Robinson's conjecture for classical groups

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Abstract. Robinson's conjecture states that the height of any irreducible ordinary character in a block of a finite group is bounded by the size of the central quotient of a defect group. This conjecture had been reduced to quasi-simple groups by Murai. The case of odd primes was settled completely in our predecessor paper. Here we investigate the 2-blocks of finite quasi-simple classical groups.

1 Introduction

The arithmetic nature of the irreducible character degrees of a finite group has been a fruitful area of research since the very beginnings of the subject. Several deep conjectures in character theory concern the p -parts occurring in character degrees, like, for example, the McKay conjecture and the Dade conjecture. In 1996, Geoffrey Robinson [13] proposed an extension of Richard Brauer's famous height zero conjecture from 1955, bounding the maximal power of a prime p dividing the degree of an irreducible character of a finite group G in terms of invariants of its p -block.

Conjecture (Robinson). Let G be a finite group, p a prime and $\chi \in \text{Irr}(G)$ lying in a p -block of G with defect group D . Then

$$p^{\text{def}(\chi)} \geq |Z(D)| \tag{RC}$$

with equality if and only if D is abelian.

Here the p -defect of an irreducible character $\chi \in \text{Irr}(G)$ is the integer $\text{def}(\chi)$ such that $|G|_p = p^{\text{def}(\chi)} \chi(1)_p$, where n_p denotes the p -part of an integer n .

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We recently succeeded in showing this conjecture for all primes $p \geq 3$ (see [8]) based on Murai's reduction of (RC) to blocks of quasi-simple groups [8, Theorem 2.3]. Here we prove (RC) for the 2-blocks of finite quasi-simple classical groups in odd characteristic.

Theorem 1. *The 2-blocks of covering groups of finite simple linear, unitary, symplectic and orthogonal groups do not provide minimal counterexamples to Robinson's conjecture.*

Thus, by the results in [8], in order to complete the proof of Robinson's conjecture in full generality, it only remains to deal with the so-called isolated 2-blocks of quasi-simple groups of exceptional Lie type in odd characteristic.

After some preparations on 2-blocks of the general linear and unitary groups, we treat the principal 2-blocks of the special linear and unitary groups in Section 3 and the remaining finite quasi-simple classical groups in Section 4; the proof of Theorem 1 is then achieved in Section 5.

Throughout the paper, we let v denote the exponential valuation associated to 2 , normalised so that $v(2) = 1$. For a finite group H , we write $v(H)$ for $v(|H|)$.

2 The general linear and unitary groups

2.1 Some notation and background

Assume $q = p^f$ is a power of a prime p . Let $\overline{\mathbb{F}}$ be an algebraic closure of the finite field $\mathbb{F}_p$. As usual, $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}})$ denotes the group of all invertible $n \times n$ matrices over $\overline{\mathbb{F}}$. Let $\gamma: \mathbf{G} \rightarrow \mathbf{G}$ be the map sending A to $(A^{-1})^t$, where t denotes the transpose of matrices. If F_p is the Frobenius map of $\overline{\mathbb{F}}$ and $F_q = (F_p)^f$, then, for $\eta \in \{\pm 1\}$, $F = \gamma^{\frac{1-\eta}{2}} F_q$ induces a Steinberg endomorphism of $\mathbf{G}$ with the finite group of fixed points $\mathbf{G}^F = \mathrm{GL}_n(\eta q)$. Recall that $\mathrm{GL}_n(-q)$ denotes the general unitary group

$$\mathrm{GU}_n(q) = \{A \in \mathrm{GL}_n(q^2) \mid F_q(A)^t A = I_n\},$$

where I_n is the identity matrix of degree n . We will use the analogous notation $\mathrm{SL}_n(\eta q)$ for $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$. Denote $\mathbb{F} = \mathbb{F}_{\eta q}$ for $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$, depending on $\eta = \pm 1$.

Let $\mathrm{Irr}(\mathbb{F}[X])$ be the set of monic irreducible polynomials in $\mathbb{F}[X]$ different from X . Denote

$$\begin{aligned} \mathcal{F}_1 &:= \{f \in \mathrm{Irr}(\mathbb{F}_{q^2}[X]) \mid f = \tilde{f}\}, \\ \mathcal{F}_2 &:= \{f \tilde{f} \mid f \in \mathrm{Irr}(\mathbb{F}_{q^2}[X]), f \neq \tilde{f}\}, \end{aligned}$$

where $\tilde{\cdot}$ is the permutation of $\text{Irr}(\mathbb{F}_{q^2}[X])$ of order 2 mapping

$$f(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0$$

to $F_q(a_0^{-1}X^m f(X^{-1}))$. Let

$$\mathcal{F} = \begin{cases} \text{Irr}(\mathbb{F}[X]) & \text{if } \eta = 1, \\ \mathcal{F}_1 \cup \mathcal{F}_2 & \text{if } \eta = -1. \end{cases}$$

As introduced in [9, § 1], the polynomials in $\mathcal{F}$ serve as the “elementary divisors” of semisimple elements of $\text{GL}_n(\eta q)$. Given a semisimple element s of $G = \text{GL}_n(\eta q)$, let $\prod_{f \in \mathcal{F}} s_f$ be the primary decomposition of s and $\prod_{f \in \mathcal{F}} L_f$ the corresponding decomposition of the Levi subgroup $L := C_G(s)$. Then we have $n = \sum_{f \in \mathcal{F}} m_f d_f$, where m_f is the multiplicity of f in the characteristic polynomial of s and d_f denotes the degree of $f \in \mathcal{F}$. Notice that L_f is isomorphic to $\text{GL}_{m_f}((\eta q)^{d_f})$. Correspondingly, the Weyl group W of L can be decomposed as $\prod_f W_f$, where W_f is the Weyl group of L_f .

From now on, we assume p is an odd prime. Let $a := v(q^2 - 1) - 1$. We give some elementary lemmas, which will be needed in the sequel.

Lemma 2.1. *Let $d = 2^\alpha m$, where m is odd and $\alpha \geq 0$.*

- (a) *If $4 \mid (q - \eta)$, then $v((\eta q)^d - 1) = a + \alpha$ and $v((\eta q)^d + 1) = 1$.*
- (b) *Suppose $4 \mid (q + \eta)$. Then, if $\alpha \geq 1$,*

$$v((\eta q)^d - 1) = a + \alpha \quad \text{and} \quad v((\eta q)^d + 1) = 1,$$

while, if $\alpha = 0$, $v((\eta q)^m + 1) = a$ and $v((\eta q)^m - 1) = 1$.

Note that $c \mid ((\eta q)^{d_f} - 1)$ if c is the multiplicative order of some root of $f \in \mathcal{F}$.

Lemma 2.2. *Let $f \in \mathcal{F}$. Suppose that f has a root of order 2^m , where $m \geq 1$. Then $d_f = 2^{m'}$, where*

$$m' = \begin{cases} m - a & \text{if } m > a, \\ 1 & \text{if } 4 \mid (q + \eta) \text{ and } 1 < m \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, we have $v((\eta q)^{2^{m'}} - 1) \geq m$, so 2^m divides $(\eta q)^{2^{m'}} - 1$. However, since d_f is the minimal integer such that $2^m \mid ((\eta q)^{d_f} - 1)$, we conclude that $d_f \mid 2^{m'}$.

We now show that $2^{m'} \mid d_f$. If $4 \mid (q - \eta)$, then $v((\eta q)^{d_f} - 1) = a + v(d_f)$. Thus $2^{m'} \mid d_f$, and the assertion follows. So we assume $4 \mid (q + \eta)$. Then we fall into the following three cases: $m > a$, $1 < m \leq a$, $m = 1$. The proof is completed by a case-by-case checking. $\square$

Lemma 2.3. *Let $a \geq 2$, and let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion of a positive integer n with $b_1 < \cdots < b_t$. Suppose $n = \sum_{i \geq 0} s_i 2^i$, where all s_i are non-negative integers. Then*

- (a) $\sum_i s_i \geq t$,
- (b) $a(\sum_i s_i - t) + \sum_{i \geq 0} i s_i \geq \sum_{i=1}^t b_i$,
- (c) if $n \geq 4$ and $s_0 \neq 0$, then $\sum_i s_i - t \geq b_1$.

In all three cases, equality occurs if and only if $n = \sum_i s_i 2^i$ is the 2-adic expansion.

- (d) Let $i_0 = \min\{i \mid s_i > 0\}$ and $k = \min\{a, i_0\}$. Then

$$a\left(\sum_i s_i - t\right) + \sum_i i s_i > k + 1$$

unless $n = \sum_i s_i 2^i$ is the 2-adic expansion and either $n = 3$ or $n = 2^h$ with $h \leq a$.

Proof. By considering $v(n)$, we have $i_0 := \min\{i \mid s_i > 0\} \leq b_1$. Then (a) and (b) follow by applying the inductive hypothesis to

$$n - 2^{i_0} = (s_{i_0} - 1)2^{i_0} + \sum_{i > i_0} s_i 2^i = 2^{i_0} + \cdots + 2^{b_1-1} + \sum_{i=2}^t 2^{b_i}.$$

We prove (c). If $b_1 = 0$, it is just (a). Thus we assume $b_1 \neq 0$. Then

$$n - 1 = 1 + \cdots + 2^{b_1-1} + 2^{b_2} + \cdots + 2^{b_t}.$$

Since $s_0 \neq 0$, we get $n - 1 = (s_0 - 1) + \sum_{i \neq 0} s_i 2^i$, and thus the assertion follows from (a).

For (d), $a(\sum_i (s_i - t) + \sum_i i s_i) > k$ follows from (b). Now note that, since $a \geq 2$, by an argument similar to the first paragraph of the proof of [8, Lemma 6.5], if $n = \sum_i s_i 2^i$ is not the 2-adic expansion, then

$$a\left(\sum_i s_i - t\right) + \sum_i i s_i \geq a + \sum_i i s_i > k + 1,$$

as claimed. Now we assume that $n = \sum_i s_i 2^i$ is the 2-adic expansion. If $t > 1$, then $\sum_i b_i > b_2 + 1 \geq k + 1$ unless $n = 3$. $\square$

2.2 Sylow 2-subgroups of $\mathrm{GL}_n(\eta q)$

Assume $4 \mid (q - \eta)$. Then, by Lemma 2.1, $v(q - \eta) = a$. Let R_+^0 be a Sylow 2-subgroup of $\mathrm{GL}_1(\eta q)$. For $b \geq 1$, let R_+^b be the Sylow 2-subgroup $R_+^0 \wr X_b$ of $\mathrm{GL}_{2^b}(\eta q)$, where X_b denotes a Sylow 2-subgroup of the group of permutation matrices of degree 2^b , which is the iterated wreath product $C_2 \wr \cdots \wr C_2$ of b copies of a cyclic group C_2 of order 2. In particular, $v(R_+^b) = a2^b + v(2^b!)$.

On the other hand, when $4 \mid (q + \eta)$, we denote by R_-^1 the Sylow 2-subgroup of $\mathrm{GL}_2(\eta q)$ as defined in [5], isomorphic to the semi-dihedral group of order 2^{a+2} . For $b \geq 2$, we denote $R_-^b = R_-^1 \wr X_{b-1} \leq \mathrm{GL}_{2^b}(\eta q)$, where X_b is as defined above. For convenience, we set $R_-^0 := \{\pm 1\}$, the Sylow 2-subgroup of $\mathrm{GL}_1(\eta q)$.

We write the 2-adic expansion of n as $n = 2^{b_1} + \cdots + 2^{b_t}$ with $b_1 < \cdots < b_t$. Recall some facts on 2-blocks and the structure of Sylow 2-subgroups of $\mathrm{GL}_n(\eta q)$; for this, recall that, according to Lusztig, the set of irreducible characters of a finite reductive group G is partitioned into *Lusztig series* $\mathcal{E}(G, s)$ labelled by semisimple elements s of the Langlands dual group G^* , up to conjugation (see, e.g., [7, § 13]). The elements of $\mathcal{E}(G, 1)$ are the *unipotent characters of G* . Lusztig's *Jordan decomposition* states that $\mathcal{E}(G, s)$ is in bijection with $\mathcal{E}(C_{G^*}(s), 1)$ whenever s has connected centraliser in the ambient algebraic group in such a way that character degrees differ by a factor of $|G^* : C_{G^*}(s)|_{p'}$. For s a semisimple $2'$ -element, one sets

$$\mathcal{E}_2(G, s) := \bigcup_t \mathcal{E}(G, st),$$

where the union runs over 2-elements $t \in C_{G^*}(s)$. This is known to be a union of 2-blocks of G . For $s = 1$, this contains the trivial character and hence the principal 2-block of G .

Theorem 2.4. *Let $G = \mathrm{GL}_n(\eta q)$, q odd, and $\epsilon \in \{\pm 1\}$ such that $q \equiv \epsilon\eta \pmod{4}$.*

- (a) *The only unipotent 2-block of G is the principal block $\mathcal{E}_2(G, 1)$.*
- (b) *A Sylow 2-subgroup of G is given by*

$$R = \prod_{i=1}^t R_\epsilon^{b_i}.$$

Thus R is abelian if and only if $n = 1$.

- (c) *The centre of R is given by*

$$Z(R) = \prod_{i=1}^t R_\epsilon^0 \otimes I_{2^{b_i}}.$$

In particular, $v(Z(R)) = t v(q - \eta)$.

Proof. Part (a) was shown by Broué [2], and (b) follows from [5]. From this, (c) can easily be derived. $\square$

2.3 Robinson's conjecture for $\mathrm{GL}_n(\eta q)$

Now we prove Robinson's conjecture for the principal 2-block of $\mathrm{GL}_n(\eta q)$.

Lemma 2.5. *Let χ be a unipotent character of $\mathrm{GL}_n(\eta q)$. Then $\mathrm{def}(\chi) \geq nv(q - \eta)$.*

Proof. For $G = \mathrm{GL}_n(q)$, all unipotent characters lie in the principal series. Thus their degree polynomials are not divisible by $q - 1$ (see [4, § 13.7]), while the order polynomial of G is divisible by $(q - 1)^n$, so we obtain the stated bound. The claim for $\mathrm{GU}_n(q)$ follows as the order polynomial as well as the degree polynomials of unipotent characters are obtained by replacing q by $-q$ in those for $\mathrm{GL}_n(q)$. $\square$

Proposition 2.6. *Let $G = \mathrm{GL}_n(\eta q)$, $n \geq 2$. Let B be the principal 2-block of G , and let R be a Sylow 2-subgroup of G . Then $\mathrm{def}(\chi) > v(Z(R))$ for any $\chi \in \mathrm{Irr}(B)$.*

Proof. There is a semisimple 2-element $s \in G$ such that $\chi \in \mathcal{E}(G, s)$ by Theorem 2.4 (a). For $b \geq 0$, define $\mathcal{F}_b = \{f \in \mathcal{F} \mid d_f = 2^b\}$. Let $\mathcal{F}(s) \subset \mathcal{F}$ be the set of elementary divisors of s , and set $\mathcal{F}_b(s) := \mathcal{F}(s) \cap \mathcal{F}_b$. Then, by Lemma 2.2, we have $\mathcal{F}(s) = \coprod_b \mathcal{F}_b(s)$ and thus

$$n = \sum_{f \in \mathcal{F}} m_f d_f = \sum_{b \geq 0} \sum_{f \in \mathcal{F}_b} m_f 2^b.$$

Let ψ be the unipotent character of $L := C_G(s)$ in Jordan correspondence with χ . Let $\prod_{f \in \mathcal{F}(s)} L_f$, $\boxtimes_{f \in \mathcal{F}(s)} \psi_f$ be the respective decompositions of L , ψ , corresponding to the primary decomposition of s . By the degree formula for the Jordan decomposition of characters, we have

$$v(\chi(1)) = v(G : L) + v(\psi(1))$$

and so, as $v(L) = \sum_{f \in \mathcal{F}(s)} v(L_f)$,

$$\mathrm{def}(\chi) = \mathrm{def}(\psi) = \sum_{f \in \mathcal{F}(s)} \mathrm{def}(\psi_f).$$

As before, write $n = 2^{b_1} + \cdots + 2^{b_t}$ for the 2-adic expansion of n .

Let first $4 \mid (q - \eta)$, so $v(q - \eta) = a$. By Lemmas 2.1 and 2.5,

$$\mathrm{def}(\psi_f) \geq m_f(a + b) \quad \text{for } f \in \mathcal{F}_b(s).$$

It follows by Lemma 2.3 that

$$\text{def}(\chi) \geq a \sum_{f \in \mathcal{F}} m_f + \sum_b b \sum_{f \in \mathcal{F}_b} m_f \geq at + \sum_{i=1}^t b_i > at, \quad (2.1)$$

as $n > 1$. By Theorem 2.4 (c), we know that $v(Z(R)) = at$, whence the claim in this case.

Now assume $4 \mid (q + \eta)$. Here, by Lemma 2.3 (a),

$$\begin{aligned} \text{def}(\chi) &\geq \sum_{f \in \mathcal{F}_0} m_f + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f (a + b) \\ &= \sum_{f \in \mathcal{F}} m_f + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f (a + b - 1) \\ &\geq t + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f (a + b - 1), \end{aligned}$$

with equality in the last line if and only if $n = \sum_b \sum_{f \in \mathcal{F}_b} m_f 2^b$ is the 2-adic expansion of n . Furthermore, as $a \geq 2$, the last sum is zero only when $\mathcal{F}_b(s) = \emptyset$ for $b \geq 1$. But, in this case, $n = 1$, which was excluded, so we conclude by Theorem 2.4 (c). $\square$

We note the following for later use.

Remark 2.7. In the notation of Proposition 2.6, assume $4 \mid (q + \eta)$. The proof of Proposition 2.6 shows that $n = 2$ if $\text{def}(\chi) - v(Z(R)) = 1$ and n is even.

3 The groups $\text{SL}_n(\eta q)$

3.1 Robinson's conjecture for $\text{SL}_n(\eta q)$, n odd

The case of $\text{SL}_n(\eta q)$ with n odd is considerably easier than the even degree case.

Proposition 3.1. *Let $H = \text{SL}_n(\eta q)$ with $n \geq 3$ odd, and let $Z \leq Z(H)$. Then Robinson's conjecture holds for the unipotent 2-blocks of H/Z .*

Proof. Set $G = \text{GL}_n(\eta q)$. Since n is odd, $H \cap \mathbf{O}_2(Z(G)) = 1$, so

$$G = H_1 \times \mathbf{O}_2(Z(G)) \quad \text{with } H_1 = \mathbf{O}^2(G).$$

Obviously, all irreducible characters of G restrict irreducibly to H_1 , and the Sylow 2-subgroups of G are the direct products of Sylow 2-subgroups of H_1 with $\mathbf{O}_2(Z(G))$. Thus, if b_1 is a 2-block of H_1 and B is the 2-block of G covering b_1 , then Robinson's conjecture holds for b_1 if and only if it holds for B .

Furthermore, $|H_1 : H|$ is odd, and so, for $\chi \in \text{Irr}(H_1)$, all constituents of $\chi|_H$ have the same defect as χ . Thus, if b is a 2-block of H and b_1 is a 2-block of H_1 covering b , then Robinson's conjecture holds for b if and only if it holds for b_1 .

Finally, $|Z(H)| = \gcd(n, q-1)$ is odd. Thus, if $\bar{b}$ is a 2-block of H/Z , where $Z \leq Z(H)$, and b is the 2-block of H dominating $\bar{b}$, then Robinson's conjecture holds for $\bar{b}$ if and only if it holds for b . The claim thus follows from Proposition 2.6. $\square$

3.2 Sylow 2-subgroups of $\text{SL}_n(\eta q)$ for n even

For $n \geq 2$ even, we first determine the centres of Sylow 2-subgroups. Denote $c = v(q - \eta)$, and, for $b \geq 0$, set $c(b) := \max\{c - b, 0\}$. Note that

$$c = \begin{cases} a & \text{if } 4 \mid (q - \eta), \\ 1 & \text{if } 4 \nmid (q - \eta). \end{cases}$$

For a subgroup $H \leq \text{GL}_n(\mathbb{F})$, we set

$$\mathcal{D}(H) := \{\det(A) \mid A \in H\} \leq \mathbb{F}^\times.$$

Lemma 3.2. *Let $b \geq 0$. In the notation of Theorem 2.4, we have*

$$|\mathcal{D}(R_\epsilon^b)| = 2^c \quad \text{and} \quad |\mathcal{D}(Z(R_\epsilon^b))| = 2^{c(b)}.$$

Proof. If $4 \mid (q - \eta)$, then R_+^b contains $\text{diag}(\zeta, 1, \dots, 1)$ with $\zeta \in \mathbb{F}^\times$ of order $o(\zeta) = 2^c$, so $|\mathcal{D}(R_+^b)| = 2^c$. Then $|\mathcal{D}(Z(R_+^b))| = 2^{c(b)}$ follows from

$$\mathcal{D}(Z(R_+^b)) = \{x^{2^b} \mid x \in \mathcal{D}(R_+^0)\}.$$

For $\epsilon = -$, this follows from the fact that R_-^1 is conjugate to $\left(\begin{smallmatrix} \zeta & \\ & 1 \end{smallmatrix}\right)_{\xi^{\eta q}}$ for $\zeta \in \mathbb{F}^\times$ of order $o(\zeta) = 2^{a+1}$. $\square$

Lemma 3.3. *Let $G = \text{GL}_n(\eta q)$ and $H = \text{SL}_n(\eta q)$ with n even. Suppose that R is a Sylow 2-subgroup of G as in Theorem 2.4 (b). Write $Q = R \cap H$. Then $|\mathcal{D}(R)| = 2^c$. Furthermore,*

$$|\mathcal{D}(Z(R))| = 2^{c(b_1)} \quad \text{and} \quad v(R : QZ(R)) = c - c(b_1) = \min\{c, b_1\}.$$

Proof. This follows easily from Lemma 3.2. $\square$

When $4 \nmid (q - \eta)$, the Weyl group part of R may not be contained in $\text{SL}_n(\eta q)$. To circumvent this problem, for $b \geq 1$, we will use $R_+^b = R_+^0 \wr X_b \cong R_+^1 \wr X_{b-1}$ and call X_{b-1} the Weyl part and $(R_+^1)^{2^{b-1}}$ the base group of R_+^b . Observe that, with this convention, the Weyl part of R_+^b is contained in $\text{SL}_n(\eta q)$.

Lemma 3.4. *Let $G = \mathrm{GL}_n(\eta q)$ and $H = \mathrm{SL}_n(\eta q)$ with n even. Suppose that R is as in Theorem 2.4 (b). Let $R_{0,i}$ be the base group and $R_{W,i} = X_{b_i-1}$ the Weyl part of $R_\epsilon^{b_i}$. Write $R_0 = R_{0,1} \times \cdots \times R_{0,t}$ and $R_W = R_{W,1} \times \cdots \times R_{W,t}$. Denote $Q = R \cap H$. Then*

$$Q = (R_0 \cap H) \rtimes R_W \quad \text{and} \quad Z(Q) = Z(R) \cap H.$$

More specifically, if $R = R_\epsilon^1$, then Q is isomorphic to the generalised quaternion group of order 2^{a+1} , and so $Z(Q) = \{\pm I_2\}$.

Proof. The Weyl part $R_{W,i}$ is generated by direct products of tensor products of matrices of the form $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$, whence $R_{W,i} \leq H$. Hence $R_W \leq H$, and so $Q = (R_0 \cap H) \rtimes R_W$.

By Lemma 3.2, $\mathcal{D}(R_\epsilon^0) = \mathcal{D}(R_\epsilon^1)$. If $t \geq 2$, then the projections $p_i: Q \rightarrow R_\epsilon^{b_i}$ are surjective for all i . Hence $Z(Q) \leq Z(R)$, and so $Z(Q) \leq Z(R) \cap H$. Since the converse inclusion is obvious, we get $Z(Q) = Z(R) \cap H$ in this case.

Now we may assume $R = R_\epsilon^{b+1}$ with $b \geq 0$. If $b = 0$, the assertion is obvious since Q is isomorphic to the generalised quaternion group of order 2^{a+1} . So we may assume $b > 0$. Let $(A_1, \dots, A_{2b})\tau$ be an element of $Z(Q)$ with $\tau \in X_b$ and $(A_1, \dots, A_{2b}) \in R_0$. Let B be any element of R_ϵ^1 with $\det(B) = \pm 1$ so that $(B, \dots, B) \in Q$. Now

$$[(A_1, \dots, A_{2b})\tau, (B, \dots, B)] = 1.$$

Hence $(A_1, \dots, A_{2b}) \in Z(R)$. Moreover, since $[(A_1, \dots, A_{2b})\tau, \pi] = 1$ for any $\pi \in X_b$ and X_b transitively permutes the A_i , we have $A_1 = \dots = A_{2b}$. It follows that $\tau \in Z(Q)$ and $\tau = 1$, and so $Z(Q) = Z(R) \cap H$, completing the proof. $\square$

3.3 Robinson's conjecture for $\mathrm{SL}_n(\eta q)$, n even

Now we prove Robinson's conjecture for unipotent 2-blocks of $\mathrm{SL}_n(\eta q)$ with even n .

Proposition 3.5. *Let $H = \mathrm{SL}_n(\eta q)$ with $n \geq 2$ even, and let Q be a Sylow 2-subgroup of H . Then, for every $\theta \in \mathcal{E}_2(H, 1)$, we have $\mathrm{def}(\theta) > v(Z(Q))$.*

Proof. Let $s \in H^*$ be a 2-element such that $\theta \in \mathcal{E}(H, s)$. Let $G = \mathrm{GL}_n(\eta q)$, and let $\chi \in \mathcal{E}_2(G, 1)$ lie above θ . By Theorem 2.4 (a), χ lies in the principal 2-block of G , so θ lies in the principal 2-block of H . Let R be a Sylow 2-subgroup of G as described in Theorem 2.4 (b) and with $R \cap H = Q$.

Now we have

$$\mathrm{def}(\theta) = v(H) - v(\theta(1)) = \mathrm{def}(\chi) - v(q - \eta) + v(\chi(1)/\theta(1))$$

and

$$\begin{aligned} v(Z(Q)) &= v(Z(R)) - v(R : Q) + v(R : QZ(R)) \\ &= v(Z(R)) - v(q - \eta) + \min\{c, b_1\} \end{aligned}$$

by Lemmas 3.3 and 3.4.

We first let $4 \mid (q - \eta)$. Then $\text{def}(\chi) - v(Z(R)) \geq \sum_{i=1}^t b_i$ by (2.1), so

$$\begin{aligned} \text{def}(\theta) &\geq v(Z(R)) - v(q - \eta) + \sum_{i=1}^t b_i + v(\chi(1)/\theta(1)) \\ &= v(Z(Q)) - \min\{c, b_1\} + \sum_{i=1}^t b_i + v(\chi(1)/\theta(1)) \geq v(Z(Q)) \end{aligned}$$

with equality only when $t = 1$, $c \geq b_1 =: b$, and so $n = 2^b$. In that case, there is exactly one element f in $\mathcal{F}(s)$, and $m_f(s) = 1$, $d_f = 2^b = n$. Therefore, in order to complete the proof, it suffices to show that $v(\chi(1)/\theta(1)) > 0$. To do this, we compute the number m of irreducible constituents of $\chi|_H$ according to the method of Denoncin [6]. Let ζ be a root of f in $\overline{\mathbb{F}}$. Since G/H is cyclic and isomorphic to $Z(G)$, it follows by [6, Proposition 3.5] that

$$m = |\{z \in Z(G) \mid zs \text{ is } G\text{-conjugate to } s\}|.$$

Now

$$\begin{aligned} m &= |\{\tau \in \mathbb{F}^\times \mid z = \text{diag}(\tau, \dots, \tau) \in Z(G), \tau\zeta \text{ is a root of } f \text{ in } \overline{\mathbb{F}}\}| \\ &= |\{\tau \in \mathbf{O}_2(\mathbb{F}^\times) \mid \tau\zeta \text{ is a root of } f \text{ in } \overline{\mathbb{F}}\}|. \end{aligned}$$

Notice that the roots of f are $\zeta, \zeta^{\eta q}, \dots, \zeta^{(\eta q)^{n-1}}$. Since $n = 2^b$, we may write

$$\zeta^{(\eta q)^{\frac{n}{2}}} = \xi\zeta \quad \text{with } \xi = \zeta^{(\eta q)^{2^{b-1}-1}} \neq 1.$$

We claim $\xi^2 = 1$. Indeed, ζ has multiplicative order 2^{a+b} . In addition, we have $v((\eta q)^{2^b} - 1) = v((\eta q)^{2^{b-1}} - 1) + 1$ by Lemma 2.1. Hence ξ has multiplicative order 2. Thus $v(m) > 0$, and we have $v(\chi(1)/\theta(1)) > 0$ by Clifford theory.

Now we assume $4 \nmid (q - \eta)$. If $Z(R) \not\leq H$, then $v(R : QZ(R)) = 0$, and hence the claim holds by Proposition 2.6. So we may assume $Z(R) \leq H$. Then we get $v(R : QZ(R)) = 1$, and so the claim holds if $\text{def}(\chi) - v(Z(R)) \geq 2$. By Remark 2.7, we have $\text{def}(\chi) - v(Z(R)) = 1$ only when $n = 2$. But, in this case, $|Z(Q)| = 2$, and so Robinson's conjecture holds by an old result of Brauer (see [8, Lemma 3.1]). $\square$

3.4 Robinson's conjecture for central quotients of $\mathrm{SL}_n(\eta q)$, n even

In this section, we investigate unipotent 2-blocks of $\mathrm{SL}_n(\eta q)/Z$ for odd q , where $Z \leq Z(\mathrm{SL}_n(\eta q))$. Throughout, $n \geq 2$ is even. As before, we first determine the centres of Sylow 2-subgroups of central quotient groups of $\mathrm{SL}_n(\eta q)$.

Lemma 3.6. *Keep the notation of Lemma 3.4; in particular, n is even. We write $\bar{Q} = Q/Z$ and $Z(\bar{Q}) = Z_0/Z$ for $1 \neq Z \leq \mathbf{O}_2(Z(H))$.*

(a) *If $t \geq 2$, then $Z_0 = Z(Q)$.*

(b) *Assume $R = R_\epsilon^b$.*

(1) *If $b = 1$, then Q/Z is isomorphic to the dihedral group of order 2^a .*

(2) *If $b > 1$, then $Z_0 = \langle Z(Q), I_{2^b} \otimes \mathrm{diag}(1, -1) \rangle$.*

Thus, if $\bar{Q}$ is not abelian, then $Z(\bar{Q}) = Z(Q)/Z$ unless either $R = R_\epsilon^1$ with $a > 2$ or $R = R_\epsilon^b$ with $b > 1$, in which cases we have

$$v(Z(\bar{Q})) = v(Z(Q)) - v(Z) + 1.$$

Proof. Recall that $|\mathcal{D}(R_-^1)| = 2$ and $\mathcal{D}(R_+^0) = \mathcal{D}(R_+^1)$ by Lemma 3.2.

(a) Suppose $t \geq 2$. We first claim that $Z_0 \subseteq R_0$, where R_0 is the base group of R as in Lemma 3.4. Otherwise, let $(A_1, A_2, \dots)\tau$ be an element of Z_0 , where $1 \neq \tau \in R_W$ and A_i is an element of $R_{0,i}$ for each i ; here, $R_{0,i}$ is the base subgroup of $R_\epsilon^{b_i}$ as in Lemma 3.4. Since R has more than one component, arguing as in the last paragraph of the proof of Lemma 3.4, we have $(A_1, A_2, \dots) \in Z(R)$. We may take a non-trivial orbit of $\langle \tau \rangle$, say $i_1, i_2 = \tau^{-1}(i_1), \dots$, and a j outside this orbit (such j exists since $t \geq 2$). In addition, take $(B_1, B_2, \dots) \in Q$, where $B_i = 1$ for all i except that B_{i_1} and B_j satisfy that the order of B_{i_1} is greater than 2 and $\det(B_{i_1})\det(B_j) = 1$. Then direct calculation shows that

$$[(A_1, A_2, \dots)\tau, (B_1, B_2, \dots)] \notin Z,$$

which is a contradiction. Thus $Z_0 \subseteq R_0$, as claimed.

Now let $A = (A_1, \dots, A_t) \in Z_0$, where $A_i \in R_{0,i}$. Then, obviously, each component of A is in the centre of the corresponding component of Q , and thus $Z_0 = Z(Q)$.

(b) The statement for $b = 1$ is well known. So assume $b \geq 2$. Then a slight modification of the argument in (a) shows that $Z_0 \subseteq R_0$, and then the claim follows as there. $\square$

Proposition 3.7. *Let $H = \mathrm{SL}_n(\eta q)$ with $n \geq 2$ even and $1 \neq Z \leq \mathbf{O}_2(Z(H))$. If a Sylow 2-subgroup $\bar{Q}$ of H/Z is non-abelian, then, for every ϕ in the principal 2-block of H/Z , we have $\mathrm{def}(\phi) > v(Z(\bar{Q}))$.*

Proof. Write θ for the inflation of ϕ to H ; then θ lies in the principal 2-block of H . It is easy to see that $\text{def}(\phi) = \text{def}(\theta) - v(Z)$. Let $G = \text{GL}_n(\eta q)$, and let R be a Sylow 2-subgroup of G and $Q = R \cap H$, a Sylow 2-subgroup of H . Let $\chi \in \mathcal{E}(G, s)$ with $s \in R$ be such that θ is an irreducible constituent of $\chi|_H$.

We first let $4 \mid (q - \eta)$. If $t \geq 2$, then, by Lemma 3.6,

$$v(Z(\bar{Q})) = v(Z(Q)) - v(Z).$$

Hence the result follows from Proposition 3.5. So now we may assume $t = 1$, that is, $R = R_+^b$. Since $\bar{Q}$ is non-abelian, we have $b > 0$. Now Lemma 3.6 yields $v(Z(\bar{Q})) = v(Z(Q)) - v(Z) + 1$.

By the proof of Proposition 3.5, $\text{def}(\theta) > v(Z(Q)) + 1$ unless

$$n = \sum_{f \in \mathcal{F}_b(s)} m_f 2^b$$

is the 2-adic expansion and $b \leq a$. Hence $\mathcal{F}_b(s) = \{f\}$, $m_f = 1$ and $o(s) = 2^{a+b}$. Notice that $L = C_G(s)$ is in fact a Coxeter torus of G , and s generates the Sylow 2-subgroup of L and hence of $H^*/[H^*, H^*]$. But then all characters in $\mathcal{E}(H, s)$ are faithful on $\mathbf{O}_2(Z(H))$. However, this contradicts the obvious fact that

$$Z \leq \ker(\theta) = \ker(\chi).$$

This achieves the proof when $4 \mid (q - \eta)$.

Now assume $4 \mid (q + \eta)$. As before, if $v(Z(\bar{Q})) = v(Z(Q)) - v(Z)$, then the assertion follows from Proposition 3.5. So, by Lemma 3.6, it remains to consider the case $v(Z(\bar{Q})) = v(Z(Q)) - v(Z) + 1$, i.e., either $R = R_+^a$ with $a > 2$ or $R = R_-^b$ with $b \geq 2$. Here $|Z(\bar{Q})| = 2$ by Lemma 3.6, and so Robinson's conjecture holds by [8, Lemma 3.1]. $\square$

4 Principal 2-blocks of quasi-simple groups of classical type

Let $\mathbf{G}$ be a simple algebraic group of symplectic or orthogonal type over an algebraically closed field of odd characteristic and F a Frobenius endomorphism of $\mathbf{G}$ with respect to an $\mathbb{F}_q$ -rational structure, and denote $G = \mathbf{G}^F$. So, in particular, q is an odd prime power. Throughout, we will fix the prime $\ell = 2$, with respect to which defects will be considered. As for the linear and unitary groups, we will need three pieces of information: defects of unipotent characters, centralisers of 2-elements and the centres of Sylow 2-subgroups.

4.1 Unipotent characters of classical groups

Here we determine lower bounds on defects of unipotent characters of classical groups. Observe that, by Lusztig's results, the classification and degrees of unipo-

tent characters are insensitive to the isogeny type (see [4, § 13.7]), so, for our purposes, we will not need to specify these here.

Lemma 4.1. *Let χ be a cuspidal unipotent character of a finite group G of classical type. Let 2^b be the precise power of 2 dividing $q + 1$. Then*

$$\text{def}(\chi) = \begin{cases} (b+1)n & \text{for types } B_n(q), C_n(q), \\ (b+1)n-1 & \text{for types } D_n(q), {}^2D_n(q). \end{cases}$$

Proof. We discuss the various types individually. If $G = B_n(q)$ or $C_n(q)$, then, by [4, Section 13.7], we have $n = s^2 + s$ for some $s \geq 1$, and, according to the formula given there,

$$\begin{aligned} \text{def}(\chi) &= s + b \sum_{i=1}^s 2i + \sum_{i=1}^s (2i-1) \\ &= s + b(s^2 + s) + s^2 \\ &= (b+1)n. \end{aligned}$$

If $G = D_n(q)$ or ${}^2D_n(q)$, then we have $n = s^2$ for some integer s (which is even in the first case, odd in the second) and

$$\begin{aligned} \text{def}(\chi) &= s-1 + b \sum_{i=1}^s (2i-1) + \sum_{i=1}^{s-1} 2i \\ &= s-1 + bs^2 + s^2 - s \\ &= (b+1)n-1. \end{aligned}$$

□

Lemma 4.2. *Let χ be a unipotent character of a finite classical group G .*

- (a) *If G is of type $B_n(q), C_n(q)$ with $n \geq 1$ or of type $D_n(q), {}^2D_n(q)$ with $n \geq 2$, then $\text{def}(\chi) > n$.*
- (b) *If G is of type $D_n(q^2)$ or ${}^2D_n(q^2)$ with $n \geq 2$, then $\text{def}(\chi) \geq 2n-1$, with equality only possibly when n is a square.*

Proof. Assume that χ lies in the Harish-Chandra series of the cuspidal unipotent character λ of a Levi subgroup $L \leq G$ for an F -stable Levi subgroup $\mathbf{L}$ of an F -stable parabolic subgroup of $\mathbf{G}$. Let $2^a, 2^b$ be the precise power of 2 dividing $q-1, q+1$, respectively. By Lusztig's classification (see [4, § 13.7]), then $[\mathbf{L}, \mathbf{L}]$ is simple of the same classical type as $\mathbf{G}$; hence λ is as considered in Lemma 4.1. Moreover, $\chi(1)$ divides the degree $|G : L|_{p'} \lambda(1)$ of the Harish-Chandra induced

character $R_L^G(\lambda)$ (as can be seen, for example, from [4, Theorem 10.11.5]). Using $\mathbf{L} = [\mathbf{L}, \mathbf{L}]Z^\circ(\mathbf{L})$, we conclude that $2^{\text{def}(\chi)} \geq 2^{\text{def}(\lambda)} |Z^\circ(\mathbf{L})^F|_2$. Now first assume that G is of type $B_n(q)$ or $C_n(q)$, and so L has type $B_u(q)$ or $C_u(q)$ for some $u \geq 1$. Then $|Z^\circ(\mathbf{L})^F| = (q-1)^{n-u}$, and so

$$\text{def}(\chi) \geq a(n-u) + (b+1)u$$

by Lemma 4.1. This is linear in u and hence at least as big as the minimum of its values at $u = 0$ and $u = n$. Hence it is larger than n unless $u = 0$ and $a = 1$. In the latter case, χ lies in the principal series. Our claim follows if $\chi(1)$ is not divisible by $(q+1)^n$. If it is divisible by $(q+1)^n$, then χ is 2-cuspidal, and, with the Ennola duality, we obtain $\text{def}(\chi) \geq (a+1)n > n$ from Lemma 4.1. If G has type $D_n(q)$ or ${}^2D_n(q)$, then L has type $D_u(q)$ or ${}^2D_u(q)$, giving

$$\text{def}(\chi) \geq a(n-u) + (b+1)u - 1,$$

and we conclude as before. For types $D_n(q^2)$ or ${}^2D_n(q^2)$, we similarly obtain $\text{def}(\chi) \geq 3n-1$ when $u = 0$ (as $8 \mid (q^2-1)$) and $\text{def}(\chi) \geq (b+1)n-1 = 2n-1$ when $u = n$ is a square. $\square$

4.2 Centralisers of semisimple elements in classical groups

We will make use of the primary decomposition of semisimple elements in classical groups. We follow the notation of [10]. As in Section 2, let $\text{Irr}(\mathbb{F}_q[X])$ be the set of non-constant monic irreducible polynomials in $\mathbb{F}_q[X]$ different from X . For $f \in \text{Irr}(\mathbb{F}_q[X])$, let f^* be the polynomial in $\text{Irr}(\mathbb{F}_q[X])$ whose roots (in $\overline{\mathbb{F}}_q$) are the inverses of the roots of f . Denote

$$\mathcal{F}_0 := \{X-1, X+1\},$$

$$\mathcal{F}_+ := \{f \in \text{Irr}(\mathbb{F}_q[X]) \mid f \notin \mathcal{F}_0, f = f^*\},$$

$$\mathcal{F}_- := \{ff^* \mid f \in \text{Irr}(\mathbb{F}_q[X]), f \neq f^*\},$$

$$\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_+ \cup \mathcal{F}_-.$$

For any integer $b \geq 1$, we also set $\mathcal{F}_b := \{f \in \mathcal{F} \mid d_f = 2^b\}$. Here, and later on, d_f denotes the degree of $f \in \mathcal{F}$. Define the *reduced degree* δ_f of $f \in \mathcal{F}$ by

$$\delta_f = \begin{cases} d_f & \text{if } f \in \mathcal{F}_0, \\ \frac{1}{2}d_f & \text{if } f \in \mathcal{F}_+ \cup \mathcal{F}_-. \end{cases}$$

Since the polynomials in $\mathcal{F}_+ \cup \mathcal{F}_-$ have even degree, δ_f is an integer. In addition, we introduce a sign ε_f for $f \in \mathcal{F}_+ \cup \mathcal{F}_-$ defined by

$$\varepsilon_f = \begin{cases} -1 & \text{if } f \in \mathcal{F}_+, \\ 1 & \text{if } f \in \mathcal{F}_-. \end{cases}$$

Let V be a finite-dimensional symplectic or orthogonal space over $\mathbb{F}_q$ with q odd and $I(V) = \mathrm{Sp}(V)$ or $\mathrm{GO}(V)$, respectively. Given a semisimple element $s \in I(V)$, there exist unique orthogonal decompositions $V = \bigoplus_{f \in \mathcal{F}} V_f$ and $s = \prod_{f \in \mathcal{F}} s_f$, where the V_f are non-degenerate subspaces of V , $s_f \in I(V_f)$ and s_f has minimal polynomial f . The above decomposition of s is often called the primary decomposition of s . Correspondingly, the centraliser of s in $I(V)$ has a decomposition $C_{I(V)}(s) = \prod_f C_f(s)$ with

$$C_f(s) := C_{I(V_f)}(s_f) = \begin{cases} I(V_f) & \text{if } f \in \mathcal{F}_0, \\ \mathrm{GL}_{m_f}(\varepsilon_f q^{\delta_f}) & \text{if } f \in \mathcal{F}_+ \cup \mathcal{F}_-, \end{cases}$$

where m_f is the multiplicity of f in the characteristic polynomial of s_f (and of s).

Now let V be an orthogonal space over $\mathbb{F}_q$ and $I^\circ(V) = \mathrm{SO}(V)$. Let $s \in I(V)$. Then $s \in I^\circ(V)$ if and only if $m_{X+1}(s)$ is even. For more details, see [10, § 1]. The following is elementary; see also Lemma 2.2.

Lemma 4.3. *Let $a := v(q^2 - 1) - 1$. If $f \in \mathcal{F}_+ \cup \mathcal{F}_-$ has a root (in $\overline{\mathbb{F}}_q$) of order 2^m for some positive integer m , then*

$$\delta_f = \begin{cases} 1 & \text{if } m \leq a, \\ 2^{m-a} & \text{if } m > a. \end{cases}$$

Moreover, $f \in \mathcal{F}_-$ unless $4 \mid (q + 1)$ and $m \leq a$.

4.3 Symplectic groups

Lemma 4.4. *Let $G = \mathrm{Sp}_{2n}(q)$ with $n \geq 2$ and q odd, and let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion. Then $\mathrm{def}(\chi) \geq t + 2$ for every character χ in the principal 2-block of G .*

Proof. As recalled earlier, the principal 2-block lies in $\mathcal{E}_2(G, 1)$. So let $s \in G^*$ be a 2-element such that $\chi \in \mathcal{E}(G, s)$, and let $\tilde{\psi} \in \mathcal{E}(C_{G^*}(s), 1)$ denote the Jordan correspondent of χ (see, for example, [3, Corollary 15.14]). Then, by the degree formula for Jordan decomposition, we have

$$\mathrm{def}(\chi) = v(|G|) - v(\chi(1)) = v(|C_{G^*}(s)|) - v(\tilde{\psi}(1)).$$

Let ψ be a unipotent character of $C_{G^*}^\circ(s)$ below $\tilde{\psi}$. Then, by Clifford theory, $\tilde{\psi}(1)/\psi(1)$ divides $|C_{G^*}(s) : C_{G^*}^\circ(s)|$, so

$$\mathrm{def}(\chi) = v(|C_{G^*}(s)|) - v(\tilde{\psi}(1)) \geq v(|C_{G^*}^\circ(s)|) - v(\psi(1)) = \mathrm{def}(\psi).$$

Thus we need to discuss the defects of unipotent characters of $C_{G^*}^\circ(s)$. By our preliminary observations, $C_{G^*}^\circ(s)$ is isogenous to (and hence has the same unipotent characters as) a product of certain orthogonal, linear and unitary groups, which we will now investigate in detail.

Here we have $G^* = \mathrm{SO}_{2n+1}(q)$. Let V be the underlying space of G^* . Let $s = \prod_f s_f$ be the primary decomposition, and let $V = \bigoplus_f V_f$ be the corresponding orthogonal decomposition of V . Then

$$C_{G^*}^\circ(s) = \prod_{f \in \mathcal{F}} C_f \quad \text{with } C_f = C_{I^\circ(V_f)}(s_f).$$

Observe that $m_2 := m_{X+1}$ is even; $m_1 := m_{X-1} \geq 1$ must be odd since d_f is even for $f \in \mathcal{F}_+ \cup \mathcal{F}_-$. Write $\mathcal{F}_b(s) := \{f \in \mathcal{F}_b \mid m_f > 0\}$. By Lemma 4.3, d_f must be a power of 2 if $m_f \neq 0$.

Now $2n + 1 = m_1 + m_2 + \sum_f m_f d_f$, and thus

$$n = \frac{m_1 - 1}{2} + \frac{m_2}{2} + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f 2^{b-1}. \quad (4.1)$$

Then, by Lemma 2.3 (a),

$$t \leq \frac{m_1 - 1}{2} + \frac{m_2}{2} + \sum_{f \in \mathcal{F} \setminus \mathcal{F}_0} m_f, \quad (4.2)$$

and equality holds if and only if (4.1) is the 2-adic expansion of n .

Write $\psi = \boxtimes_f \psi_f$, where ψ_f is a unipotent character of C_f . By Lemma 4.2 (a),

$$\mathrm{def}(\psi_{X-1}) \geq \frac{m_1 - 1}{2} \quad (4.3)$$

with equality only when $m_1 = 1$ and, also by Lemma 4.2 (a), if $m_2 > 0$, then

$$\mathrm{def}(\psi_{X+1}) \geq \frac{m_2}{2} \quad (4.4)$$

with equality only when $m_2 = 2$. If $f \in \bigcup_{b \geq 1} \mathcal{F}_b(s)$, then $\delta_f = 2^{b-1} \geq 2$ is a power of 2 by Lemma 4.3 and $C_f \cong \mathrm{GL}_{m_f}(q^{\delta_f})$. By Lemma 2.5,

$$\mathrm{def}(\psi_f) \geq v(q^{\delta_f} - 1)m_f \geq (b + 1)m_f \geq 3m_f \geq m_f + 2. \quad (4.5)$$

If $f \in \mathcal{F}_1(s)$, then $4 \mid (q + 1)$, $\delta_f = 1$ and $C_f \cong \mathrm{GL}_{m_f}(-q)$, so, by Lemma 2.5,

$$\mathrm{def}(\psi_f) \geq m_f v(q + 1) \geq 2m_f \geq m_f + 1. \quad (4.6)$$

By (4.2), (4.3), (4.4), (4.5) and (4.6), we can now compare $\text{def}(\chi)$ with $t + 2$. If $\bigcup_{b>1} \mathcal{F}_b(s) \neq \emptyset$, then $\text{def}(\chi) \geq t + 2$ by (4.5). So now assume $\mathcal{F}_b(s) = \emptyset$ for $b > 1$. Then $n = \frac{m_1-1}{2} + \frac{m_2}{2} + \sum_{f \in \mathcal{F}_1(s)} m_f$, and hence equality does not hold in (4.2). If $n \geq 4$, then $n - t \geq 3$, and then $\text{def}(\chi) \geq t + 3$ holds. On the other hand, when $n = 2, 3$, then $n - t = 1$. If $\mathcal{F}_1(s) \neq \emptyset$, then $\text{def}(\chi) \geq t + 2$ holds by (4.6). So we assume $\mathcal{F}_1(s) = \emptyset$. If $m_1 > 1$, then the result follows from (4.3). Hence we may further assume $m_1 = 1$. Then $n = \frac{m_2}{2}$, and $\text{def}(\chi) \geq t + 2$ holds by (4.4). This completes the proof. $\square$

Let W be a Sylow 2-subgroup of $\text{Sp}_2(q) = \text{SL}_2(q)$. Clearly, $v(Z(W)) = 1$. For any positive integer b , we let $W_b = W \wr X_b$, where X_b is a Sylow 2-subgroup of the symmetric group of degree 2^b (i.e., X_b is isomorphic to $C_2 \wr \cdots \wr C_2$ with b factors). Then $v(Z(W_b)) = 1$.

Now we let $n = 2^{b_1} + \cdots + 2^{b_t}$, with $0 \leq b_1 < \cdots < b_t$, be the 2-adic expansion of n . Then, by [5, Theorem 1], a Sylow 2-subgroup R of $G = \text{Sp}_{2n}(q)$ is isomorphic to $W_{b_1} \times \cdots \times W_{b_t}$, and thus $v(Z(R)) = t$. The next result follows by a similar proof as for Lemma 3.6

Lemma 4.5. *In the notation above, let*

$$S = G/Z(G), \quad R \in \text{Syl}_2(G) \quad \text{and} \quad Q = R/Z(G) \in \text{Syl}_2(S).$$

Let $Z_0 \leq R$ be such that $Z_0/Z(G) = Z(Q)$. Then $Z_0 = Z(R)$ if $t \geq 2$, and $v(Z_0) = v(Z(R)) + 1$ if $t = 1$.

From Lemmas 4.4 and 4.5, we immediately deduce the following result.

Proposition 4.6. *Let $G = \text{Sp}_{2n}(q)$ with $n \geq 2$ and q odd, and let $Z \leq Z(G)$. Then (RC) holds for the principal 2-block of G/Z .*

4.4 Odd-dimensional orthogonal groups

Lemma 4.7. *Let $G = \text{SO}_{2n+1}(q)$, $n \geq 3$ and q odd, and let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion. Then $\text{def}(\chi) \geq t + 2$ for every character χ in the principal 2-block of G .*

Proof. Let $s \in G^* \cong \text{Sp}_{2n}(q)$ be a semisimple 2-element such that $\chi \in \mathcal{E}(G, s)$. Let $\psi \in \mathcal{E}(C_{G^*}(s), 1)$ be a unipotent character in Jordan correspondence with χ . As in the proof of Lemma 4.4, we have $\text{def}(\chi) \geq \text{def}(\psi)$.

Let V be the underlying space of G^* , $s = \prod_f s_f$, $V = \bigoplus_f V_f$ and

$$C_{G^*}(s) = \prod_{f \in \mathcal{F}(s)} C_f \quad \text{with} \quad C_f = C_{I(V_f)}(s_f).$$

Keep the notation used in the proof of Lemma 4.4. Then m_1 and m_2 are both even. Also, $n = \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f 2^{b-1}$, and so

$$t \leq \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f \in \mathcal{F} \setminus \mathcal{F}_0} m_f.$$

By Lemma 4.2 (a), $\text{def}(\psi_i) \geq \frac{m_i}{2}$ for $i = 1, 2$, with equality only when $m_i \leq 2$. For $f \in \mathcal{F}(s) \setminus \mathcal{F}_0$, the formulas for $\text{def}(\psi_f)$ are the same as in (4.5) and (4.6).

Now first assume $n \geq 4$. If $\mathcal{F}_b(s) = \emptyset$ for $b > 1$, then

$$n = \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f \in \mathcal{F}_1} m_f.$$

Hence $\text{def}(\chi) \geq n \geq t + 3$ as $n \geq 4$. So now assume $\bigcup_{b > 1} \mathcal{F}_b(s) \neq \emptyset$. Also, by (4.5), it is easy to see that the claim holds unless $|\bigcup_{b > 1} \mathcal{F}_b(s)| = 1$ and $m_f = 1$ for the unique polynomial f in $\bigcup_{b > 1} \mathcal{F}_b(s)$. If $\mathcal{F}_1(s) \neq \emptyset$, then, by (4.6), we get the result. Hence $\mathcal{F}_1(s) = \emptyset$, and then $\mathcal{F}(s) \setminus \mathcal{F}_0 = \{f\}$. Again by (4.5), $\text{def}(\psi_f) \geq m_f + 2$, so $n = \frac{m_1}{2} + \frac{m_2}{2} + \delta_f$ must be the 2-adic expansion, and then $m_1 + m_2 \leq 2$. Hence $\delta_f \geq n - 1 \geq 3$, and then $\delta_f \geq 4$ since it is a power of 2. Thus, by (4.5), $\text{def}(\psi_f) \geq 4m_f$, and so $\text{def}(\chi) \geq t + 3$, completing the proof in case $n \geq 4$.

Now assume $n = 3$. By Lemma 4.2 (a), for $i \in \{1, 2\}$, if $m_i > 0$, then

$$\text{def}(\psi_i) > \frac{m_i}{2}. \quad (4.7)$$

Similar to the proof of Lemma 4.4, if $\bigcup_{b > 1} \mathcal{F}_b(s) \neq \emptyset$, then, by (4.5), we have $\text{def}(\chi) \geq t + 2$. So we may assume $\mathcal{F}_b(s) = \emptyset$ for $b > 1$, and then

$$n = 3 = \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f \in \mathcal{F}_1} m_f.$$

Note that here $n - t = 1$. Moreover, by (4.7), we may assume $m_1 = m_2 = 0$, and then $\mathcal{F}(s) = \mathcal{F}_1(s)$. Thus $\text{def}(\chi) \geq t + 2$ follows from (4.6), and we are done. $\square$

To discuss the Sylow 2-subgroups, let W be a Sylow 2-subgroup of $\text{GO}_2^\eta(q)$ with $q \equiv \eta 1 \pmod{4}$, where $\eta \in \{\pm\}$. Then W is isomorphic to the dihedral group of order 2^{a+1} , where $a = v(q^2 - 1) - 1$, and so $v(Z(W)) = 1$. For any positive integer b , we let $W_b = W \wr X_b$, where X_b is defined as in Section 2.2. Then $v(Z(W_b)) = 1$.

Now let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion of n , and let

$$\iota: \text{GO}_{2n}^\eta(q) \rightarrow \text{SO}_{2n+1}(q), \quad A \mapsto \text{diag}(A, \det(A))$$

be an embedding of $\mathrm{GO}_{2n}^\eta(q)$ into $\mathrm{SO}_{2n+1}(q)$ with $q^n \equiv \eta 1 \pmod{4}$. Then, by [5, Theorem 2], the image of $W_{b_1} \times \cdots \times W_{b_t}$ under ι is a Sylow 2-subgroup of $\mathrm{SO}_{2n+1}(q)$.

Now we consider the simple group $\Omega_{2n+1}(q) = [\mathrm{SO}_{2n+1}(q), \mathrm{SO}_{2n+1}(q)]$. Let $R \cong W_{b_1} \times \cdots \times W_{b_t}$ be a Sylow 2-subgroup of $\mathrm{SO}_{2n+1}(q)$ as above, and let $Q = R \cap \Omega_{2n+1}(q)$. We give an explicit description of Q . Let

$$\theta: \mathrm{GO}_2^\eta(q) \rightarrow \mathbb{F}^\times / \mathbb{F}^{\times 2}$$

be the spinor norm on $\mathrm{GO}_2^\eta(q)$. Let W be the Sylow 2-subgroup of $\mathrm{GO}_2^\eta(q)$ as before. Then $W_0 = \ker \theta|_W$ is isomorphic to the dihedral group of order 2^a . In particular, $Z(W) \subseteq W_0$. Denote by $B = W \times \cdots \times W$ the base subgroup of R . Then $R = B \rtimes A$ with $A = X_{b_1} \times \cdots \times X_{b_t}$, so $Q = B_0 \rtimes A$, where B_0 is the subgroup of B consisting of all elements $(w_1, \dots, w_n)$ satisfying $\theta(w_1) \cdots \theta(w_n) = \bar{1}$; see, for example, [14, § 4].

Proposition 4.8. *Let $S = \Omega_{2n+1}(q)$ with $n \geq 3$ and q odd. Then (RC) holds for the principal 2-blocks of all covering groups of S .*

Proof. For $\Omega_{2n+1}(q)$, we let R, Q be as above. Then $Z(Q) = Z(R)$ by our description of Q . Then, by Lemma 4.7, (RC) holds for the principal 2-block of S .

For $2.S = \mathrm{Spin}_{2n+1}(q)$, the assertion follows directly from [8, Lemma 3.1] by the fact that the centre of a Sylow 2-subgroup of $\mathrm{Spin}_{2n+1}(q)$ has order 2 by [11, Lemma 4.4]. $\square$

4.5 Even-dimensional orthogonal groups

The simply connected group of type D_n is the spin group $\mathrm{Spin}_{2n}^\pm(q)$, and its dual group is the projective conformal special orthogonal group $\mathrm{PCSO}_{2n}^\pm(q)$, the quotient of the conformal special orthogonal group $\mathrm{CSO}_{2n}^\pm(q)$ modulo its central torus. We thus need some control on centralisers of 2-elements in conformal special orthogonal groups. Recall that q is an odd prime power.

Lemma 4.9. *Let $s \in G^* := \mathrm{CSO}_{2n}^\pm(q)$ be a 2-element. Then $C_{G^*}^\circ(s)$ is a product of groups $D_{k_i}(\epsilon_i q)$ (two factors) or $D_{k_i}(\epsilon_i q^2)$ (one factor) with groups of type $\mathrm{GU}_{k_i}(q)$ and $\mathrm{GL}_{k_i}(q^{2^{d_i}})$ for suitable $k_i, d_i \geq 0$ and $\epsilon_i \in \{\pm 1\}$, where $\mathrm{GU}_{k_i}(q)$ only occurs when $4 \mid (q+1)$.*

Proof. Let $s \in G^* = \mathrm{CSO}_{2n}$. Since $\mathrm{CSO}_{2n} = \mathrm{SO}_{2n} \cdot Z(\mathrm{CSO}_{2n})$, we can write $s = s_1 z$ with elements $s_1 \in \mathrm{SO}_{2n}$ and $z \in Z(\mathrm{CSO}_{2n})$ a scalar matrix. Clearly, $C_{G^*}(s) = C_{G^*}(s_1)$. Note that s_1 and z need not be F -stable, but as

$$\mathrm{SO}_{2n} \cap Z(\mathrm{CSO}_{2n}) = \{\pm 1\}$$

and both groups are F -stable, we have $F(s_1) = \pm s_1$, and s_1 is F^2 -stable. So $C_{G^*}(s)^{F^2}$ has the structure described above, corresponding to an orthogonal decomposition of $\mathbb{F}_{q^2}^{2n}$ into the s_1 -eigenspaces. Now F permutes these eigenspaces according to whether $F(s_1) = s_1$ or $F(s_1) = -s_1$. Thus the two orthogonal factors of $C_{G^*}(s)^{F^2}$ are either fixed or permuted, and we obtain a collection of type A -factors, as claimed. See also [12, Lemma 2.5] for a more precise statement. $\square$

Lemma 4.10. *Let $G = \text{Spin}_{2n}^\pm(q)$, $n \geq 4$ and q odd, and let $n = 2^{b_1} + \dots + 2^{b_t}$ be the 2-adic expansion with $b_1 < \dots < b_t$. Then*

$$\text{def}(\chi) \geq \max\{t + 2, t + b_1 + 1\}$$

for every character χ in the principal 2-block of G .

Proof. Let $G = \text{Spin}_{2n}^\pm(q)$ with $n \geq 4$. Assume $\chi \in \mathcal{E}(G, s)$ for some semisimple 2-element $s \in G^* = \text{PCSO}_{2n}^\pm(q)$. Let $\psi \in \mathcal{E}(C_{G^*}^\circ(s), 1)$ be a unipotent character in Jordan correspondence with χ . Then, as in the proof of Lemma 4.4, we have $\text{def}(\chi) \geq \text{def}(\psi)$.

Let $\tilde{s} \in \tilde{G}^* := \text{CSO}_{2n}^\pm(q)$ be a preimage of s under the natural map. Then the structure of $C_{\tilde{G}^*}^\circ(\tilde{s})$ is described in Lemma 4.9, and thus also the structure of $C_{G^*}^\circ(s)$. We write $\frac{m_1}{2}, \frac{m_2}{2}$ for the ranks of the two D -factors, or $\frac{m_0}{2}$ if there is just one over $\mathbb{F}_{q^2}$, and m_f for the ranks of the various GL- and GU-factors with $f \in \mathcal{F} \setminus \mathcal{F}_0$. Then

$$n = m_0 + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f 2^{b-1}, \quad (4.8)$$

$t \leq m_0 + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f \in \mathcal{F} \setminus \mathcal{F}_0} m_f$, $\text{def}(\psi_f) \geq m_f(b + 1)$ for every $f \in \mathcal{F}_b$, and $\text{def}(\psi_0) \geq m_0 - 1$ for the unipotent character ψ_0 from a possible D -factor over $\mathbb{F}_{q^2}$. Thus we have

$$\text{def}(\psi) \geq (m_0 - 1) + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} (b + 1)m_f \geq t - 1 + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} b m_f$$

(where the summand $(m_0 - 1)$ is only present when $m_0 > 0$). By the same argument as in Lemma 4.7, we obtain $\text{def}(\chi) \geq t + 2$ (noting that $n \geq 4$) and even $\text{def}(\chi) \geq t + 3$ when $m_0 = 0$.

For the second bound, by (4.5) and (4.6) in the proof of Lemma 4.4, it suffices to show that

$$(m_0 - 1) + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f(b + 1) \geq t + b_1 + 1.$$

If $m_0 = m_1 = m_2 = 0$, then, by Lemma 2.3 (b),

$$\begin{aligned} & \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f(b+1) - (t + b_1 + 1) \\ &= 2 \left(\sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f - t \right) + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} (b-1)m_f + t - (b_1 + 1) \\ &\geq \sum_{i=1}^t b_i + t - (b_1 + 1) \geq 0. \end{aligned}$$

If $m_1 + m_2 > 0$ (and so $m_0 = 0$), then, by Lemma 2.3 (c),

$$\frac{m_1}{2} + \frac{m_2}{2} + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f(b+1) \geq t + b_1 + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f b \geq t + b_1.$$

Equality only holds when (4.8) is the 2-adic expansion of n . Hence $m_1 + m_2 = 2$ and $\mathcal{F}_1(s) = \emptyset$; then $\bigcup_{b \geq 1} \mathcal{F}_b(s) \neq \emptyset$ since $n \geq 4$. Thus $\sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f b > 0$. So equality does not hold.

Finally, if $m_0 > 0$ (and hence $m_1 = m_2 = 0$), then the 2-adic expansion of $n - 1 = m_0 - 1 + \sum_{b,f} m_f 2^{b-1}$ has $t + b_1 - 1$ terms, so, by Lemma 2.3 (a), we are done unless $\sum_{b,f} b m_f < 2$. But note that $m_0 - 1$ is odd, so, unless $m_0 = 2$, its 2-adic expansion has at least 2 terms. Thus either $m_0 = 2$ or $n = m_0$. Both cases are easily dealt with. $\square$

Note that $n \geq 4$ implies either $t + 2 \geq 4$ or $t + b_1 + 1 \geq 4$, so $\text{def}(\chi) \geq t + 3$ in Lemma 4.10 when $t = 1$.

We next discuss Sylow 2-subgroups. Let W_b be as in the previous subsection. First assume $q^n \equiv -\eta 1 \pmod{4}$. Then we have $\text{SO}_{2n}^\eta(q) = S \times C_2$ with $S = \Omega_{2n}^\eta(q)$ simple, and $\text{Spin}_{2n}^\eta(q) = 2.S$ is the only proper covering group of S .

Lemma 4.11. *Let R be a Sylow 2-subgroup of $\Omega_{2n}^\eta(q)$ with $q^n \equiv -\eta 1 \pmod{4}$, and let $n - 1 = 2^{b_1} + \cdots + 2^{b_r}$ be the 2-adic expansion. Then $v(Z(R)) = r$.*

Proof. By [14, Theorem 7], a Sylow 2-subgroup of $\Omega_{2n}^\eta(q)$ is isomorphic to a Sylow 2-subgroup of $\text{GO}_{2n-2}^{\eta'}(q)$ with $q^{n-1} \equiv \eta' 1 \pmod{4}$, i.e., to $W_{b_1} \times \cdots \times W_{b_r}$. The assertion follows. $\square$

Proposition 4.12. *Let $S = \Omega_{2n}^\eta(q)$ with $n \geq 4$ and $q^n \equiv -\eta 1 \pmod{4}$. Then (RC) holds for the principal 2-blocks of all covering groups of S .*

Proof. If $q^n \equiv -\eta 1 \pmod{4}$, then, by Lemma 4.11, $v(Z(R)) = r = t + b_1 - 1$ in the notation of Lemmas 4.10, 4.11. If χ is a character of $G = 2.S = \text{Spin}_{2n}^\eta(q)$

with $Z(G)$ in its kernel, then, considering it as a character of S , $\text{def}(\chi) \geq t + b_1$, which shows the claim. If $\hat{R}$ denotes a Sylow 2-subgroup of G , then clearly $v(Z(\hat{R})) \leq r + 1 = t + b_1$. Again, the claim follows with Lemma 4.10. $\square$

We now turn to the more difficult case $q^n \equiv \eta 1 \pmod{4}$. Here we have

$$\begin{aligned} \text{SO}_{2n}^\eta(q) &= 2.S.2 \quad \text{with } S = \text{P}\Omega_{2n}^\eta(q) \text{ simple,} \\ \text{Spin}_{2n}^\eta(q) &= \begin{cases} 2^2.S & \text{if } n \text{ is even,} \\ 4.S & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

First note that a Sylow 2-subgroup R of $\text{GO}_{2n}^\eta(q)$ is isomorphic to $W_{b_1} \times \cdots \times W_{b_t}$, where $n = 2^{b_1} + \cdots + 2^{b_t}$. Let $Q = R \cap \text{SO}_{2n}^\eta(q)$. Then $v(Z(Q)) = t$.

Lemma 4.13. *Let $H = \Omega_{2n}^\eta(q)$ with $q^n \equiv \eta 1 \pmod{4}$, and keep all the notation above.*

- (a) *Let $Q_0 = R \cap H$. Then $v(Z(Q_0)) = t$.*
- (b) *Let $\overline{Q}_0 = Q_0/Z(H)$, a Sylow 2-subgroup of $\text{P}\Omega_{2n}^\eta(q)$, $Z(\overline{Q}_0) = Z_0/Z(H)$. Then $Z_0 = Z(Q_0)$ if $t > 1$, while, if $t = 1$, $v(Z_0) = v(Z(Q_0)) + 1 = 2$ and $|Z(\overline{Q}_0)| = 2$.*

Proof. (a) Let θ be the spinor norm on $\text{GO}_2^\eta(q)$. Write $R = B \rtimes A$, where B is the base subgroup, a direct product of copies of W and $A = X_{b_1} \times \cdots \times X_{b_t}$. Then an element $(w_1, \dots, w_n)a$ with $w_i \in W$ and $a \in A$ is in Q_0 if and only if $\det(w_1) \cdots \det(w_n) = 1$ and $\theta(w_1) \cdots \theta(w_n) = \bar{1}$. Thus $v(Z(Q_0)) = t$ follows easily. Part (b) follows by a similar proof as for Lemma 3.6. $\square$

Proposition 4.14. *Let $S = \text{P}\Omega_{2n}^\eta(q)$ with $n \geq 4$ and $q^n \equiv \eta 1 \pmod{4}$. Then (RC) holds for the principal 2-blocks of all covering groups of S .*

Proof. First assume that n is even. Then, by [8, Corollary 2.4], we just need to consider the three groups S , $2.S = \Omega_{2n}^\eta(q)$ and $2'.S = \text{HSpin}_{2n}^\eta(q)$. (The two half-spin groups are isomorphic under the graph automorphism of order 2.) In the notation of Lemma 4.13, the centres of Sylow 2-subgroups R of these groups satisfy $v(Z(R)) \leq t, t, t + 1$, respectively, while, by Lemma 4.10, the defects of characters belonging to the principal 2-block of G that descend to these groups are at least $t + 1, t + 2, t + 2$, respectively. So (RC) holds in all cases.

Now assume that n is odd. Then the groups to consider are S , $2.S = \Omega_{2n}^\eta(q)$ and $4.S$. Here again, the centres of Sylow 2-subgroups R satisfy $v(Z(R)) \leq t, t, t + 1$, while the defects are bounded below by $t + 1, t + 2, t + 3$, respectively. $\square$

5 Proof of the main result

Proof of Theorem 1. Let B be a 2-block of a quasi-simple covering group G of a finite simple classical group S . If S is defined in characteristic 2, our claim is in [8, Theorem 5.1]. So S is a classical group in odd characteristic. By [8, Theorem 3.6], we may assume that G is not an exceptional covering group. Then G is one of the groups considered in the previous sections. According to [3, Theorem 21.14], then G has only one unipotent 2-block, the principal 2-block $\mathcal{E}_2(G, 1)$. This is not a counterexample to (RC) by Propositions 3.1, 3.5, 3.7, 4.6, 4.8, 4.12 and 4.14. If B is not unipotent, then $\text{Irr}(B) \subseteq \mathcal{E}_2(G, s)$ for some semi-simple $2'$ -element $1 \neq s \in G^*$. If G is of symplectic or orthogonal type, then centralisers of non-trivial $2'$ -elements in G^* are proper Levi subgroups. If G is special linear or unitary, then at least the connected components of these centralisers are proper Levi subgroups. In either case, by the reduction theorem of Bonnafé–Rouquier [1], then B is Morita equivalent to a 2-block of a strictly smaller group and thus cannot be a minimal counterexample to (RC) either. $\square$

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