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IPCW Estimator for Kendall's Tau under Bivariate Censoring

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Abstract

We investigate the nonparametric estimation of Kendall's coefficient of concordance, τ, for measuring the association between two variables under bivariate censoring. The proposed estimator is a modification of the estimator introduced by Oakes (1982), using a Horvitz-Thompson-type correction for the pairs that are not orderable. With censored data, a pair is orderable if one can establish whether the uncensored pair is discordant or concordant using the data available for that pair. Our estimator is shown to be consistent and asymptotically normally distributed. A simulation study shows that the proposed estimator performs well when compared with competing alternatives. The various methods are illustrated with a real data set.

KEYWORDS: Kendall's tau, dependence, Horvitz-Thompson estimator, Kaplan-Meier estimator, martingales

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1 Introduction

In many biomedical experiments, the prime interest of the study is to investigate the relationship between two random variables. Kendall's coefficient of concordance, τ, is a simple measure of association between a pair of lifetime random variables (X, Y). This measure is independent of the marginal distributions for X and Y. Its rank invariance property makes it particularly suitable, especially when only the association between the random variables is of interest. Semi-parametric models for bivariate data, see for instance Clayton (1978), Oakes (1986), and Genest (1987), use a copula to model the dependency between the variables. Kendall's τ is a function of the copula parameter; thus estimators of Kendall's τ naturally yield estimators of the copula parameter in semi-parametric models (Genest & Rivest, 1993, Wang & Wells, 2000b). Nonparametric estimation of τ from n identical independent replications $\{(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\}$ of (X, Y) has been extensively studied, see for instance Kendall & Gibbons (1990) and Gibbons (1971).

In recent years, a substantial research effort has been devoted to the estimation of the dependence between lifetime random variables from incomplete data, see for instance Lin, Sun & Ying (1999), Wang & Wells (1997), Wang & Wells (1998) and Fine, Jiang & Chappell (2001). This paper focuses on the nonparametric estimation of τ under bivariate censoring, that is, when only $(\tilde{X}, \tilde{Y}, \delta_X, \delta_Y)$ are observable, where $\tilde{X} = \min(X, C_X)$, $\delta_X = I_{\{X<C_X\}}$ is a censoring indicator, C_X is a censoring random variable independent of X, and $\tilde{Y}$, C_Y, and δ_Y are defined in a similar way for Y. Brown, Hollander & Korwar (1974), Weier & Basu (1980) and Oakes (1982) modified the estimator of τ to account for censoring in both coordinates. It turns out that none of these estimators is consistent when $\tau \neq 0$. Alternatively, Wang & Wells (2000a) derived an estimator for τ expressed as an integral of an estimate of the bivariate survival function.

In this paper, we propose a new nonparametric estimator for τ under bivariate censoring based on a modification of the one proposed in Oakes (1982). The contribution of each orderable pair to the coefficient is weighted by the inverse probability that the pair is orderable. A pair is orderable if its concordance-discordance status can be established using the information available in the censored sample. Our estimator takes different expressions depending on the censoring scheme. Four situations are considered: independent censoring variables (C_X and C_Y independent), censoring on X only ($C_Y = \infty$), univariate censoring ($C_X = C_Y = C$) and the general case (C_X and C_Y dependent). Wang & Wells (1997) investigated the estimation of the bivariate survival function under these simplified censoring schemes. The proposed

Horvitz-Thompson-type estimator is an extension of the Inverse Probability Censoring Weighted (IPCW) estimators class of Robins & Rotnitzky (1992) to multivariate selection probabilities. This class of estimators allows the estimation of the joint survival function for successive events (Lin, Sun & Ying, 1999), the estimation of the mean quality adjusted lifetime with censored data (Zhao & Tsiatis 1997 and Zhao & Tsiatis 2000) and the estimation of regression parameters in a multiplicative intensity model (van der Laan & Robins, 2003).

This new estimator for τ is shown to be consistent under suitable regularity conditions. It is also asymptotically normally distributed for the first three cases. Simulations comparing the proposal with existing estimators show its good performance. In Section 2, we present our estimator and investigate its asymptotic behavior in Section 3. Numerical investigations are presented in Section 4 and we conclude with a discussion in Section 5.

2 Estimation of τ

Let (X_1, Y_1) and (X_2, Y_2) be two independent replications of (X, Y), a bivariate lifetime random variable with continuous marginals $S_X(x) = P(X > x)$ and $S_Y(y) = P(Y > y)$. This pair is said to be concordant if $(X_1 - X_2)(Y_1 - Y_2) > 0$ and discordant if $(X_1 - X_2)(Y_1 - Y_2) < 0$. Kendall's tau (Kendall & Gibbons, 1990) is defined by

$$\begin{aligned}\tau &= P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - P\{(X_1 - X_2)(Y_1 - Y_2) < 0\} \\ &= 2P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - 1 \\ &= E(a_{12}b_{12}) \\ &= 4\int_0^\infty \int_0^\infty \pi(x, y)\frac{\partial^2 \pi(x, y)}{\partial x \partial y} dx dy - 1 \qquad (1)\end{aligned}$$

where $a_{ij} = 2 \times I_{\{X_i - X_j > 0\}} - 1$, $b_{ij} = 2 \times I_{\{Y_i - Y_j > 0\}} - 1$ and $\pi(.,.)$ is the bivariate survival function of (X, Y) defined by $\pi(x, y) = P(X > x, Y > y)$.

In the absence of censoring, one can estimate τ by its sample version

$$\hat{\tau}_K = \binom{n}{2}^{-1} \sum_{i<j} a_{ij} b_{ij}. \qquad (2)$$

2.1 Previous estimators under bivariate censoring

With censored values, the concordance/discordance status can be established only for orderable pairs. Oakes (1982) showed that a pair is orderable if

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$\{\tilde{X}_{ij} < \tilde{C}_X^{ij}, \tilde{Y}_{ij} < \tilde{C}_Y^{ij}\}$ where $\tilde{X}_{ij} = \min(X_i, X_j)$, $\tilde{Y}_{ij} = \min(Y_i, Y_j)$, $\tilde{C}_X^{ij} = \min(C_X^i, C_X^j)$ and $\tilde{C}_Y^{ij} = \min(C_Y^i, C_Y^j)$. Denote the indicator of this event by L_{ij}. Several alternatives to (2) have been proposed to estimate τ under bivariate censoring.

Brown, Hollander & Korwar (1974) modified the definitions of a_{ij} and b_{ij} for non orderable pairs. A pair of points is not orderable as soon as one of the events $(\tilde{X}_{ij} > \tilde{C}_X^{ij})$ or $(\tilde{Y}_{ij} > \tilde{C}_Y^{ij})$ is true. The condition on the X's holds if any of the following mutual exclusive events is true: $(\tilde{X}_i > \tilde{X}_j; \delta_X^i = 1; \delta_X^j = 0), (\tilde{X}_i > \tilde{X}_j; \delta_X^i = 0; \delta_X^j = 0), (\tilde{X}_i < \tilde{X}_j; \delta_X^i = 0; \delta_X^j = 1)$ or $(\tilde{X}_i < \tilde{X}_j; \delta_X^i = 0; \delta_X^j = 0)$. It is easy to see that

$$P(X_i - X_j > 0 | \tilde{X}_i > \tilde{X}_j; \delta_X^i = 1; \delta_X^j = 0) = 1 - \frac{S_X(\tilde{x}_i)}{S_X(\tilde{x}_j)} \tag{3}$$

$$P(X_i - X_j > 0 | \tilde{X}_i > \tilde{X}_j; \delta_X^i = 0; \delta_X^j = 0) = 1 - \frac{S_X(\tilde{x}_i)}{2\ S_X(\tilde{x}_j)} \tag{4}$$

$$P(X_i - X_j > 0 | \tilde{X}_i < \tilde{X}_j; \delta_X^i = 0; \delta_X^j = 1) = \frac{S_X(\tilde{x}_j)}{S_X(\tilde{x}_i)} \tag{5}$$

$$P(X_i - X_j > 0 | \tilde{X}_i < \tilde{X}_j; \delta_X^i = 0; \delta_X^j = 0) = \frac{S_X(\tilde{x}_j)}{2\ S_X(\tilde{x}_i)}, \tag{6}$$

where $\tilde{x}_i$ and $\tilde{x}_j$ are the observed values of $\tilde{X}_i$ and $\tilde{X}_j$.

The coefficients a'_{ij} for non orderable pairs, defined by

$$a'_{ij} = 2\hat{P}(X_i - X_j > 0 | \tilde{X}_i, \tilde{X}_j, \delta_X^i, \delta_X^j) - 1,$$

are obtained by substituting the probabilities in the right-hand side of (3), (4), (5) and (6) by their Kaplan-Meier estimates based on $\{(\tilde{X}_k, \delta_X^k),\ k = 1, \cdots, n\}$. For orderable pairs, set $a'_{ij} = a_{ij}$. The b'_{ij} are defined analogously.

This yields the estimator

$$\hat{\tau}_B = \frac{\sum_{i,j} a'_{ij} b'_{ij}}{(\sum_{i,j} a'^2_{ij} \sum_{i,j} b'^2_{ij})^{1/2}}. \tag{7}$$

Oakes (1982) modified (2) by summing over orderable pairs only:

$$\hat{\tau}_O = \binom{n}{2}^{-1} \sum_{i<j} L_{ij} a_{ij} b_{ij}. \tag{8}$$

It turns out that none of these estimators is consistent when $\tau \neq 0$, since they ignore the dependence between X and Y when estimating the concordance-discordance status of a non orderable pair.

Alternatively, Wang & Wells (2000a) used (1) to estimate τ by

$$\begin{aligned}\hat{\tau}_W &= 4\int_0^\infty\int_0^\infty \hat{\pi}(x,y)\frac{\partial^2\hat{\pi}(x,y)}{\partial x\partial y}dxdy - 1\\ &= 4\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{\pi}(x_{(i)},y_{(j)})\hat{\pi}(\Delta x_{(i)},\Delta y_{(j)}) - 1, \qquad (9)\end{aligned}$$

where $\{x_{(0)} = 0 < x_{(1)} < x_{(2)} < \cdots < x_{(n)}\}$ and $\{y_{(0)} = 0 < y_{(1)} < y_{(2)} < \cdots < y_{(n)}\}$ are the ordered samples of $\{\tilde{X}_k,\ k = 1,\cdots,n\}$ and $\{\tilde{Y}_k,\ k = 1,\cdots,n\}$, $\hat{\pi}(.,.)$ is a nonparametric estimator for $\pi(.,.)$ and $\hat{\pi}(\Delta x_{(i)},\Delta y_{(j)}) = \hat{\pi}(x_{(i)},y_{(j)}) - \hat{\pi}(x_{(i-1)},y_{(j)}) - \hat{\pi}(x_{(i)},y_{(j-1)}) + \hat{\pi}(x_{(i-1)},y_{(j-1)})$ is the estimated probability mass of the rectangle $[x_{(i-1)},x_{(i)}]\times[y_{(j-1)},y_{(j)}]$. Several nonparametric estimators for the bivariate survival function can be substituted in (9); see for instance those of Campbell (1981), Dabrowska (1988), and Prentice & Cai (1992).

2.2 New estimators

One can consider the orderable pairs as a sample selected from the population of $\binom{n}{2}$ possible pairs. A common technique in survey sampling to correct the bias consists of weighting the pairs in (8) by the inverse estimated probabilities $\hat{p}_{ij}$ of being selected (Horvitz & Thompson,1952). This yields the new estimator referred to as the 'modified Oakes estimator'

$$\hat{\tau}_{mo} = \binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}a_{ij}b_{ij}}{\hat{p}_{ij}} \qquad (10)$$

When $|\tau|$ is close to 1, $\hat{\tau}_{mo}$ might lie outside $[-1,1]$. An alternative to address this issue is to consider τ_{mo2} defined by

$$\hat{\tau}_{mo2} = \left[\sum_{i<j}\frac{a_{ij}b_{ij}}{\hat{p}_{ij}}\right]^{-1}\sum_{i<j}\frac{L_{ij}a_{ij}b_{ij}}{\hat{p}_{ij}}. \qquad (11)$$

It is easy to see that $\hat{\tau}_{mo2}$ always lies inside $[-1,1]$.

Note that $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ are observed for all orderable pairs so the individual selection probabilities are expressed as

$$\begin{aligned}p_{ij} &= P\{L_{ij}|\tilde{X}_{ij},\tilde{Y}_{ij}\}\\ &= P\{\tilde{C}_X^{ij} > \tilde{X}_{ij}, \tilde{C}_Y^{ij} > \tilde{Y}_{ij}|\tilde{X}_{ij},\tilde{Y}_{ij}\}\\ &= P\{C_X^i > \tilde{X}_{ij}, C_X^j > \tilde{X}_{ij}, C_Y^i > \tilde{Y}_{ij}, C_Y^j > \tilde{Y}_{ij}|\tilde{X}_{ij},\tilde{Y}_{ij}\}\\ &= P\{C_X > \tilde{X}_{ij}, C_Y > \tilde{Y}_{ij}|\tilde{X}_{ij},\tilde{Y}_{ij}\}^2 \qquad (12)\end{aligned}$$

The estimation of p_{ij} for orderable pairs depends on the censoring scheme.

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2.2.1 Independent censoring variables

In this case, (C_X, C_Y) are independent so that the selection probability p_{ij} is written as

$$p_{ij} = \{S_X^c(\tilde{X}_{ij})\ S_Y^c(\tilde{Y}_{ij})\}^2, \tag{13}$$

where S_X^c and S_Y^c are the survival functions of C_X and C_Y, respectively. These can be estimated by the Kaplan-Meier estimators based on $\{(\tilde{X}_k, 1-\delta_X^k), k = 1,\dots,n\}$ and $\{(\tilde{Y}_k, 1-\delta_Y^k), k = 1,\dots,n\}$, respectively. This yields

$$\hat{p}_{ij} = \{\hat{S}_X^c(\tilde{X}_{ij})\ \hat{S}_Y^c(\tilde{Y}_{ij})\}^2. \tag{14}$$

2.2.2 Censoring on X only

When only one coordinate, say X, is censored, $C_Y = \infty$. Therefore, this scenario can be viewed as a particular case of Section (2.2.1) where $S_Y^c(\tilde{Y}_{ij}) = 1$. Clearly, the individual selection probability can thus be written as

$$p_{ij} = \{S_X^c(\tilde{X}_{ij})\}^2, \tag{15}$$

which can be estimated by the Kaplan-Meier estimator based on $\{(\tilde{X}_k, 1-\delta_X^k), k = 1,\dots,n\}$.

2.2.3 Univariate censoring

In this case, $C_X = C_Y = C$ so that the selection probability p_{ij} is expressed as

$$\begin{aligned} p_{ij} &= P\{\tilde{X}_{ij} < C, \tilde{Y}_{ij} < C\}^2 \\ &= [S^c\{\max\ (\tilde{X}_{ij}, \tilde{Y}_{ij})\}]^2, \end{aligned}$$

where S^c is the survival function of the common censoring variable C. It can be estimated by the Kaplan-Meier estimator based on $\{(\max\ (\tilde{X}_k, \tilde{Y}_k), 1-\delta_X^k\delta_Y^k), k = 1,\dots,n\}$.

2.2.4 General case

In the general case, the relationship between C_X and C_Y is left completely unspecified. The selection probability is expressed in terms of the bivariate survival function $\pi^c(.,.)$ of the censoring variables:

$$p_{ij} = \{\pi^c(\tilde{X}_{ij}, \tilde{Y}_{ij})\}^2. \tag{16}$$

As for $\hat{\tau}_W$, several nonparametric estimators are available for the bivariate survival function . In our numerical investigations, we shall use the Dabrowska estimator for both $\hat{\tau}_{mo}$ and $\hat{\tau}_W$.

3 Asymptotic behavior

In this section, we investigate the asymptotic properties of our estimators. We distinguish two cases: one where the p_{ij}'s are strictly bounded away from zero for all pairs with probability one and one where the p_{ij}'s can be equal to zero for some pairs.

3.1 Case I: Bounded away from zero

Define $\tilde{\tau}$ by:

$$\tilde{\tau} = \binom{n}{2}^{-1} \sum_{i<j} \frac{L_{ij}}{p_{ij}} a_{ij} b_{ij} \tag{17}$$

and write $\sqrt{n}(\hat{\tau}_{mo} - \tau)$ as $\sqrt{n}(\hat{\tau}_{mo} - \tilde{\tau}) + \sqrt{n}(\tilde{\tau} - \tau)$.

In this case, we assume that all original pairs (X, Y) fall inside $[0, L_X] \times [0, L_Y]$, where L_X and L_Y are such that

$$\pi^c(L_X, L_Y) > \delta > 0, \tag{18}$$

for some constant δ. It is easy to see that in this case, one has

$$P(p_{ij} > \delta \text{ for all } (i,j)) = 1.$$

Under this assumption, we show in Appendix A that $\sqrt{n}(\tilde{\tau} - \tau)$ is a zero-mean U-statistic of order 2. On the other hand, we show in Appendix B that $\sqrt{n}(\hat{\tau}_{mo} - \tilde{\tau})$ is asymptotically equivalent to a sum of independent and identically distributed zero mean terms for the cases presented in Sections (2.2.1), (2.2.2) and (2.2.3). Thus, $\sqrt{n}(\hat{\tau}_{mo} - \tau)$ is asymptotically equivalent to a zero mean U-statistic of order 2. Consistency and asymptotic normality for these cases follow (van der Vaart, 1998). However, the computation of the variance involves complex formulas so we use the jackknife procedure. Similar results are available for $\hat{\tau}_{mo2}$; see Appendix C.

Condition (18) fails if either X or Y has an infinite support. In this case, the probability that a pair is orderable is strictly positive provided that the censoring variables have infinite supports as well. This probability is not bounded away from 0 and the proofs presented in the appendices do not apply. However, extensive simulations suggest that both $\hat{\tau}_{mo}$ and $\hat{\tau}_{mo2}$ are still consistent in this case.

3.2 Case II: Censoring variables with short supports

The supports of the censoring variables can be shorter than that of the failure times of interest. This occurs when censoring is due to the termination of the study. In this case, $p_{ij} = 0$ for some (i, j) and all the observed points belong to $\mathcal{B}$, the support of the censoring variables. The concordance/discordance relationship outside $\mathcal{B}$ is missing. Wang & Wells (2000a) discuss this issue and show that the parameter estimated by their estimator is

$$\tau_W^* = \int_{\mathcal{B}} \pi_{\mathcal{B}}(x,y) \frac{\partial^2 \pi_{\mathcal{B}}(x,y)}{\partial x \partial y} dxdy - 1,$$

where $\pi_{\mathcal{B}}(x,y) = \Pr(X > x; Y > y | (X,Y) \in \mathcal{B})$.

The new estimators are also vulnerable to this problem because the missing information is not recoverable. Let's derive expressions for the parameters estimated by $\hat{\tau}_{mo}$ and $\hat{\tau}_{mo2}$ respectively under these conditions.

Without loss of generality, assume that X and Y are both uniformly distributed on $[0,1]$, and that the supports of C_X and C_Y are $[0, A]$ and $[0, B]$ respectively, where $0 < A, B < 1$. The observable region $\mathcal{B}$ consists of $\{0 \leq x \leq A, 0 \leq y \leq B\}$. A pair of observations $\{(X_i, Y_i), (X_j, Y_j)\}$ has a positive probability of being orderable, e.g. $p_{ij} > 0$, if and only if $(\tilde{X}_{ij}, \tilde{Y}_{ij}) \in \mathcal{B}$ (Fan et al. 2000). Denote the indicator of this event by O_{ij}. Kendall's tau is then expressed as:

$$\begin{aligned} \tau &= E(a_{12}b_{12}) \\ &= E(a_{12}b_{12}|O_{12})P(O_{12}) + E(a_{12}b_{12}|O_{12}^c)P(O_{12}^c) \end{aligned} \tag{19}$$

where O_{ij}^c is O_{ij}'s complement.

Thus, we have no information about $E(a_{12}b_{12}|O_{12}^c)$. In this case, $\hat{\tau}_{mo}$ and $\hat{\tau}_{mo2}$ estimate τ^c and τ_2^c respectively instead of estimating τ, where

$$\tau^c = E(a_{12}b_{12}|O_{12})P(O_{12}) \quad \text{and} \quad \tau_2^c = E(a_{12}b_{12}|O_{12}) \tag{20}$$

Fan et al. (2000) refer to τ_2^c as *a finite region censored data version of Kendall's tau.*

The biases of $\hat{\tau}_{mo}$ and $\hat{\tau}_{mo2}$ are then expressed as:

$$\begin{aligned} \tau^c - \tau &= -E(a_{12}b_{12}|O_{12}^c)P(O_{12}^c) \\ \tau_2^c - \tau &= P(O_{12}^c)\{E(a_{12}b_{12}|O_{12}) - E(a_{12}b_{12}|O_{12}^c))\}. \end{aligned}$$

In general, the magnitude and direction of these biases are difficult to evaluate. To illustrate these biases, we consider the special cases where

(X, Y) follows a Clayton (Clayton 1978) and a Frank (Genest 1987) copula. These Archimedean copulas are indexed by a one-dimensioned parameter α measuring the dependence between X and Y. For these copula families, this parameter is related to Kendall's tau by

$$\tau_{Cl}(\alpha) = \frac{\alpha}{\alpha + 2}$$

and

$$\tau_F(\alpha) = 1 + \frac{4}{\alpha}\{\int_0^\alpha \frac{t}{e^t - 1} dt - 1\}, \tag{21}$$

respectively.

For simplicity, assume that only X is subject to censoring ($C_Y = \infty$). Under these conditions, $P(O_{12}^c) = (1 - A)^2$, $P(O_{12}) = 1 - P(O_{12}^c)$ and $E(a_{12}b_{12}|O_{12}^c)$ corresponds to *the truncated Kendall's tau* that measures the association of the conditional copula for the dependency between X and Y given X is larger than A. Manatunga & Oakes (1996) have shown that when the underlying copula is Clayton with parameter α, then the conditional copula is also Clayton with parameter α. Thus for Clayton's copula, one has $E(a_{12}b_{12}|O_{12}^c) = \tau_{Cl}$ and $E(a_{12}b_{12}|O_{12}) = \tau_{Cl}$. Therefore the relative biases are given by $(\tau^c - \tau_{Cl})/\tau_{Cl} = -P(O_{12}^c) = -(1 - A)^2$ and $(\tau_2^c - \tau_{Cl})/\tau_{Cl} = 0$

For Frank's copula, Manatunga & Oakes (1996) showed that the conditional copula is Frank with parameter $\alpha(1 - A)$. Therefore

$$E(a_{12}b_{12}|O_{12}^c) = \tau_F\{\alpha(1 - A)\}.$$

The relative biases $(\tau^c - \tau_F)/\tau_F$ and $(\tau_2^c - \tau_F)/\tau_F$ are derived from (19) and (21).

In Figures 1 & 2, we present these relative biases for both copulas, along with those of $\hat{\tau}_W$, for $A = 0.7$. These figures suggest that, for both copula families, $\hat{\tau}_{mo2}$ is less biased than $\hat{\tau}_{mo}$, which in turn is less biased than $\hat{\tau}_W$. Further computations not shown here suggest that these conclusions remain valid for both copula families when A varies. Future investigations are needed to ascertain the relative biases in the general case.

4 Numerical examples

4.1 Simulations

A series of simulations were carried out to assess the finite-sample performance of the proposal and to compare it to current alternatives. The pairs (X, Y)

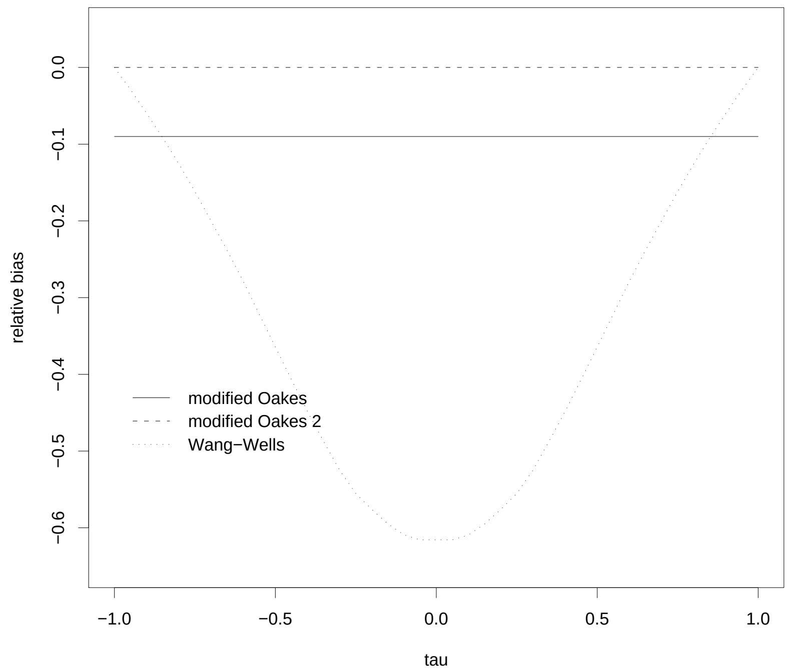

Figure 1: Relative biases as a function of τ for **Clayton**'s copula when only X is subject to censoring and $A = 0.7$.

were generated from a Clayton copula (Clayton, 1978) with unit exponential marginals. The censoring variables (C_X, C_Y) were generated with exponential marginals with a parameter controlling the censoring fraction. Samples were simulated with a sample size of 100, 200 and 400, a Kendall's tau of 0.2, 0.5 and 0.8 and a censoring fraction (% C) of 20% and 40%. We considered the four censoring schemes presented in Sections 2.2.1 to 2.2.4. For the general case (Section 2.2.4), (C_X, C_Y) were simulated from a Clayton copula with $\tau = 0.5$. For each combination of the parameters below, 1000 replications were performed. For each simulated sample, we computed $\hat{\tau}_B$, $\hat{\tau}_W$ and $\hat{\tau}_{mo}$. Results are reported in Tables 1-4. In each cell, we report the mean bias $(\times 10^3)$ and the root mean squared error, r.m.s.e, $(\times 10^3)$ computed over the 1000 replications.

The estimator $\hat{\tau}_B$ lacks theoretical support. It is not convergent. The simulations in Tables 1-4 highlight its large bias (often larger than 10%). Nevertheless, it has the smallest r.m.s.e. in several cases, especially when $\tau = 0.8$.

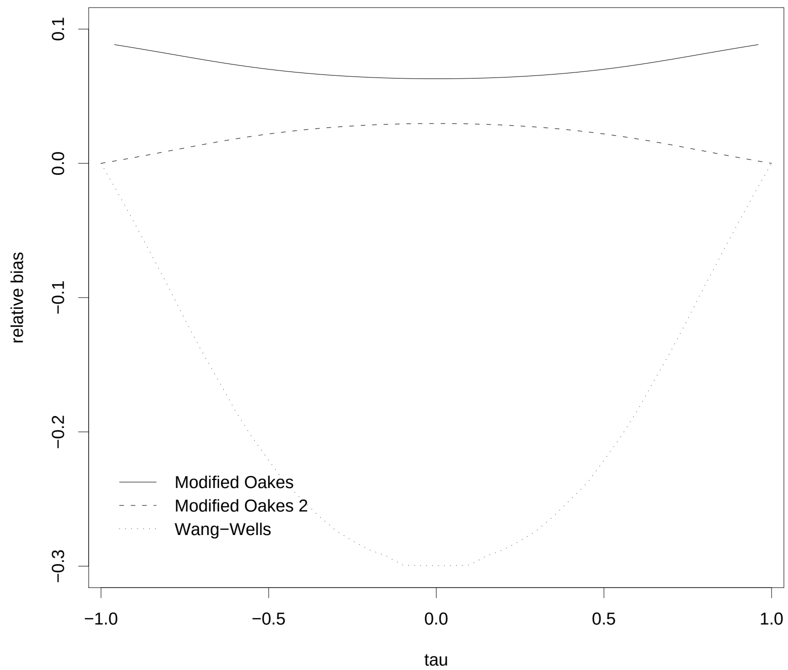

Figure 2: Relative biases as a function of τ for **Frank**'s copula when only X is subject to censoring and $A = 0.7$.

Tables 1-3 confirm the substantial bias reduction associated with our estimator when compared to $\hat{\tau}_W$ for the cases presented in Sections (2.2.1), (2.2.2) and (2.2.3). It is noticeable that the bias of $\hat{\tau}_W$ is always negative in Tables 1-4. This agrees with Figures 1-2 and with the empirical findings reported by Wang & Well (2000a). On the other hand, the r.m.s.e. of $\hat{\tau}_{mo}$ decreases when either the sample size or the Kendall's tau increases or the censoring fraction decreases. In these tables, this r.m.s.e. is smaller than that of $\hat{\tau}_W$ except for the line ($\tau = 0.8$, $n = 400$, $\%C = 20\%$) in Table 1.

In the general case (Table 4), even though $\hat{\tau}_W$ performs slightly better than $\hat{\tau}_{mo}$, the r.m.s.e.'s of both estimators are large and can be as high as 50%, especially when $\%C = 40$. Thus, in this case, both $\hat{\tau}_W$ and $\hat{\tau}_{mo}$ can give only a rough idea on the association level.

Finally, simulations, not reported here, show that in general $\hat{\tau}_{mo}$ outperforms $\hat{\tau}_{mo2}$.

% C	n	τ	Modified Oakes	Brown et al	Wang & Wells
20	100	0.2	-0.18 (65.60)	9.77 (66.75)	-20.80 (69.30)
		0.5	0.62 (58.24)	23.50 (61.02)	-20.90 (62.52)
		0.8	-4.22 (38.25)	20.38 (32.92)	-27.91 (49.52)
	200	0.2	-1.71 (52.24)	8.18 (53.04)	-11.35 (53.74)
		0.5	0.39 (41.36)	22.67 (45.45)	-10.07 (42.43)
		0.8	-1.10 (27.64)	21.43 (28.34)	-11.74 (30.19)
	400	0.2	1.05 (34.36)	10.93 (36.17)	-4.03 (34.66)
		0.5	-0.21 (28.18)	22.53 (35.33)	-4.76 (29.02)
		0.8	-0.93 (20.86)	21.25 (24.89)	-4.35 (19.59)
40	100	0.2	2.94 (95.75)	29.52 (84.99)	-20.82 (99.43)
		0.5	-8.74 (91.79)	44.88 (76.58)	-29.76 (97.28)
		0.8	-14.63 (88.73)	-12.11 (35.66)	-24.54 (116.25)
	200	0.2	-2.37 (64.59)	26.70 (59.80)	-11.77 (65.42)
		0.5	0.79 (62.78)	48.80 (64.25)	-7.23 (66.76)
		0.8	-6.01 (60.04)	-11.16 (26.74)	-9.35 (76.22)
	400	0.2	-1.74 (44.11)	26.44 (46.45)	-5.82 (45.50)
		0.5	-1.62 (40.72)	46.79 (55.07)	-3.72 (41.15)
		0.8	-1.94 (42.83)	-10.32 (19.77)	-0.48 (46.68)

Table 1: Independent censoring: Biases ($\times 10^3$) and root mean squared errors ($\times 10^3$, in parentheses) for three estimators of τ.

4.2 Example: Tremin Trust data

We use the Tremin Trust data (Lisabeth et al. 2004) containing bleeding and hormonal information of 562 women followed up to 40 years, beginning in 1935. Our analysis considers women who were at most 25 at enrollment and who were still under observation at age 35; see Nan et al. (2006) for a description of the study.

The analysis focuses on three markers for entry into *early transition years*, a pre-menopausal stage of the woman's reproductive life. The first one, denoted CYC45, is the age at which the menstrual cycle is at least 45 days long for the first time. The last two, denoted SD6 and SD8, are ages at which the cycle length standard deviation, computed over the last 12 months, becomes larger than 6 and 8 days for the first time respectively. Lisabeth et al. (2004) investigated whether these markers identify the same event in a woman's reproductive life cycle. This can be achieved by estimating an

% C	n	τ	Modified Oakes	Brown et al	Wang & Wells
20	100	0.2	-1.81 (67.65)	3.45 (67.95)	-17.52 (70.21)
		0.5	-2.09 (56.35)	9.03 (56.44)	-21.11 (60.80)
		0.8	-2.04 (30.05)	7.67 (27.60)	-23.78 (40.04)
	200	0.2	-1.31 (47.44)	3.87 (47.68)	-9.26 (48.45)
		0.5	0.24 (39.45)	11.30 (40.46)	-9.30 (40.86)
		0.8	-1.52 (21.88)	7.58 (20.41)	-11.87 (25.65)
	400	0.2	-1.00 (33.21)	4.01 (33.52)	-5.08 (33.62)
		0.5	-0.98 (27.14)	10.04 (28.87)	-5.52 (27.80)
		0.8	-0.15 (15.00)	8.25 (15.50)	-5.47 (16.42)
40	100	0.2	4.94 (74.01)	18.31 (72.86)	-15.44 (77.36)
		0.5	-3.50 (64.31)	15.64 (58.67)	-24.52 (71.90)
		0.8	-6.12 (38.38)	-29.70 (43.42)	-25.99 (58.81)
	200	0.2	-0.47 (54.71)	12.34 (51.83)	-10.48 (56.69)
		0.5	-0.02 (44.45)	18.87 (43.44)	-11.46 (47.50)
		0.8	-4.03 (27.58)	-29.42 (36.89)	-14.14 (38.67)
	400	0.2	1.42 (37.05)	14.45 (38.05)	-3.58 (37.88)
		0.5	-1.53 (31.02)	18.15 (32.92)	-6.69 (33.36)
		0.8	-0.80 (17.90)	-27.82 (31.66)	-5.23 (25.15)

Table 2: Censoring on X only: Biases ($\times 10^3$) and root mean squared errors ($\times 10^3$, in parentheses) for three estimators of τ.

association measure such as Kendall's tau. In this dataset, occurrence of the markers are all censored by the same random variable, i.e., the minimum of *loss-to-followup* and *end-of-study* (Section 2.2.3). The censoring fractions for the three markers are equal to 36.48%, 29.89% and 38.43% respectively. Lisabeth et al. (2004) derived a parametric estimate of Kendall's tau under a Clayton copula. They obtained $\hat{\tau}(CYC45, SD6) = 0.65$, $\hat{\tau}(CYC45, SD8) = 0.62$ and $\hat{\tau}(SD6, SD8) = 0.59$. These estimates can be biased if the assumed copula is misspecified.

We estimated Kendall's tau using the modified Oakes estimator, and those of Wang & Wells (2000a) and Brown et al. (1974). The results are presented in Table 5.

The three estimates are different. Without knowledge of the true τ's, it is very difficult to compare these results. We simulated 1000 samples with conditions similar to the ones encountered in the data set ($n = 562$, $\tau = 0.65$, a single censoring variable yielding two censoring fractions of 0.35 and a Clay-

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% C	n	τ	Modified Oakes	Brown et al	Wang & Wells
20	100	0.2	0.88 (71.61)	11.11 (72.64)	-20.23 (74.34)
		0.5	-1.08 (60.27)	22.76 (62.55)	-25.74 (66.40)
		0.8	-1.78 (31.93)	26.19 (37.31)	-28.38 (45.19)
	200	0.2	-0.80 (49.11)	8.71 (50.30)	-11.20 (50.90)
		0.5	1.34 (40.66)	24.08 (46.91)	-11.03 (42.13)
		0.8	-1.29 (22.23)	26.06 (32.52)	-15.32 (28.32)
	400	0.2	0.61 (35.05)	10.56 (37.18)	-4.44 (35.61)
		0.5	0.31 (28.99)	23.12 (36.82)	-5.61 (29.58)
		0.8	-1.01 (15.19)	25.97 (29.11)	-8.03 (18.09)
40	100	0.2	3.32 (83.67)	33.86 (86.89)	-28.66 (91.16)
		0.5	0.12 (73.55)	70.95 (95.74)	-36.99 (87.70)
		0.8	-7.99 (42.29)	61.30 (67.36)	-46.43 (70.60)
	200	0.2	-2.05 (61.74)	28.96 (64.22)	-18.36 (65.43)
		0.5	-0.55 (49.99)	69.30 (82.24)	-18.98 (55.75)
		0.8	-3.53 (28.94)	63.14 (66.05)	-24.11 (43.50)
	400	0.2	-0.57 (41.77)	30.53 (50.35)	-8.83 (43.01)
		0.5	-1.28 (34.94)	68.03 (74.45)	-9.60 (38.01)
		0.8	-2.08 (19.59)	62.23 (63.66)	-12.66 (26.57)

Table 3: Univariate censoring: Biases ($\times 10^3$) and root mean squared errors ($\times 10^3$, in parentheses) for three estimators of τ.

ton copula). The biases and r.m.s.e. of the three estimators are $(-.002, .027)$, $(-.010, .026)$, $(-.009, .027)$, for $\hat{\tau}_{mo}$ $\hat{\tau}_W$, and $\hat{\tau}_B$ respectively. Thus, in this example, $\hat{\tau}_{mo}$ has the smallest bias while having a comparable root mean squared error. The parametric estimator of τ computed by Lisabeth et al. (2004) seems to underestimate the association level of the pair $(CYC45, SD6)$. This may be due to the misspecification of the assumed copula. On the other hand, we found results similar to theirs for the pairs $(CYC45, SD8)$ and $(SD6, SD8)$. Our finding suggests that the two markers $CYC45$ and $SD6$ are more likely to identify the same event than the $SD8$ marker.

Finally, note that the Jackknife method provides reasonable standard error estimates for this example, close to the empirical root mean squared error of the simulations conducted under similar conditions to those of the data at hand.

% C	n	τ	Modified Oakes	Brown et al	Wang & Wells
20	100	0.2	-1.34 (72.01)	9.57 (71.25)	-20.95 (73.62)
		0.5	-0.91 (67.12)	22.46 (60.46)	-24.15 (62.99)
		0.8	17.64 (66.01)	25.76 (36.58)	-27.42 (45.18)
	200	0.2	0.20 (49.43)	10.39 (49.28)	-9.88 (49.95)
		0.5	2.64 (43.70)	23.45 (45.52)	-10.94 (41.63)
		0.8	21.69 (46.18)	25.36 (31.57)	-12.79 (28.01)
	400	0.2	-0.73 (35.16)	8.71 (35.48)	-6.23 (34.80)
		0.5	0.45 (32.62)	21.79 (35.78)	-7.02 (29.98)
		0.8	22.14 (37.58)	25.15 (28.22)	-6.33 (17.86)
40	100	0.2	-0.23 (128.97)	27.04 (83.96)	-37.58 (95.09)
		0.5	46.47 (157.49)	60.71 (85.70)	-38.83 (88.71)
		0.8	18.58 (117.75)	36.87 (47.51)	-40.00 (82.94)
	200	0.2	11.83 (92.34)	32.59 (62.81)	-14.13 (60.48)
		0.5	67.66 (122.95)	65.27 (79.01)	-15.14 (57.55)
		0.8	4.72 (113.80)	39.29 (44.32)	-15.33 (52.98)
	400	0.2	12.97 (91.70)	30.61 (49.34)	-7.13 (42.28)
		0.5	66.42 (99.15)	63.77 (70.91)	-6.64 (37.85)
		0.8	-27.81 (110.27)	39.11 (41.66)	-6.90 (32.75)

Table 4: Dependent censoring (Clayton, $\tau(C_X, C_Y) = 0.5$): Biases ($\times 10^3$) and root mean squared errors ($\times 10^3$, in parentheses) for three estimators of τ.

5 Discussion

In this paper, we derived a Horvitz-Thompson-type estimator for Kendall's tau under bivariate censoring. Under suitable regularity conditions, this estimator is consistent and asymptotically normal. The consistency of the proposal is confirmed by the simulation results in Tables 1, 2 and 3. Simulations conducted to compare the proposed estimator with alternatives show clearly that there is no uniformly best estimator for all situations. Except in the simulations of Table 4 the proposed modified Oakes estimator has the smallest bias and a relatively small r.m.s.e. It seems to outperform its competitors if the relationship between the two random censoring variables is simple (Sections 2.2.1, 2.2.2 and 2.2.3).

The general case where the relationship between the censoring variables is unspecified (Section 2.2.4) is much more difficult and all existing estimators fail to perform well in this case. More research is warranted here.

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	SD6	SD8
CYC45	$\hat{\tau}_{mo} = 0.729$ (s.e. 0.029) $\hat{\tau}_W = 0.694$ (s.e. 0.031) $\hat{\tau}_B = 0.596$ (s.e. 0.029)	$\hat{\tau}_{mo} = 0.642$ (s.e. 0.033) $\hat{\tau}_W = 0.643$ (s.e. 0.035) $\hat{\tau}_B = 0.561$ (s.e. 0.032)
SD6	*** *** ***	$\hat{\tau}_{mo} = 0.582$ (s.e. 0.032) $\hat{\tau}_W = 0.642$ (s.e. 0.032) $\hat{\tau}_B = 0.535$ (s.e. 0.031)

Table 5: Kendall's tau estimates for the markers.

Finally, this paper assumes that the censoring variables are independent of the variables of interest. This may fail to hold in some complex situations such as serial events gap times. Extending the proposal to handle such situations is a subject of current research.

Appendix A:
A U-Statistic expression for $\sqrt{n}(\tilde{\tau} - \tau)$

It is clear that $\tilde{\tau}$ is a U-statistic of order 2 whose expectation is equal to

$$\begin{aligned} E(\tilde{\tau}) &= \binom{n}{2}^{-1} \sum_{i<j} E\{\frac{L_{ij}}{p_{ij}} a_{ij} b_{ij}\} \\ &= \binom{n}{2}^{-1} \sum_{i<j} E\{E[\frac{L_{ij}}{p_{ij}} a_{ij} b_{ij} | \tilde{X}_{ij}, \tilde{Y}_{ij}]\} \\ &= \binom{n}{2}^{-1} \sum_{i<j} E\{\frac{1}{p_{ij}} E[L_{ij} a_{ij} b_{ij} | \tilde{X}_{ij}, \tilde{Y}_{ij}]\}. \end{aligned}$$

By Oakes (1986), the concordance/discordance status and the orderability event are unconditionally independent. Once $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ are fixed, by (12), the orderability event depends only on the censoring variables C_X and C_Y while the concordance/discordance status depends only on the original pairs and hence these events are conditionally independent. So we have

$$E\{L_{ij} a_{ij} b_{ij} | \tilde{X}_{ij}, \tilde{Y}_{ij}\} = E\{L_{ij} | \tilde{X}_{ij}, \tilde{Y}_{ij}\} E\{a_{ij} b_{ij} | \tilde{X}_{ij}, \tilde{Y}_{ij}\}$$

and, if $p_{ij} > 0$ for all pairs,

$$E(\tilde{\tau}) = \binom{n}{2}^{-1} \sum_{i<j} E\{E[a_{ij} b_{ij} | \tilde{X}_{ij}, \tilde{Y}_{ij}]\}$$

$$= \binom{n}{2}^{-1} \sum_{i<j} E\{a_{ij}b_{ij}\} = \tau$$

And thus $\sqrt{n}(\tilde{\tau} - \tau)$ is a zero-mean U-statistic of order 2.

Appendix B: An asymptotic expression for $\sqrt{n}(\hat{\tau} - \tilde{\tau})$

Consider the simple case where only X is censored (Section 2.2.2). Following the lines of Andrei & Murray (2006), we write

$$\begin{aligned} \frac{1}{\hat{p}_{ij}} - \frac{1}{p_{ij}} &= \frac{1}{\hat{S_X^c}^2(\tilde{X}_{ij})} - \frac{1}{{S_X^c}^2(\tilde{X}_{ij})} \\ &= \frac{{S_X^c}^2(\tilde{X}_{ij}) - \hat{S_X^c}^2(\tilde{X}_{ij})}{{S_X^c}^2(\tilde{X}_{ij})\hat{S_X^c}^2(\tilde{X}_{ij})} \end{aligned} \tag{22}$$

The absolute value of the right hand side of (22) is bounded from above by

$$\frac{\mathrm{Sup}_{0\le t\le L_X}|{S_X^c}^2(t) - \hat{S_X^c}^2(t)|}{{S_X^c}^2(L_X)\hat{S_X^c}^2(L_X)}$$

and thus converges to zero in probability by uniform convergence of the Kaplan-Meier estimator on $[0, L_X]$ and continuity of the mapping : $t \mapsto t^2$. The convergence of $\sqrt{n}(\hat{\tau} - \tilde{\tau})$ follows.

On the other hand, one has

$$\begin{aligned} \sqrt{n}\left[\frac{{S_X^c}^2(t) - \hat{S_X^c}^2(t)}{{S_X^c}^2(t)}\right] &= 2\frac{\sqrt{n}\{\hat{S}_X^c(t) - S_X^c(t)\}}{S_X^c(t)} + o_p(1) \\ &= 2\sqrt{n}\int_0^t \frac{\hat{S_X^c}(u^-)}{S_X^c(u)}\frac{\sum_{k=1}^n dM_k^c(u)}{\sum_{k=1}^n 1_{\{\tilde{X}_k\ge u\}}} + o_p(1), \end{aligned}$$

where $M^c(u)$ is the standard martingale associated to the censoring variable C_X defined by $M^c(u) = 1_{\{C_X\le u;\delta_X=0\}} - \int_0^u 1_{\{\tilde{X}\ge s\}}d\Lambda^c(s)$ and Λ^c is the cumulative hazard function associated to C.

Thus, one has

$$\sqrt{n}\left[\frac{{S_X^c}^2(\tilde{X}_{ij}) - \hat{S_X^c}^2(\tilde{X}_{ij})}{{S_X^c}^2(\tilde{X}_{ij})}\right] = 2\sqrt{n}\int_0^{L_X} 1_{\{\tilde{X}_{ij}\ge u\}}\frac{\hat{S_X^c}(u^-)}{S_X^c(u)}\frac{\sum_{k=1}^n dM_k^c(u)}{\sum_{k=1}^n 1_{\{\tilde{X}_k\ge u\}}} + o_p(1)$$

and

$$\sqrt{n}(\hat{\tau}-\tilde{\tau}) = \frac{1}{\sqrt{n}}\int_0^{L_X}\left[\frac{1}{\binom{n}{2}}\sum_{i<j} L_{ij}a_{ij}b_{ij}1_{\{\tilde{X}_{ij}\geq u\}}\right]\frac{\hat{S_X^c}(u^-)}{S_X^c(u)}\frac{\sum_{k=1}^n dM_k^c(u)}{\sum_{k=1}^n 1_{\{\tilde{X}_k\geq u\}}} + o_p(1) \tag{23}$$

Since

$$\frac{\hat{S_X^c}(u^-)}{S_X^c(u)}\frac{1}{\sum_{k=1}^n 1_{\{\tilde{X}_k\geq u\}}} \rightarrow \frac{1}{P(\tilde{X}\geq u)}$$

almost surely, the right hand side of (23) can be expressed as

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\int_0^{L_X}\left[\frac{1}{\binom{n}{2}}\sum_{i<j}\frac{L_{ij}a_{ij}b_{ij}1_{\{\tilde{X}_{ij}\geq u\}}}{\hat{S}_X^c(\tilde{X}_{ij})}\right]\frac{2}{P(\tilde{X}\geq u)}\sum_{k=1}^n dM_k^c(u) + o_p(1)\\
&= \frac{1}{\sqrt{n}}\int_0^{L_X} E\left[\frac{L_{12}a_{12}b_{12}1_{\{\tilde{X}_{12}\geq u\}}}{\hat{S}_X^c(\tilde{X}_{12})}\right]\frac{2}{P(\tilde{X}\geq u)}\sum_{k=1}^n dM_k^c(u) + o_p(1)\\
&= \frac{1}{\sqrt{n}}\sum_{k=1}^n\int_0^{L_X} E\left[\frac{L_{12}a_{12}b_{12}1_{\{\tilde{X}_{12}\geq u\}}}{S_X^c(\tilde{X}_{12})}\right]\frac{2}{P(\tilde{X}\geq u)} dM_k^c(u) + o_p(1)
\end{aligned}$$

Finally, $\sqrt{n}(\hat{\tau}-\tilde{\tau})$ is asymptotically equivalent to a sum of zero-mean independent and identically distributed terms. By using the same arguments, one can show that this also holds for cases in Sections (2.2.1) and (2.2.3). Unfortunately, similar results are not available for the general case presented in Section (2.2.4).

Appendix C: An asymptotic expression for $\sqrt{n}(\hat{\tau}_{mo2}-\tilde{\tau})$

One has:

$$\begin{aligned}
\sqrt{n}\{\hat{\tau}_{mo2}-\tilde{\tau}\} &= \sqrt{n}\left[\frac{\binom{n}{2}^{-1}\sum_{i<j}\frac{L_{ij}a_{ij}b_{ij}}{\hat{p}_{ij}}}{\binom{n}{2}^{-1}\sum_{i<j}\frac{a_{ij}b_{ij}}{\hat{p}_{ij}}} - \tilde{\tau}\right]\\
&= \sqrt{n}\left[\hat{\tau}_{mo2}-\tilde{\tau}\right] - \sqrt{n}\tilde{\tau}\left[\binom{n}{2}^{-1}\sum_{i<j}\frac{a_{ij}b_{ij}}{\hat{p}_{ij}} - 1\right] + o_p(1)\\
&= \sqrt{n}\left[\hat{\tau}_{mo2}-\tilde{\tau}\right] - \sqrt{n}\tilde{\tau}\left[\binom{n}{2}^{-1}\sum_{i<j}\frac{a_{ij}b_{ij}}{p_{ij}} - 1\right]
\end{aligned}$$

$$
\begin{aligned}
&- \quad \sqrt{n}\tilde{\tau}\left[\binom{n}{2}^{-1}\left\{\sum_{i<j}\frac{a_{ij}b_{ij}}{\hat{p}_{ij}} - \sum_{i<j}\frac{a_{ij}b_{ij}}{p_{ij}}\right\}\right] + o_p(1) \\
&= \quad A + B + C + o_p(1)
\end{aligned}
$$

By appendix B, A is equivalent to a sum of independent and identically distributed zero-mean terms. The term B is a sum of n independent and identically distributed zero-mean terms. Finally, a similar development in appendix A yields

$$
C = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\int_0^{L_X}\frac{2}{P(\tilde{X}\geq u)}E\left[\frac{a_{12}b_{12}1_{\{\tilde{X}_{12}\geq u\}}}{S_X^C(\tilde{X}_{12})}\right]dM_k^c(u) + o_p(1)
$$

Asymptotic expression for $\sqrt{n}(\hat{\tau}_{mo2} - \tilde{\tau})$ follows.

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