Property (T), finite-dimensional representations, and generic representations

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Abstract. Let G be a discrete group with Property (T). It is a standard fact that, in a unitary representation of G on a Hilbert space $\mathcal{H}$, almost invariant vectors are close to invariant vectors, in a quantitative way. We begin by showing that, if a unitary representation has some vector whose coefficient function is close to a coefficient function of some finite-dimensional unitary representation σ , then the vector is close to a sub-representation isomorphic to σ : this makes quantitative a result of P. S. Wang. We use that to give a new proof of a result by D. Kerr, H. Li and M. Pichot, that a group G with Property (T) and such that $C^*(G)$ is residually finite-dimensional, admits a unitary representation which is generic (i.e. the orbit of this representation in $\text{Rep}(G, \mathcal{H})$ under the unitary group $U(\mathcal{H})$ is comeager). We also show that, under the same assumptions, the set of representations equivalent to a Koopman representation is comeager in $\text{Rep}(G, \mathcal{H})$.

1 Introduction

Let G be a discrete group and π a unitary representation of G on some Hilbert space $\mathcal{H}$. For a finite set $F \subset G$ and $\varepsilon > 0$, a vector $\xi \in \mathcal{H}$ is (F, ε) -invariant if $\max_{g \in F} \|\pi(g)\xi - \xi\| < \varepsilon$. Recall that π almost has invariant vectors if, for every pair (F, ε) , the group G has (F, ε) -vectors; and that the group G has *Kazhdan's Property (T)* or is a *Kazhdan group* if every unitary representation of G almost having invariant vectors, has non-zero invariant vectors; see e.g. [3] for Property (T). The definition can be reformulated in terms of weak containment of representations: G has Property (T) if every unitary representation weakly containing the trivial representation of G , contains it strongly (see [3, Remark 1.1.2]). Crucial for us is an equivalent characterization due to P. S. Wang [12, Corollary 1.9 and Theorem 2.1]: the group G has Property (T) if and only if for some (hence every) irreducible finite-dimensional unitary representation σ of G , every unitary representation π of G that contains σ weakly, contains it strongly.

It is a simple but useful fact that, if G has Property (T) and π is a unitary representation almost having invariant vectors, “almost invariant vectors are close to invariant vectors”. More precisely, we have the following.

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Proposition 1.1 ([3, Proposition 1.1.9]). *Let G be a Kazhdan group. If S is a finite generating set of G and ε_0 is the corresponding Kazhdan constant, then for every $\delta \in]0, 1[$ and every unitary representation π of G , any $(S, \varepsilon_0\delta)$ -invariant vector ξ satisfies $\|\xi - P\xi\| \leq \delta\|\xi\|$, where P is the orthogonal projection on the subspace of $\pi(G)$ -invariant vectors.*

For a Kazhdan group G and a unitary representation π of G , fix a unit vector ξ and look at the coefficient function

$$\phi_{\pi,\xi}(g) = \langle \pi(g)\xi, \xi \rangle \quad (g \in G).$$

The question we first address in this paper is: if $\phi_{\pi,\xi}$ is close to some coefficient of an irreducible finite-dimensional unitary representation σ of G , must ξ be close to a finite-dimensional invariant subspace of π carrying a sub-representation isomorphic to σ ? We will see that, in analogy to Proposition 1.1, the answer is positive – with some care.

Definition 1.2. Let G be a finitely generated group with a symmetric finite generating set $S \subseteq G$ containing e and let ϕ be some normalized positive definite function on G associated with a unitary irreducible representation σ , of finite dimension d . Let π be some unitary representation of G on $\mathcal{H}$. Let $\varepsilon > 0$. Say that a unit vector $\xi \in \mathcal{H}$ is (π, ϕ, ε) -good if for every $s \in S^{2d^2+1}$ we have $|\phi_{\pi,\xi}(s) - \phi(s)| < \varepsilon$.

Note that S^k is just the ball of radius k centered at the identity in G . So there is a certain lack of uniformity in Definition 1.2: we require an approximation of $\phi_{\pi,\xi}$ by ϕ on a ball whose size depends on the dimension of the representation d . Our main result, proved in Section 2, can be viewed as a quantitative version of Wang's result.

Theorem 1.3. *Let G be a discrete Kazhdan group, S a finite symmetric generating set with $e \in S$, and let ϕ be a normalized positive definite function on G associated with a finite-dimensional unitary irreducible representation σ of G . For every $\delta \in]0, 1[$ there exists $\varepsilon_{\phi,\delta} > 0$ such that for every unitary representation π of G on a Hilbert space $\mathcal{H}$, and every unit vector $x \in \mathcal{H}$ that is $(\pi, \phi, \varepsilon_{\phi,\delta})$ -good, there exists a unit vector $x' \in \mathcal{H}$ with $\|x - x'\| \leq \delta$ such that the restriction of π to the span of $\pi(G)x'$ is isomorphic to σ .*

In Section 3, we apply Theorem 1.3 to the study of the global structure of the space of unitary representations of Kazhdan groups. Let us start with the notation. Let G be an arbitrary countable group and let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. The set $\text{Rep}(G, \mathcal{H})$ of all homomorphisms from G into the unitary group $U(\mathcal{H})$ can be viewed as a closed subset of the product

space $U(\mathcal{H})^G$, when we equip $U(\mathcal{H})$ with the strong operator topology. With this identification, $\text{Rep}(G, \mathcal{H})$ is a Polish (i.e. separable and completely metrizable) space. We refer the reader to the work [8], especially to the section on the spaces of unitary representations, for more information about this point of view on unitary representations. Recall that two unitary representations $\pi_1, \pi_2 \in \text{Rep}(G, \mathcal{H})$ are *isomorphic*, or *unitarily equivalent* if there is a unitary operator $\phi \in U(\mathcal{H})$ such that $\pi_1(g) = \phi\pi_2(g)\phi^*$ for every $g \in G$. Notice that this is an orbit equivalence relation given by the action of the unitary group $U(\mathcal{H})$ on the space $\text{Rep}(G, \mathcal{H})$ by conjugation. Kechris raised the question (see again the section on the space of unitary representations in [8]) of whether or not there are countable groups with a *generic unitary representation*, where “generic” here means its conjugacy class is large in the sense of Baire category, i.e. a representation whose class under the unitary equivalence contains a dense G_δ subset. As a matter of fact, we note that it follows from the topological zero-one law that for every countable group G either there is a generic representation in $\text{Rep}(G, \mathcal{H})$, or all conjugacy classes are meager (see e.g. [7, Theorem 8.46]; to apply it, note that there is a dense conjugacy class in $\text{Rep}(G, \mathcal{H})$ – indeed, take some countable dense set of representations from $\text{Rep}(G, \mathcal{H})$ and consider their direct sum).

Here as an application of Theorem 1.3 we prove the following result.

Theorem 1.4. *Let G be a discrete Kazhdan group such that finite-dimensional representations are dense in the unitary dual $\hat{G}$. Then there is a generic unitary representation of G .*

We note that, although not explicitly stated there, this result already follows from a more general result of Kerr, Li and Pichot from [9], where they prove (see Theorem 2.5 there) that if A is a separable C^* -algebra where finite-dimensional representations are dense in $\hat{A}$, then there is a dense G_δ class in $\text{Rep}(A, \mathcal{H})$. Theorem 1.4 is then a special case for $A = C^*(G)$. Our proof is nevertheless done by more elementary means, in particular it does not invoke Voiculescu’s theorem (see [9, proof of Theorem 2.5] for details).

The converse of Theorem 1.4 can be also derived from [9], see Remark 3.1 below.

Another open question posed by Kechris in [8, Problem H.16] is whether the subset of those representations $\pi \in \text{Rep}(G, \mathcal{H})$, where G is still a countable group, that are equivalent to Koopman representations, is meager in $\text{Rep}(G, \mathcal{H})$ or not. Such representations are called *realizable by an action* in [8]. Let us recall the terminology first. Let (X, μ) be a standard probability space (i.e. a space isomorphic to the unit interval $[0, 1]$ equipped with the Lebesgue measure). Let $\alpha : G \curvearrowright (X, \mu)$ be an action of a countable group G on X by measure-preserving measurable

transformations. Consider the unitary representation $\pi_\alpha : G \rightarrow L^2(X, \mu)$ defined by $\pi_\alpha(g)f(x) = f(\alpha(g^{-1}, x))$ for every $f \in L^2(X, \mu)$. The *Koopman representation* of α is the restriction of π_α to the invariant subspace $L^2_0(X, \mu)$, which is the orthogonal complement of the invariant subspace of constant functions.

In Section 4 we prove the following result addressing the question of Kechris.

Theorem 1.5. *Let G be a discrete Kazhdan group such that finite-dimensional representations are dense in the unitary dual $\hat{G}$. Then the set of representations realizable by an action is comeager in $\text{Rep}(G, \mathcal{H})$.*

Let us record that the condition that finite-dimensional representations are dense in the unitary dual $\hat{G}$ is, by the result of Archbold from [1], equivalent with the statement that the full group C^* -algebra $C^*(G)$ is residually finite-dimensional. That is in turn, by the result of Exel and Loring from [6] (see also [11]), equivalent with the statement that finite-dimensional representations are dense in $\text{Rep}(G, \mathcal{H})$, which we shall use in the proof. Note that we call a representation $\pi \in \text{Rep}(G, \mathcal{H})$ finite-dimensional if the subalgebra $\pi(G)$ generates in $B(\mathcal{H})$ is finite-dimensional.

The existence of infinite discrete Kazhdan groups with residually finite-dimensional C^* -algebras seems to be open – see [3, Question 7.10] and also Lubotzky and Shalom [10, Question 6.5] where they ask if there are infinite discrete Kazhdan groups with property FD, which is strictly stronger than having a residually finite-dimensional C^* -algebra (a group has *property FD* if representations factoring through finite groups are dense in the unitary dual).

Question 1.6. It is known that being residually finite is not a sufficient condition to have a residually finite-dimensional C^* -algebra by a result of Bekka [2]. However how about being LERF? (Recall that a finitely generated group is LERF if any finitely generated subgroup is the intersection of the finite index subgroups containing it.) Ershov and Jaikin-Zapirain constructed in [5] a Kazhdan group which is LERF. Is its group C^* -algebra residually finite-dimensional?

Remark 1.7. We note that on the other hand we cannot exclude the possibility of proving by a different argument that for every infinite group G , all classes in $\text{Rep}(G, \mathcal{H})$ are meager. That would together with Theorem 1.4 give that there are no infinite Kazhdan groups with a residually finite-dimensional C^* -algebra.

2 A quantitative version of Wang's theorem

Let G be an infinite, finitely generated group. Let S be a finite, symmetric, generating set of G , with $e \in S$. Let $\mathbb{C}G$ be the complex group ring of G .

2.1 Quantifying the Burnside theorem

Let σ be an irreducible unitary representation of dimension d , i.e. a homomorphism $\sigma : G \rightarrow U_d(\mathbb{C})$ such that $\sigma(G)$ has no proper invariant subspace. The classical Burnside theorem says that $\sigma(\mathbb{C}G) = M_d(\mathbb{C})$, i.e. $\sigma(G)$ contains a basis of $M_d(\mathbb{C})$.

Definition 2.1. Set $k(\sigma) = \min\{k > 0 : \dim_{\mathbb{C}} \text{span } \sigma(S^k) = d^2\}$.

Lemma 2.2. *There is a constant $C > 0$ (only depending on S) such that*

$$C \log d \leq k(\sigma) \leq d^2.$$

Proof. We have

$$d^2 = \dim_{\mathbb{C}} \text{span } \sigma(S^{k(\sigma)}) \leq |\sigma(S^{k(\sigma)})| \leq |S^{k(\sigma)}| \leq |S|^{k(\sigma)}.$$

Taking logarithms,

$$\frac{2}{\log |S|} \log d \leq k(\sigma).$$

To prove the upper bound, observe that the sequence $\text{span } \sigma(S^k)$ of subspaces of $M_d(\mathbb{C})$ is strictly increasing for $k < k(\sigma)$. Indeed, assume that k is such that $\text{span } \sigma(S^k) = \text{span } \sigma(S^{k+1})$: this means that $\text{span } \sigma(S^k)$ is stable by left multiplication by $\sigma(S)$, hence by $\sigma(G)$ as S is generating. Since the identity matrix is in $\sigma(S^k)$, we have

$$\sigma(G) \subset \text{span } \sigma(S^k),$$

hence $k \geq k(\sigma)$. From this it is clear that $k(\sigma) \leq d^2$. $\square$

Let v be a unit vector in $\mathbb{C}^d$. Since v is cyclic for $\sigma(G)$, the map

$$T_v : \mathbb{C}S^{k(\sigma)} \rightarrow \mathbb{C}^d, \quad f \mapsto \sigma(f)v$$

is onto. Let $(\ker T_v)^\perp$ denote the orthogonal of $\ker T_v$ in $\mathbb{C}S^{k(\sigma)}$ and let U_v be the inverse of the map $T_v|_{(\ker T_v)^\perp}$. Endow $\mathbb{C}S^{k(\sigma)}$ with the ℓ^1 -norm, and let $\|U_v\|_{2 \rightarrow 1}$ be the corresponding operator norm of U_v . So for every w a unit vector in $\mathbb{C}^d$, there exists $f \in \mathbb{C}S^{k(\sigma)}$ with $\|f\|_1 \leq \|U_v\|_{2 \rightarrow 1}$ such that $\sigma(f)v = w$.

Lemma 2.3. *There exists $M > 0$ such that for every two unit vectors $v, w \in \mathbb{C}^d$, there exists $f \in \mathbb{C}S^{k(\sigma)}$ with $\|f\|_1 \leq M$ such that $\sigma(f)v = w$.*

Proof. This is the preceding observation plus compactness of the unit sphere in $\mathbb{C}^d$: the constant $M = \max_{\|v\|=1} \|U_v\|_{2 \rightarrow 1}$ does the job. $\square$

2.2 From weak containment to weak containment à la Zimmer

Recall that, if π, ρ are unitary representations of a discrete group G , the representation π is weakly contained in the representation ρ (i.e. $\pi \preceq \rho$) if every function of positive type associated with π can be pointwise approximated by finite sums of positive definite type associated with ρ . If π is irreducible, this is equivalent to requiring that every normalized function of positive type associated with π can be pointwise approximated by normalized functions of positive type associated with ρ (see [3, Proposition F.1.4]).

Zimmer introduced in [13, Definition 7.3.5] a different, inequivalent notion of weak containment. A $n \times n$ -submatrix of π is a function

$$G \rightarrow M_n(\mathbb{C}), \quad g \mapsto (\langle \pi(g)e_i, e_j \rangle)_{1 \leq i, j \leq n},$$

where $\{e_1, \dots, e_n\}$ is an orthonormal family in $\mathcal{H}_\pi$. Say that π is weakly contained in ρ in Zimmer's sense (i.e. $\pi \preceq_Z \rho$) if, for every $n > 0$, every $n \times n$ -submatrix of π can be pointwise approximated by $n \times n$ -submatrices of ρ . The exact relation with the classical notion recalled above, is worked out in [3, Remark F.1.2 (ix)]; in particular, when π is irreducible, $\pi \preceq \rho$ implies $\pi \preceq_Z \rho$. Our first goal will be to make the latter statement quantitative. For this we need a definition.

Let ϕ be associated with σ , as in Definition 1.2. Let v be a unit vector in $\mathcal{H}_\sigma$ such that $\phi = \phi_{\sigma, v}$. Let $e_1, \dots, e_d$ be an orthonormal basis of $\mathbb{C}^d$; by Lemma 2.3, we find functions $f_1, \dots, f_d \in \mathbb{C}S^{k(\sigma)}$ with $\max_i \|f_i\|_1 \leq M$ such that

$$\sigma(f_i)v = e_i \quad (i = 1, \dots, d).$$

Lemma 2.4. *Let $\pi \in \text{Rep}(G, \mathcal{H})$ be a unitary representation. Assume that there is $\varepsilon > 0$ and a unit vector $\eta \in \mathcal{H}$ such that for $s \in S^{2k(\sigma)+1}$ we have*

$$|\langle \pi(s)\eta, \eta \rangle - \langle \sigma(s)v, v \rangle| < \varepsilon.$$

Set $\eta_i = \pi(f_i)\eta$. Then for $i, j = 1, \dots, d$ and $g \in S$,

$$|\langle \sigma(g)e_i, e_j \rangle - \langle \pi(g)\eta_i, \eta_j \rangle| \leq \varepsilon M^2.$$

Proof. For $g \in S$,

$$\begin{aligned} & |\langle \sigma(g)e_i, e_j \rangle - \langle \pi(g)\eta_i, \eta_j \rangle| \\ &= |\langle \sigma(g)\sigma(f_i)v, \sigma(f_j)v \rangle - \langle \pi(g)\pi(f_i)\eta, \pi(f_j)\eta \rangle| \\ &= \left| \sum_{s, t \in G} f_i(s) \overline{f_j(t)} (\langle \sigma(t^{-1}gs)v, v \rangle - \langle \pi(t^{-1}gs)\eta, \eta \rangle) \right| \\ &\leq \sum_{s, t \in G} |f_i(s)| |f_j(t)| |\langle \sigma(t^{-1}gs)v, v \rangle - \langle \pi(t^{-1}gs)\eta, \eta \rangle|. \end{aligned}$$

Since the supports of the functions f_i are contained in $S^{k(\sigma)}$, and since we have $t^{-1}gs \in S^{2k(\sigma)+1}$ for $s, t \in S^{k(\sigma)}$, we find, using the assumption, that

$$|\langle \sigma(g)e_i, e_j \rangle - \langle \pi(g)\eta_i, \eta_j \rangle| \leq \varepsilon \sum_{s,t \in G} |f_i(s)| |f_j(t)| = \varepsilon \|f_i\|_1 \|f_j\|_1 \leq \varepsilon M^2,$$

as desired. $\square$

In the previous proof, by applying the Gram–Schmidt orthonormalization process to the η_i , it is possible to show that the $d \times d$ -submatrix $(\langle \sigma(\cdot)e_i, e_j \rangle)_{1 \leq i, j \leq d}$ of σ is close on S to some $d \times d$ -submatrix of α , with an explicit bound; but we do not need that fact at this point.

2.3 Quantifying Wang’s theorem

Let $\mathcal{H}_\sigma$ be the (d -dimensional) Hilbert space of σ , and let $\mathcal{H}_{\bar{\sigma}}$ be the conjugate Hilbert space (with complex conjugate scalar multiplication and complex conjugate inner product), equipped with the conjugate representation $\bar{\sigma}$. Form the tensor product $\mathcal{H}_{\bar{\sigma}} \otimes \mathcal{H}_\pi$, carrying the representation $\bar{\sigma} \otimes \pi$. Set $\xi_i = e_i \otimes \eta_i$ and $\xi = \sum_{i=1}^d \xi_i \in \mathcal{H}_{\bar{\sigma}} \otimes \mathcal{H}_\pi$, where the e_i and η_i are as in the section above; observe that the ξ_i are pairwise orthogonal. We need an estimate on how ξ is moved by $\bar{\sigma} \otimes \pi$:

$$\begin{aligned} \|\xi - (\bar{\sigma} \otimes \pi)(g)\xi\|^2 &= 2\|\xi\|^2 - 2\operatorname{Re}\langle (\bar{\sigma} \otimes \pi)(g)\xi, \xi \rangle \\ &= 2 \sum_{i=1}^d \|\xi_i\|^2 - 2 \sum_{i,j=1}^d \operatorname{Re}\langle (\bar{\sigma} \otimes \pi)(g)\xi_i, \xi_j \rangle \\ &= 2 \sum_{i=1}^d \|\eta_i\|^2 - 2 \sum_{i,j=1}^d \operatorname{Re}\langle e_j, \sigma(g)e_i \rangle \langle \pi(g)\eta_i, \eta_j \rangle. \end{aligned}$$

Observe that for every $g \in G$ we have

$$d = \sum_{i,j=1}^d \langle \sigma(g)e_i, e_j \rangle \langle e_j, \sigma(g)e_i \rangle$$

as the e_i form an orthonormal basis. Subtracting and adding $2d$ to the previous formula, we get

$$\begin{aligned} \|\xi - (\bar{\sigma} \otimes \pi)(g)\xi\|^2 &= 2 \left[\sum_{i=1}^d (\|\eta_i\|^2 - 1) \right] - 2 \sum_{i,j=1}^d \operatorname{Re}\langle e_j, \sigma(g)e_i \rangle (\langle \pi(g)\eta_i, \eta_j \rangle - \langle \sigma(g)e_i, e_j \rangle) \end{aligned}$$

hence, using Cauchy–Schwarz,

$$\begin{aligned} & \|\xi - (\bar{\sigma} \otimes \pi)(g)\xi\|^2 \\ & \leq 2 \sum_{i=1}^d \|\eta_i\|^2 - 1 + 2 \sum_{i,j=1}^d |\langle \pi(g)\eta_i, \eta_j \rangle - \langle \sigma(g)e_i, e_j \rangle|. \end{aligned} \quad (2.1)$$

Theorem 1.3 will follow immediately from the next proposition, together with Lemma 2.2

Proposition 2.5. *Let G be a discrete Kazhdan group, S a finite symmetric generating set with $e \in S$, and let ϕ be a normalized positive definite function on G associated with a finite-dimensional unitary irreducible representation σ of G . For every $\delta \in]0, 1[$ there exists $\varepsilon_{\phi, \delta} > 0$ such that for every $\pi \in \text{Rep}(G, \mathcal{H})$, and every unit vector $x \in \mathcal{H}$ such that $|\phi(s) - \phi_{\pi, x}(s)| < \varepsilon_{\phi, \delta}$ for $s \in S^{2k(\sigma)+1}$, there exists a unit vector $x' \in \mathcal{H}$ with $\|x - x'\| \leq \delta$ such that the restriction of π to the span of $\pi(G)x'$ is isomorphic to σ .*

Proof. Set $d = \dim \sigma$. Let v be a unit vector in $\mathcal{H}_\sigma$ such that $\phi(g) = \langle \sigma(g)v, v \rangle$ for every $g \in G$. As in Section 2.2, for an orthonormal basis $e_1, \dots, e_d$ of $\mathbb{C}^d$, we find functions $f_1, \dots, f_d \in \mathbb{C}S^{k(\sigma)}$ with $\max_i \|f_i\|_1 \leq M$ such that

$$\sigma(f_i)v = e_i \quad (i = 1, \dots, d).$$

Let $0 < \varepsilon_0 < 2$ be such that (S, ε_0) is a Kazhdan pair for G . Fix $\delta \in]0, 1[$, and set

$$\varepsilon_{\phi, \delta} = \varepsilon = \frac{\delta^2 \varepsilon_0^2}{24d(d+1)M^2}.$$

Let $\pi \in \text{Rep}(G, \mathcal{H})$ and let $x \in \mathcal{H}$ be a unit vector with $|\phi_{\pi, x}(s) - \phi(s)| < \varepsilon$ for $s \in S^{2k(\sigma)+1}$. Set $\eta_i = \pi(f_i)x$. We may assume that $e_1 = v$ and the function f_1 is δ_e , so that $\eta_1 = x$. We want to prove that the vector

$$\xi = \sum_{i=1}^d (e_i \otimes \eta_i) \in \mathcal{H}_{\bar{\sigma}} \otimes \mathcal{H}$$

is $(S, t\varepsilon_0)$ -invariant for some $0 < t < 1$, in order to apply Proposition 1.1.

For $g \in S$ we have, by Lemma 2.4 and inequality (2.1),

$$\|\xi - (\bar{\sigma} \otimes \pi)(g)\xi\|^2 \leq 2d\varepsilon M^2 + 2d^2\varepsilon M^2 = 2d(d+1)\varepsilon M^2 = \frac{\delta^2 \varepsilon_0^2}{12}.$$

Again by Lemma 2.4, evaluated at $g = e$, we have

$$\|\eta_i\|^2 - 1 \leq \varepsilon M^2 < \frac{1}{2},$$

hence $\frac{1}{2} \leq \|\eta_i\|^2 \leq \frac{3}{2}$ and $\frac{d}{2} \leq \|\xi\|^2 = \sum_{i=1}^d \|\eta_i\|^2 \leq \frac{3d}{2}$. So that, for $g \in S$,

$$\|\xi - (\bar{\sigma} \otimes \pi)(g)\xi\|^2 \leq \frac{\delta^2 \varepsilon_0^2}{6d} \|\xi\|^2.$$

By Proposition 1.1, there exists a G -fixed $\xi' \in \mathcal{H}_{\bar{\sigma}} \otimes \mathcal{H}$ such that

$$\|\xi - \xi'\|^2 \leq \frac{\delta^2}{6d} \|\xi\|^2.$$

Write $\xi' = \sum_{i=1}^d e_i \otimes \zeta_i$, so that

$$\|\xi - \xi'\|^2 = \sum_{i=1}^d \|\eta_i - \zeta_i\|^2.$$

Identify $\mathcal{H}_{\bar{\sigma}} \otimes \mathcal{H}$ with the space of linear operators from $\mathcal{H}_{\sigma}$ to $\mathcal{H}$ (endowed with the Hilbert–Schmidt norm), via $u \otimes y \mapsto (w \mapsto \langle w, u \rangle y)$. Then ξ' identifies with the operator $w \mapsto \sum_{i=1}^d \langle w, e_i \rangle \zeta_i$, which is therefore an intertwining operator between σ and π . The image of this operator, which is $\text{span}\{\zeta_1, \dots, \zeta_d\}$, carries a sub-representation of π unitarily equivalent to σ (by Schur’s lemma). Set $x'' = \zeta_1$. Then

$$\begin{aligned} \|x - x''\|^2 &= \|\eta_1 - \zeta_1\|^2 \leq \sum_{i=1}^d \|\eta_i - \zeta_i\|^2 = \|\xi - \xi'\|^2 \\ &\leq \frac{\delta^2}{6d} \|\xi\|^2 \leq \frac{\delta^2}{6d} \frac{3d}{2} = \frac{\delta^2}{4}, \end{aligned}$$

i.e. $\|x - x''\| \leq \frac{\delta}{2}$. Finally, set $x' = \frac{x''}{\|x''\|}$, a unit vector in $\mathcal{H}$. Then by the triangle inequality,

$$\begin{aligned} \|x - x'\| &\leq \|x - x''\| + \|x'' - x'\| = \|x - x''\| + \|x''\| \left| 1 - \frac{1}{\|x''\|} \right| \\ &= \|x - x''\| + |\|x''\| - \|x\|| \leq 2\|x - x''\| \leq \delta. \end{aligned}$$

This concludes the proof. $\square$

Question 2.6. In the previous proof, the constant $\varepsilon_{\phi, \delta}$ depends on σ through the dimension d and the constant M from Lemma 2.3. By [12, Theorem 2.6], a discrete Kazhdan group has finitely many unitary irreducible representations of a given finite dimension (up to unitary equivalence), so Theorem 1.3 can be made uniform over all unitary irreducible representations σ with dimension less than a given dimension. Can it be made uniform over *all* finite-dimensional unitary representations?

3 Proof of Theorem 1.4

Let $\{U_n\}$ be a countable basis of open sets in the unit sphere K of $\mathcal{H}$, and let Φ be the set of all positive definite functions on G defining irreducible finite-dimensional representations. Notice that the set $X' \subseteq \text{Rep}(G, \mathcal{H})$ of all representations π such that for every $n \in \mathbb{N}$ and every $\delta > 0$ there exist $m > 0$, $x \in U_n$, $x_i \in K$, $c_i \in \mathbb{C} \setminus \{0\}$, and $\phi_i \in \Phi$, $i \leq m$, such that $x = \sum c_i x_i$, the x_i are pairwise orthogonal and each x_i is $(\pi, \phi_i, \varepsilon_{\phi_i, \delta'_i})$ -good, where $\delta'_i = \frac{\delta}{|c_i|.m}$, and $\varepsilon_{\phi_i, \delta'_i}$ is given by Theorem 1.3, is a G_δ set. Indeed, for fixed $n, \delta, m, x, \bar{x} = (x_1, \dots, x_m)$, $\bar{c} = (c_1, \dots, c_m)$, $\bar{\phi} = (\phi_1, \dots, \phi_m)$ as above, the set

$$V_{x, \bar{x}, \bar{c}, \bar{\phi}}^{n, \delta, m} = \{\pi \in \text{Rep}(G, \mathcal{H}) : \text{each } x_i \text{ is } (\pi, \phi_i, \varepsilon_{\phi_i, \delta'_i})\text{-good}\}$$

is clearly open. We also set $V_{x, \bar{x}, \bar{c}, \bar{\phi}}^{n, \delta, m}$ to be the empty set if the x_i are not pairwise orthogonal or $x \neq \sum c_i x_i$. Now we can define X' by

$$X' = \bigcap_{n \in \mathbb{N}} \bigcap_{\delta \in \mathbb{Q}^+} \bigcup_{m \in \mathbb{N}} \bigcup_{x \in U_n} \bigcup_{\bar{x} \in K^m} \bigcup_{\bar{c} \in \mathbb{C}^m} \bigcup_{\bar{\phi} \in \Phi^m} V_{x, \bar{x}, \bar{c}, \bar{\phi}}^{n, \delta, m},$$

which is a G_δ condition.

Moreover, X' is dense in $\text{Rep}(G, \mathcal{H})$ as it contains all direct sums of finite-dimensional representations, which, by our assumption, are dense in $\text{Rep}(G, \mathcal{H})$. This is because it is easy to see that for every such sum π there are densely many elements $x \in K$ of the form $\sum c_i x_i$, where x_i are pairwise orthogonal unit vectors, and each x_i is (π, ϕ_i, δ) -good for some ϕ_i and every $\delta > 0$.

Now we show that every representation in X' is a direct sum of finite-dimensional representations. Fix $\pi \in X'$. Using Zorn's lemma, we can decompose $\mathcal{H}$ into $\mathcal{H}_0$ and $\mathcal{H}_1$ such that $\mathcal{H}_0$ is the direct sum of all finite-dimensional representations contained in π . For $i = 0, 1$, let $P_{\mathcal{H}_i}$ be the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}_i$. Suppose that $\mathcal{H}_1$ is not trivial, and fix $0 < \delta < 1$, $x \in K$, pairwise orthogonal $x_i \in K$ and $c_i \in \mathbb{C} \setminus \{0\}$, $i \leq m$, such that $x = \sum c_i x_i$, each x_i is $(\pi, \phi_i, \varepsilon_{\phi_i, \frac{\delta}{|c_i|.m}})$ -good for some $\phi_i \in \Phi$, and $\|x - P_{\mathcal{H}_0}x\| > \delta$ (the last condition can be satisfied by choosing x in an appropriate U_n): By Theorem 1.3, there exist $x'_i \in K$, $i \leq m$, inducing irreducible finite-dimensional representations, and such that

$$\|x_i - x'_i\| < \frac{\delta}{|c_i|.m},$$

that is, $\|x - \sum c_i x'_i\| < \delta$. But then, clearly, $x'_{i_0} \notin \mathcal{H}_0$ for some $i_0 \leq m$, as if this were not the case, we would get that

$$\|x - \sum c_i x'_i\| \geq \|x - P_{\mathcal{H}_0}x\| > \delta.$$

Since $P_{\mathcal{H}_1}$ is a G -intertwiner, the image under $P_{\mathcal{H}_1}$ of the linear span of $\pi(G)x'_{i_0}$, is an invariant subspace of $\mathcal{H}_1$, which is a contradiction.

Now let X'' be the set of all representations that contain every finite-dimensional representation with infinite multiplicity. As G is a Kazhdan group, we can see that X'' is given by a G_δ condition. Indeed, for $[\sigma]$ the isomorphism class of a finite-dimensional unitary irreducible representation of G , and $n > 0$, let $V_{[\sigma],n}$ be the set of representations $\pi \in \text{Rep}(G, \mathcal{H})$ such that $[\sigma]$ appears in π with multiplicity at least n . Clearly, $V_{[\sigma],n}$ is open and $X'' = \bigcap_{[\sigma]} \bigcap_n V_{[\sigma],n}$, where the intersection is countable because there are countably many $[\sigma]$.

By our assumption on $C^*(G)$, the set X'' is dense. Thus, $X = X' \cap X''$ is a dense G_δ set, all the representations of which are direct sums of finite-dimensional representations, each appearing with infinite multiplicity. Clearly, all elements in X are conjugate. $\square$

Remark 3.1. The converse of Theorem 1.4 also follows from [9, Theorem 2.5]. That is, if either G does not have Property (T), or $C^*(G)$ is not residually finite-dimensional, then all classes in $\text{Rep}(G, \mathcal{H})$ are meager. We will now prove this. By [9, Theorem 2.5], if for a separable C^* -algebra A the set of isolated points in $\hat{A}$ is not dense, then the restriction of the action of $U(\mathcal{H})$ by conjugation on a dense G_δ invariant subset of $\text{Rep}(A, \mathcal{H})$ is turbulent. That, by the definition of turbulence, in particular implies that every class in $\text{Rep}(G, \mathcal{H})$ is meager. Now take $A = C^*(G)$ and notice that isolated points in $\hat{G}$ correspond to finite-dimensional representations. Indeed, suppose that A is an arbitrary unital separable C^* -algebra and $\pi \in \hat{A}$ is isolated. Then we can decompose A as $I \oplus J$, where $\hat{J} = \{\pi\}$. As J is a separable C^* -algebra with a one-point spectrum, it is isomorphic to the compact operators on the Hilbert space $\mathcal{H}_\pi$. Since A is unital, so is J , meaning that J and π are finite-dimensional. It follows that when G does not have Property (T), $\hat{G}$ does not have isolated points, by [12, Theorem 2.1]; when $C^*(G)$ is not residually finite-dimensional, then the isolated points in $\hat{G}$ are not dense by Archbold's main result in [1].

4 Proof of Theorem 1.5

For a unitary representation π , we denote by $\infty \cdot \pi$ the ℓ^2 -direct sum of countably many copies of π .

Lemma 4.1. *Let H be a locally compact group. Assume that H has (up to unitary equivalence) countably many finite-dimensional irreducible unitary representations $\sigma_1, \sigma_2, \dots$. Then the representation $\bigoplus_{n=1}^{\infty} \infty \cdot \sigma_n$ is unitarily equivalent to a Koopman representation.*

Proof. View σ_n as a continuous homomorphism $H \rightarrow U(N_n)$. Let K_n denote the closure of $\sigma_n(H)$ in $U(N_n)$, so that K_n is a compact group (on which H acts by

left translations by elements of $\sigma_n(H)$). Let m_n denote normalized Haar measure on K_n , and let λ_n denote the regular representation of K_n on $L^2(K_n, m_n)$. For $p \geq 1$, let $K_{n,p}$ denote a copy of K_n endowed with the measure $2^{-n-p}m_n$. Set $X = \coprod_{n,p} K_{n,p}$, endowed with the H -invariant probability measure

$$\mu = \bigoplus_{n,p} 2^{-n-p}m_n.$$

Note that the H -representations on $L^2(X, \mu)$ and on $L^2_0(X, \mu)$ are equivalent, as $L^2(X, \mu)$ contains the trivial representation with infinite multiplicity.

So it is enough to prove that the H -representation π on $L^2(X, \mu)$ is equivalent to $\bigoplus_{n=1}^{\infty} \infty \cdot \sigma_n$. To see this, first observe that π is equivalent to $\bigoplus_n \infty \cdot \pi_n$, where $\pi_n = \lambda_n \circ \sigma_n$. By the Peter–Weyl theorem, π_n decomposes as a direct sum of finite-dimensional irreducible representations of H , hence of certain representations σ_k , and moreover σ_n is a sub-representation of π_n (because the natural representation of K_n on $\mathbb{C}^{N_n}$ is irreducible, hence appears as a sub-representation of λ_n). This shows that $\bigoplus_n \infty \cdot \pi_n$ is equivalent to $\bigoplus_n \infty \cdot \sigma_n$. $\square$

To prove Theorem 1.5, observe that a discrete Kazhdan group G satisfies the assumption of Lemma 4.1 (by [12, Theorem 2.6]). Let $(\sigma_n)_{n \in \mathbb{N}}$ be an enumeration of all finite-dimensional irreducible unitary representations of G . By Theorem 1.4 and its proof, the representation $\bigoplus_{n=1}^{\infty} \infty \cdot \sigma_n$ has a comeager conjugacy class.

In particular, we get the following statement which was proved in [4] for finite abelian groups.

Corollary 4.2. *Let G be a finite group. Then the set of unitary representations realizable by an action is comeager in $\text{Rep}(G, \mathcal{H})$.*

Remark 4.3. Kechris proves (see [8, Section (F) in Appendix H]) that, if G is torsion-free abelian, then the set of representations realizable by an action is meager in $\text{Rep}(G, \mathcal{H})$.

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