap E_\alpha(Q)| = \aleph_0\}| = \mathfrak{c}$. $\square$

Lemma 5.21. *The following inequalities hold:*

- (i) $(\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_{cd}) \setminus (\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_f) \neq \emptyset$,
- (ii) $(\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_n) \setminus (\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_{cd}) \neq \emptyset$,
- (iii) $(\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_c) \setminus (\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_n) \neq \emptyset$,
- (iv) $(\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})) \setminus (\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_c) \neq \emptyset$.

Proof. Consider any $V \in \mathcal{V}(Q)$.

(i) Indeed, let Z be an infinite closed discrete subset of $\mathbb{R}$ consisting of elements of Q . Since $Z \in \mathcal{J}_{cd}$, we have $V \cup Z \in \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_{cd}$. But $|(V \cup Z) \cap Q| = \aleph_0$, and hence $(V \cup Z) \cap Q \notin \mathcal{J}_f$. So, by Lemma 5.19, we have $V \cup Z \notin \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_f$.

(ii) Let K be an infinite countable subset of Q with only one limit point $0 \in K$. Since $K \in \mathcal{J}_n$, we have $V \cup K \in \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_n$. Note that $(V \cup K) \cap Q \supseteq K$ and $K \notin \mathcal{J}_{cd}$. Hence we have $(V \cup K) \cap Q \notin \mathcal{J}_{cd}$. It follows from Lemma 5.19 that $V \cup K \notin \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_{cd}$.

(iii) Let us note that $V \cup Q \in \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_c$. But $(V \cup Q) \cap Q = Q \notin \mathcal{J}_n$. So, by Lemma 5.19, $V \cup Q \notin \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_n$.

(iv) Note that, for each $B \in \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_c$, we have $|\{\alpha \in I : |B \cap E_\alpha(Q)| = \aleph_0\}| \leq \aleph_0$. Let A be the set from Lemma 5.20. Then $V \cup A \in \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})$ and $|\{\alpha \in I : |(V \cup A) \cap E_\alpha(Q)| = \aleph_0\}| = \mathfrak{c}$. Hence $V \cup A \notin \mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_c$. $\square$

Corollary 5.13, Propositions 5.17, 5.18 and Lemma 5.21 imply the following statement.

Proposition 5.22. *The semigroups of sets $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_f$, $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_{cd}$, $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_c$, $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{J}_n$, $\mathcal{S}_{\mathcal{V}(Q)}\Delta\mathcal{M}(\mathbb{R})$, $\mathcal{S}_{\mathcal{V}(Q)} = \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_f$, $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_{cd}$, $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_c$, $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J}_n$, $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{M}(\mathbb{R})$ are pairwise distinct.*

Lemma 5.23. *The following statements hold.*

- (i) *Let $A \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma\Delta\mathcal{J}$, where $\mathcal{J}$ is an ideal of sets such that $\mathcal{J} \neq \{\emptyset\}$. Then $|\{\alpha \in I : A \cap E_\alpha(Q) = \emptyset\}| \leq |I(A)|$ for some $I(A) \in \mathcal{J}$.*
- (ii) *There exists an infinite closed discrete subset S of the real line which can be extended to a Vitali Q -selector V_S of $\mathbb{R}$.*
- (iii) *There exists a nowhere dense subset K of the real line of cardinality $\mathfrak{c}$ which can be extended to a Vitali set V_K of $\mathbb{R}$.*

Proof. (i) Let $A \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma\Delta\mathcal{J}$. Hence there exist $U \in \mathcal{S}_{\mathcal{V}(Q)}^\sigma$ and $I_1, I_2 \in \mathcal{J}$ such that $A = (U \setminus I_1) \cup I_2$. Note that $|\{\alpha \in I : A \cap E_\alpha(Q) = \emptyset\}| \leq |\{\alpha \in I : (U \setminus I_1) \cap E_\alpha(Q) = \emptyset\}| \leq |I_1|$. Put $I(A) = I_1$.

(ii) Let $\{\alpha_i\}_{i=1}^\infty$ be an infinite countable subset of I , the indexed set of the equivalence classes. For each positive integer i , consider a point x_i from $E_{\alpha_i}(Q) \cap (i, i+1)$. The set $S = \{x_1, x_2, \dots\}$ is required.

(iii) Choose as K the set $(V + q_0) \cap \mathbb{C}$ from Lemma 5.20, and put $V_K = V + q_0$. $\square$

Proposition 5.24. $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_f \not\subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_{cd}$ and $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_n \setminus \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_c \neq \emptyset$.

Proof. Note that $V_S \setminus S \in \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{J}_{cd} \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_{cd}$, where S and V_S are from Lemma 5.23 (ii). Since, for each $I \in \mathcal{J}_f$, we have $|I| < \aleph_0$, by Lemma 5.23 (i), we get $V_S \setminus S \notin \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_f$. The inclusion $\mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_f \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_{cd}$ is evident. Note that $V_K \setminus K \in \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{J}_n \subseteq \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_n$, where K and V_K are from Lemma 5.23 (iii). Since, for each $I \in \mathcal{J}_c$, we have $|I| \leq \aleph_0$, by Lemma 5.23 (i), we get $V_K \setminus K \notin \mathcal{S}_{\mathcal{V}(Q)}^\sigma \Delta \mathcal{J}_c$. $\square$

5.3 Countable dense subgroups of $\mathbb{R}$ and related semigroups of sets

Let $\mathcal{F}$ be the family of all countable dense, in the real line $\mathbb{R}$, subgroups of the additive group $(\mathbb{R}, +)$.

Proposition 5.25 (cf. [11]). *For each $Q_1 \in \mathcal{F}$, there exists some $Q_2 \in \mathcal{F}$ such that $Q_1 \subsetneq Q_2$ and $|Q_2/Q_1| = \aleph_0$.*

Proposition 5.26 ([11]). *Let Q_1 and Q_2 be elements of $\mathcal{F}$ such that $Q_1 \subsetneq Q_2$ and $|Q_2/Q_1| = \aleph_0$. Then we have $\mathcal{S}_{\mathcal{V}(Q_1)} \cap \mathcal{S}_{\mathcal{V}(Q_2)} = \emptyset$.*

Corollary 5.27 ([11]). *For each $Q_1 \in \mathcal{F}$, there exists a $Q_2 \in \mathcal{F}$ such that $\mathcal{S}_{\mathcal{V}(Q_2)} \cap \mathcal{S}_{\mathcal{V}(Q_1)} = \emptyset$. In particular, the family $\{\mathcal{S}_{\mathcal{V}(Q)} : Q \in \mathcal{F}\}$ has no element which contains all others.*

Proof. Let $Q_1 \in \mathcal{F}$. By Proposition 5.25, one can find $Q_2 \in \mathcal{F}$ such that $Q_1 \subsetneq Q_2$ and $|Q_2/Q_1| = \aleph_0$. It follows from Proposition 5.26 that $\mathcal{S}_{\mathcal{V}(Q_2)} \cap \mathcal{S}_{\mathcal{V}(Q_1)} = \emptyset$. So there is no $Q^* \in \mathcal{F}$ such that the generated semigroup $\mathcal{S}_{\mathcal{V}(Q^*)}$ contains the semigroup $\mathcal{S}_{\mathcal{V}(Q)}$ for each $Q \in \mathcal{F}$. $\square$

5.4 Supersemigroup of sets based on all Vitali selectors

Put $\mathcal{V}^{\text{sup}} = \{V : V \in \mathcal{V}(Q), Q \in \mathcal{F}\}$, and consider the family $\mathcal{S}_{\mathcal{V}^{\text{sup}}}$. Let us note that $\mathcal{S}_{\mathcal{V}^{\text{sup}}}$ is a semigroup of sets on $\mathbb{R}$. Then $\mathcal{S}_{\mathcal{V}^{\text{sup}}}$ is called the *supersemigroup of sets based on all Vitali selectors* of $\mathbb{R}$ [1].

Remark 5.28. For the families $\mathcal{V}^{\text{sup}}$, $\mathcal{S}_{\mathcal{V}^{\text{sup}}}$, we have the following simple observations.

- (i) The families $\mathcal{V}^{\text{sup}}$, $\mathcal{S}_{\mathcal{V}^{\text{sup}}}$ are invariant under translations of $\mathbb{R}$.
- (ii) For each $Q \in \mathcal{F}$ and each ideal of sets $\mathcal{J}$ on $\mathbb{R}$, the following inclusions hold:

$$\mathcal{V}(Q) \subsetneq \mathcal{V}^{\text{sup}}, \quad \mathcal{S}_{\mathcal{V}(Q)} \subsetneq \mathcal{S}_{\mathcal{V}^{\text{sup}}}, \quad \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{J} \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{J}, \quad \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{J} \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}.$$

Similarly to Proposition 5.8, one can get the following.

Proposition 5.29. *Let $\mathcal{J}$ be an ideal of sets on $\mathbb{R}$ such that $\mathcal{J} \neq \{\emptyset\}$. Then the following statements hold.*

- (i) *The families $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{J}$ and $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}$ are semigroups of sets, and $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{J} \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}$. If $\mathcal{J}$ contains some Vitali selector, then $\mathcal{J} \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}$. In particular, $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{J} \neq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}$.*
- (ii) *If $\mathcal{J}$ is invariant under translations of $\mathbb{R}$, then the families $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{J}$ and $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}$ are also invariant under translations of $\mathbb{R}$.*

Lemma 5.30 ([1]). *For each set $U \in \mathcal{S}_{\mathcal{V}^{\text{sup}}}$ and each non-empty open set O of $\mathbb{R}$, there is a set $V \in \mathcal{V}^{\text{sup}}$ such that $V \subseteq O \setminus U$. In particular, the set U cannot contain $O \setminus M$, where M is any meager set.*

Similarly to Proposition 5.10, one can prove the following.

Proposition 5.31. *Let $\mathcal{J}$ be an ideal of subsets on $\mathbb{R}$ and $\mathcal{V}^{\text{sup}} \cap \mathcal{J} = \emptyset$. Then $(\mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}) \cap (\tau_{\mathbb{R}} \Delta \mathcal{J}) = \emptyset$. In particular, $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \cap (\tau_{\mathbb{R}} \Delta \mathcal{J}) = \emptyset$. Moreover, each element of the semigroup $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{J}$ is zero-dimensional. In particular, each element of $\mathcal{S}_{\mathcal{V}^{\text{sup}}}$ is zero-dimensional.*

Since $\mathcal{V}^{\text{sup}} \cap \mathcal{M}(\mathbb{R}) = \emptyset$, we have the following.

Corollary 5.32 ([1]). *The families $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{M}(\mathbb{R})$, $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{M}(\mathbb{R})$ are semigroups of sets, and*

$$\mathcal{S}_{\mathcal{V}^{\text{sup}}} \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{M}(\mathbb{R}) \subseteq \mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{M}(\mathbb{R}).$$

Moreover, $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \cup \mathcal{M}(\mathbb{R})$, $\mathcal{S}_{\mathcal{V}^{\text{sup}}} \Delta \mathcal{M}(\mathbb{R})$ are invariant under translations of $\mathbb{R}$ and consist of zero-dimensional sets without the Baire property.

Corollary 5.33 (cf. [11]). *Let $\mathcal{I}$ be $\mathcal{I}_f$, $\mathcal{I}_c$, $\mathcal{I}_{cd}$ or $\mathcal{I}_n$. Then the families $\mathcal{S}_{\mathcal{V}^{\sup}} \cup \mathcal{I}$, $\mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{I}$ are semigroups of sets, and $\mathcal{S}_{\mathcal{V}^{\sup}} \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \cup \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{I} \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{M}(\mathbb{R})$. The considered semigroups are invariant under translations of $\mathbb{R}$ and consist of zero-dimensional sets without the Baire property.*

Question 5.34. What is the relationship between the semigroups mentioned in Corollary 5.33?

5.5 Semigroups of Lebesgue non-measurable sets

In [6], Kharazishvili proved that every non-empty union of finitely many Vitali sets is not Lebesgue measurable, i.e., $\mathcal{S}_{\mathcal{V}(\mathbb{Q})} \subseteq \mathcal{L}^c(\mathbb{R})$. Let us generalize the fact by the use of his method.

Proposition 5.35. *Each element $U \in \mathcal{S}_{\mathcal{V}^{\sup}}$ is non-measurable in the Lebesgue sense.*

Proof. Let $\mathcal{A}$ be the ring of all bounded subsets of $\mathbb{R}$ which are measurable in the Lebesgue sense. Note that the Lebesgue measure m restricted to the ring $\mathcal{A}$ is finitely additive, translation invariant, and $m([0, 1]) = 1$. By Theorem 4.6, there exists a finitely additive translation invariant extension m' of m from $\mathcal{A}$ onto the ring $\mathcal{B}$ of all bounded subsets of $\mathbb{R}$. It is easy to see that m' is monotone, i.e., if $C, D \in \mathcal{B}$ and $C \subseteq D$, then $m'(C) \leq m'(D)$. Assume now that some $U \in \mathcal{S}_{\mathcal{V}^{\sup}}$ is a measurable set in the Lebesgue sense. It is easy to see that U is no null set. So $m(U) \neq 0$. Without loss of generality, we can assume that $U \cap [0, 1] \neq \emptyset$ and $m(U \cap [0, 1]) > 0$. Let $U = \bigcup_{i=1}^k V_i$, where $V_i \in \mathcal{V}(Q_i)$ and $Q_i \in \mathcal{F}$. Since $U \cap [0, 1] = \bigcup_{i=1}^k (V_i \cap [0, 1])$, there exists a number i such that $m'(V_i \cap [0, 1]) = a > 0$. Choose a positive integer n such that $n \cdot a > 2$. Then consider n different elements $r_j, j = 1, \dots, n$, from $Q_i \cap (0, 1)$. Note that the set $S = \bigcup_{j=1}^n (r_j + (V_i \cap [0, 1]))$ is a disjoint union of subsets $r_j + (V_i \cap [0, 1]), j = 1, \dots, n$. Since m' is finitely additive and translation invariant, $m'(S) = n \cdot a > 2$. On the other hand, $S \subseteq (0, 2)$, and hence $m'(S) \leq m'((0, 2)) = 2$. We have a contradiction with the monotonicity. $\square$

Corollary 5.36. *For each $Q \in \mathcal{F}$, we have $\mathcal{S}_{\mathcal{V}(Q)} \subseteq \mathcal{L}^c(\mathbb{R})$.*

Let us also note the following:

- (i) for each $Q \in \mathcal{F}$, the families $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{N}(\mathbb{R})$ and $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{N}(\mathbb{R})$ are semigroups of sets invariant under translation of $\mathbb{R}$; moreover, by Proposition 5.8 (i), we have $\mathcal{S}_{\mathcal{V}(Q)} \not\subseteq \mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{N}(\mathbb{R}) \not\subseteq \mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{N}(\mathbb{R})$;
- (ii) the families $\mathcal{S}_{\mathcal{V}^{\sup}} \cup \mathcal{N}(\mathbb{R})$ and $\mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{N}(\mathbb{R})$ are semigroups of sets invariant under translation of $\mathbb{R}$; moreover, $\mathcal{S}_{\mathcal{V}^{\sup}} \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \cup \mathcal{N}(\mathbb{R}) \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{N}(\mathbb{R})$;
- (iii) for each $Q \in \mathcal{F}$, we have $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{N}(\mathbb{R}) \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \cup \mathcal{N}(\mathbb{R})$ and $\mathcal{S}_{\mathcal{V}(Q)} \Delta \mathcal{N}(\mathbb{R}) \subseteq \mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{N}(\mathbb{R})$.

Propositions 5.35 and 2.4 imply the following statement.

Proposition 5.37. $\mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{N}(\mathbb{R}) \subseteq \mathcal{L}^c(\mathbb{R})$.

Let us note that $\mathcal{I}_{cd} \subseteq \mathcal{I}_c \subseteq \mathcal{N}(\mathbb{R})$.

Corollary 5.38. *Let $\mathcal{I}$ be either $\mathcal{I}_{cd}$ or $\mathcal{I}_c$. Then the semigroups $\mathcal{S}_{\mathcal{V}^{\sup}} \cup \mathcal{I}$ and $\mathcal{S}_{\mathcal{V}^{\sup}} \Delta \mathcal{I}$ consist of non-measurable sets in the Lebesgue sense.*

Remark 5.39. It follows from Proposition 3.6 that, for every $Q \in \mathcal{F}$, the semigroup of sets $\mathcal{S}_{\mathcal{V}(Q)} \cup \mathcal{N}(\mathbb{R})$ contains sets with the Baire property.

6 Bernstein construction and semigroups of sets on the real line

Let κ be infinite cardinal less than $\mathfrak{c}$.

Lemma 6.1. *Let $A \subseteq \mathbb{R}$ be such that $|A| \leq \kappa$, and let B be a Bernstein set on $\mathbb{R}$. Then $B \setminus A$, $A \cup B$ and $A \Delta B$ are Bernstein sets.*

Let $\mathcal{S}_{\mathcal{F}}$ be the family of Bernstein sets from Example 1.3.

Lemma 6.2. Let $F_i \in \mathcal{S}_{\mathcal{F}}$ and $A_i \subseteq \mathbb{R}$ such that $|A_i| \leq \kappa$, $i = 1, 2$. If $F_1 \cup A_1 = F_2 \cup A_2$ ($F_1 \cup A_1 = F_2 \Delta A_2$ or $F_1 \Delta A_1 = F_2 \Delta A_2$), then $F_1 = F_2$.

Proposition 2.6 and Lemmas 6.1 and 6.2 imply the following.

Proposition 6.3. Let $\mathcal{J} = \mathcal{J}_f$ ($\mathcal{J}_{cd}$ or $\mathcal{J}_c$). Then the families $\mathcal{S}_{\mathcal{F}} \cup \mathcal{J}$, $\mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}$ are invariant under translations semigroups of Bernstein sets such that $\mathcal{S}_{\mathcal{F}} \cup \mathcal{J} \not\subseteq \mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}$. Moreover,

$$\mathcal{S}_{\mathcal{F}} \cup \mathcal{J}_f \not\subseteq \mathcal{S}_{\mathcal{F}} \cup \mathcal{J}_{cd} \not\subseteq \mathcal{S}_{\mathcal{F}} \cup \mathcal{J}_c \quad \text{and} \quad \mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}_f \not\subseteq \mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}_{cd} \not\subseteq \mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}_c.$$

All the mentioned semigroups are pairwise different.

Let us note that the standard Cantor set $\mathbb{C}$ on the interval $[0, 1]$ is nowhere dense in $\mathbb{R}$, and it is a null set. Furthermore, any set containing $\mathbb{C}$ is not a Bernstein set.

Corollary 4.10 and Proposition 2.4 imply the following.

Lemma 6.4. Let B be a Bernstein set on $\mathbb{R}$, and let A be a meager set (resp. a null set) of $\mathbb{R}$. Then $A \cup B$, $B \setminus A$ and $A \Delta B$ are elements of $\mathcal{B}_p^c(\mathbb{R})$ (resp. $\mathcal{L}^c(\mathbb{R})$).

Remark 6.5. It follows from Proposition 3.6 that $\mathcal{S}_{\mathcal{F}} \cup \mathcal{M}(\mathbb{R})$ contains measurable, in the sense of Lebesgue, sets, and $\mathcal{S}_{\mathcal{F}} \cup \mathcal{N}(\mathbb{R})$ contains sets with the Baire property. Let us note that $\mathcal{M}(\mathbb{R}) \cup \mathcal{N}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$.

Proposition 2.6, Lemma 6.4 and Remark 6.5 imply the following.

Proposition 6.6. Let $\mathcal{J} = \mathcal{M}(\mathbb{R})$ or $\mathcal{N}(\mathbb{R})$. Then the families $\mathcal{S}_{\mathcal{F}} \cup \mathcal{J}$, $\mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}$ are invariant under translations semigroups of sets such that $\mathcal{S}_{\mathcal{F}} \cup \mathcal{J} \not\subseteq \mathcal{S}_{\mathcal{F}} \Delta \mathcal{J}$. Moreover, $\mathcal{S}_{\mathcal{F}} \Delta \mathcal{M}(\mathbb{R}) \subseteq \mathcal{B}_p^c(\mathbb{R})$ and $\mathcal{S}_{\mathcal{F}} \Delta \mathcal{N}(\mathbb{R}) \subseteq \mathcal{L}^c(\mathbb{R})$. All the mentioned semigroups are pairwise different.

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