Isomorphism questions for metric ultraproducts of finite quasisimple groups

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Abstract. New results on metric ultraproducts of finite simple groups are established. We show that the isomorphism type of a simple metric ultraproduct of groups $X_{n_i}(q)$ ($i \in I$) for $X \in \{\text{PGL}, \text{PSp}, \text{PGO}^{(\varepsilon)}, \text{PGU}\}$ ($\varepsilon = \pm$) along an ultrafilter $\mathcal{U}$ on the index set I for which $n_i \rightarrow_{\mathcal{U}} \infty$ determines the type X and the field size q up to the possible isomorphism of a metric ultraproduct of groups $\text{PSp}_{n_i}(q)$ and a metric ultraproduct of groups $\text{PGO}_{n_i}^{(\varepsilon)}(q)$. This extends results of [A. Thom and J. Wilson, Metric ultraproducts of finite simple groups, *Comp. Rend. Math.* **352** (2014), no. 6, 463–466].

1 Introduction

In [5, 6], Thom and Wilson discussed various properties of metric ultraproducts of finite simple groups. In particular, they asked which such ultraproducts can be isomorphic. In [5, Theorem 2.2], a metric ultraproduct of alternating groups is distinguished from a metric ultraproduct of classical groups of Lie type, where the permutation degrees resp. dimensions of the natural module tend to infinity. This is done by considering the structure of centralizers of torsion elements in these groups (see [5, Theorems 2.8 and 2.9]). In the case of a metric ultraproduct of classical groups of Lie type, in [5, Theorem 2.8], investigating the structure of such centralizers of semisimple and unipotent torsion elements, Thom and Wilson even extract the “limit characteristic” of the group. At the end of [5, Section 2], they ask which metric ultraproducts of classical groups of different types can be isomorphic.

In this note, we will give an almost complete answer to this question in the case when the field sizes are bounded. We will show that, for a metric ultraproduct of alternating or classical groups of Lie type of unbounded rank over fields of bounded size, one can extract the Lie type (up to one exception). Also one can extract the “limit field size”. Our results are summed up in Theorem 1 below. To state it, we first need to introduce some notation which we fix for the whole article.

Let $\mathcal{H} = (H_i)_{i \in I}$ be a sequence of groups where either $H_i = S_{n_i}$ is a symmetric group or $H_i = X_i(q_i)$ is a classical group of Lie type X_i over the finite field $\mathbb{F}_{q_i}$ with q_i elements (resp. $\mathbb{F}_{q_i^2}$ in the unitary case, $i \in I$). In the latter case, we let each X_i be one of GL_{n_i} , Sp_{2m_i} , $\mathrm{GO}_{2m_i}^\pm$, GO_{2m_i+1} (q_i odd), or GU_{n_i} for suitable $m_i, n_i \in \mathbb{Z}_+$ ($i \in I$).

Recall that a *norm* ℓ on a group H is a function $H \rightarrow [0, \infty]$ such that $\ell(h) = 0$ iff $h = 1_H$, $\ell(h) = \ell(h^{-1}) = \ell(h^g)$, and $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in H$. Call a pair (H, ℓ) , where H is a group and ℓ a norm on it, a *normed group*. Recall that the *metric ultraproduct* of a sequence of normed groups $(H_i, \ell_i)_{i \in I}$ along an ultrafilter $\mathcal{U}$ on the set I is defined as the quotient $\prod_{i \in I} H_i / N$, where $N := \{(h_i)_{i \in I} \in \prod_{i \in I} H_i \mid \lim_{\mathcal{U}} \ell_i(h_i) = 0\}$ is the normal subgroup of $\mathcal{U}$ -null sequences.

Throughout, let $G := \mathcal{H}_{\mathcal{U}}^{\mathrm{met}}$ be the metric ultraproduct of the groups H_i from above equipped with the *normalized Hamming norm*

$$\ell_H(\sigma) := |\mathrm{supp}(\sigma)|/n = |\{x \in \{1, \dots, n\} \mid x.\sigma \neq x\}|/n$$

resp. the *normalized rank norm* $\ell_{\mathrm{rk}}(g) := \mathrm{rk}(1 - g)/n$ when H_i is a symmetric resp. a classical linear group of Lie type along the ultrafilter $\mathcal{U}$. Assume that the permutation degrees resp. dimensions of the natural module n_i of H_i ($i \in I$) tend to infinity along $\mathcal{U}$.

Note that, since $\mathcal{U}$ is an ultrafilter, we may assume that each group H_i is of the same type, i.e., all groups H_i are either symmetric, linear, symplectic, orthogonal, or unitary groups. In these five distinct cases, we write $S_{\mathcal{U}}$, $\mathrm{GL}_{\mathcal{U}}$, $\mathrm{Sp}_{\mathcal{U}}$, $\mathrm{GO}_{\mathcal{U}}$, or $\mathrm{GU}_{\mathcal{U}}$ for G . Also, when the field sizes q_i are bounded, we may assume that $q_i = q$ is constant ($i \in I$), setting $q := \lim_{\mathcal{U}} q_i$. Throughout the article, set $Z := \mathbf{Z}(G)$ to be the center of G and $\overline{G} := G/Z$.

If the groups H_i ($i \in I$) are symmetric groups, then $Z = \mathbf{1}$ and $\overline{G} = G$. Now assume that all groups H_i are of type $X(q_i)$ ($i \in I$; i.e., they are not symmetric groups). Then $\overline{G} = G/Z = \overline{\mathcal{H}}_{\mathcal{U}}^{\mathrm{met}}$ is the metric ultraproduct of the groups $\overline{H}_i := H_i/\mathbf{Z}(H_i)$ with respect to the projective rank norm

$$\ell_{\mathrm{pr}}(\overline{h}) := \inf\{\ell_{\mathrm{rk}}(h) \mid h \text{ is a lift of } \overline{h} \text{ in } H\},$$

which is defined on the general projective linear group. By the results from [3], $\overline{G}$ is the unique simple quotient of G .

Similarly to the above, write $\mathrm{PGL}_{\mathcal{U}}$, $\mathrm{PSp}_{\mathcal{U}}$, $\mathrm{PGO}_{\mathcal{U}}$, or $\mathrm{PGU}_{\mathcal{U}}$ for $\overline{G}$ when all the groups H_i ($i \in I$) are linear, symplectic, orthogonal, or unitary groups. Moreover, in this case, if all the fields $\mathbb{F}_{q_i}$ ($i \in I$) are equal to $\mathbb{F}_q$ (or $\mathbb{F}_{q^2}$ in the unitary case), we write $\mathrm{GL}_{\mathcal{U}}(q)$, $\mathrm{Sp}_{\mathcal{U}}(q)$, $\mathrm{GO}_{\mathcal{U}}(q)$, $\mathrm{GU}_{\mathcal{U}}(q)$ resp. $\mathrm{PGL}_{\mathcal{U}}(q)$, $\mathrm{PSp}_{\mathcal{U}}(q)$, $\mathrm{PGO}_{\mathcal{U}}(q)$, $\mathrm{PGU}_{\mathcal{U}}(q)$ for G resp. $\overline{G}$. Write $\mathrm{GL}_{\mathcal{U}}(k)$ resp. $\mathrm{PGL}_{\mathcal{U}}(k)$ for the metric

ultraproduct of groups $\mathrm{GL}_{n_i}(k)$ resp. $\mathrm{PGL}_{n_i}(k)$ over the field k ($i \in I$) along $\mathcal{U}$. Write $\mathbf{M}_n(k)$ for the matrix ring of degree n over the field k and $\mathbf{PM}_n(k)$ for the associated projective space $(\mathbf{M}_n(k) \setminus \{0\})/k^\times$. Set $\mathbf{M}_n(q) := \mathbf{M}_n(\mathbb{F}_q)$ and $\mathbf{PM}_n(q) := \mathbf{PM}_n(\mathbb{F}_q)$. Also write $\mathbf{M}_{\mathcal{U}}, \mathbf{M}_{\mathcal{U}}(q)$ resp. $\mathbf{PM}_{\mathcal{U}}, \mathbf{PM}_{\mathcal{U}}(q)$ for the metric ultraproduct of the spaces $\mathbf{M}_{n_i}(q_i), \mathbf{M}_{n_i}(q)$ resp. $\mathbf{PM}_{n_i}(q_i), \mathbf{PM}_{n_i}(q)$ with respect to the metrics

$$d_{\mathrm{rk}}(g, h) := \mathrm{rk}(g - h)/n,$$

$$d_{\mathrm{pr}}(\bar{g}, \bar{h}) := \inf\{d_{\mathrm{rk}}(g, h) \mid g, h \text{ are lifts of } \bar{g}, \bar{h}\} \quad (i \in I)$$

so that $\mathrm{GL}_{\mathcal{U}} \subseteq \mathbf{M}_{\mathcal{U}}, \mathrm{GL}_{\mathcal{U}}(q) \subseteq \mathbf{M}_{\mathcal{U}}(q), \mathrm{PGL}_{\mathcal{U}} \subseteq \mathbf{PM}_{\mathcal{U}}, \mathrm{PGL}_{\mathcal{U}}(q) \subseteq \mathbf{PM}_{\mathcal{U}}(q)$.

The main result of this article is now as follows.

Theorem 1. *Let $\bar{G} \cong \bar{G}_1 \cong \bar{G}_2$ with $G_j = X_j \mathcal{U}_j(q_j)$, where*

$$X_j \in \{\mathrm{GL}, \mathrm{Sp}, \mathrm{GO}, \mathrm{GU}\} \quad (j = 1, 2).$$

Then it holds that $q_1 = q_2$. Also we must have $X_1 = X_2$ or $\{X_1, X_2\} = \{\mathrm{Sp}, \mathrm{GO}\}$. Moreover, an ultraproduct $\bar{X}_1 \mathcal{U}_1$ where the sizes q_i of the finite fields $\mathbb{F}_{q_i}$ ($i \in I_1$) tend to infinity along $\mathcal{U}_1$ cannot be isomorphic to an ultraproduct $\bar{X}_2 \mathcal{U}_2(q)$.

Let us conclude this introduction by saying some words about the proof of Theorem 1. Our strategy is to compute double centralizers of semisimple torsion elements of a fixed order $o \in \mathbb{Z}_+$ in the above metric ultraproducts. If the sizes q_i ($i \in I$) of the fields $\mathbb{F}_{q_i}$ are bounded, it turns out that these are always finite abelian groups. Then we consider the maximal possible exponent which such a double centralizer can have. It turns out that this data is enough to determine the limit field size q and the Lie type (up to the exception mentioned in Theorem 1). If the field sizes q_i ($i \in I$) tend to infinity, a double centralizer of such a torsion element of order $o > 2$ is always infinite. This separates this case from the former one. Throughout the article, we will make frequent use of the exposition given in [4, Subsection 3.4.2, § 1]. This paragraph in the author's PhD thesis provides a concise presentation of the classification of the possible indecomposable blocks of elements from classical groups of Lie type stabilizing a form (which is, of course, well known). For the convenience of the reader, we will quote the facts needed from it in the subsequent section.

2 Notation and auxiliary results from [4]

In this section, we introduce the necessary notation and quote the facts from [4] which we need for the proof of Theorem 1. Keep the notation from the introduction. Subsequently, G will be one of $S_{\mathcal{U}}, \mathrm{GL}_{\mathcal{U}}(q), \mathrm{Sp}_{\mathcal{U}}(q), \mathrm{GO}_{\mathcal{U}}(q)$, or $\mathrm{GU}_{\mathcal{U}}(q)$

(so that $\overline{G}$ is either $S_{\mathcal{U}}$, $\mathrm{PGL}_{\mathcal{U}}(q)$, $\mathrm{PSp}_{\mathcal{U}}(q)$, $\mathrm{PGO}_{\mathcal{U}}(q)$, or $\mathrm{PGU}_{\mathcal{U}}(q)$, respectively). We refer to these five distinct cases as the *symmetric case*, the *linear case*, the *symplectic case*, the *orthogonal case*, and the *unitary case*, respectively. The last four cases are grouped together as the *classical case*. The symplectic and orthogonal case are grouped together as the *bilinear case*.

For the rest of this section, we assume that we are in the classical case. Then set $k = \mathbb{F}_q$ in the non-unitary case and $k = \mathbb{F}_{q^2}$ in the unitary case. Write $p := \mathrm{char}(k)$ for the characteristic of the field k .

Subsequently, assume that we are not in the linear case. Then we let $\sigma: k \rightarrow k$ be the q -Frobenius map $x \mapsto x^q$, i.e., $\sigma = \mathrm{id}_k$ in the bilinear case, whereas σ is the unique involution of $k = \mathbb{F}_{q^2}$ in the unitary case. For a monic polynomial

$$\chi = a_0 + a_1 X + \cdots + a_{k-1} X^{k-1} + X^k \in k[X],$$

write $\chi^* = a_0^{-\sigma} X^{\deg(\chi)} \chi^{\sigma}(X^{-1})$ for the *dual polynomial* of χ . Here

$$\chi^{\sigma} = a_0^{\sigma} + a_1^{\sigma} X + \cdots + a_{k-1}^{\sigma} X^{k-1} + X^k$$

is the polynomial χ with coefficients twisted by the automorphism σ . Call χ *self-dual* iff $\chi = \chi^*$. Furthermore, write f_i for the sesquilinear forms stabilized by the group H_i (and Q_i for the quadratic forms in the orthogonal case for $p = 2$, $i \in I$). Note that a metric ultraproduct of groups $\mathrm{GO}_{2m_i+1}(q)$ for q a power of $p = 2$ with respect to the projective rank norm where the m_i tend to infinity along the chosen ultrafilter is isomorphic to such an ultraproduct of groups $\mathrm{Sp}_{2m_i}(q)$. Actually, $\mathrm{GO}_{2m_i+1}(q) \cong \mathrm{Sp}_{2m_i}(q)$, but the natural module of $\mathrm{GO}_{2m_i+1}(q)$ has one-dimensional radical $\mathrm{rad}(f_i)$, whereas $\mathrm{rad}(f_i) = 0$ in the second case. Hence we may and do assume that the forms f_i ($i \in I$) are all non-singular and exclude groups of type $\mathrm{GO}_{2m+1}(q)$ for q even. Write

$$F(\chi) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & \cdots & \cdots & \cdots & -a_{k-1} \end{pmatrix}$$

for the *Frobenius block* associated to the monic polynomial

$$\chi = a_0 + a_1 X + \cdots + a_{k-1} X^{k-1} + X^k \in k[X],$$

i.e., multiplication by $\overline{X}$ in the quotient ring $k[X]/(\chi)$ represented in the basis

$$1, \overline{X}, \dots, \overline{X}^{k-1}.$$

Recall that, by the existence of the *generalized Jordan normal* (or *primary rational canonical form*) form, each linear transformation $g \in \mathbf{M}_n(k)$ can be written as

$$g \cong \bigoplus_{\substack{\chi \in k[X] \\ \chi \text{ monic primary}}} F(\chi)^{\oplus c_\chi(g)},$$

where the $c_\chi(g)$ are uniquely determined. A polynomial $\chi \in k[X]$ is here called *primary* iff it is the power i^e ($e \in \mathbb{Z}_+$) of an irreducible polynomial $i \in k[X]$.

Here are the results needed from [4]. Subsequently, the group H is one of the groups H_i ($i \in I$), and the form f is alternating bilinear, symmetric bilinear, or σ -sesquilinear in the symplectic, orthogonal, and unitary case, respectively. Hence the sign ε of f is -1 in the symplectic case, and $\varepsilon = +1$ in the orthogonal and unitary case. The form Q is a compatible quadratic form in the orthogonal case for $p = 2$, i.e., $Q(u + v) = Q(u) + f(u, v) + Q(v)$ for all u, v . Recall that two sesquilinear forms f and f' (or quadratic forms Q and Q') on a vector space V are called *linearly equivalent* if $f(u, v) = f'(u.g, v.g)$ (resp. $Q(v) = Q'(v.g)$) for all $u, v \in V$ and some $g \in \text{GL}(V)$.

Fact 1 ([4, Fact 3.40]). *The following hold.*

- (i) *Whenever $\chi \in k[X]$ is (monic) self-dual and is not divided by $X \pm 1$ in the bilinear case, then there exists a non-singular form f (coming from a quadratic form Q when $p = 2$) which is preserved by $F(\chi)$ (and Q as well). The form f (and Q) is even unique up to linear equivalence.*
- (ii) *Assume now that $\chi = (X \pm 1)^e$ (which is self-dual) and we are in the bilinear case.*
 - (a) *At first, assume that p is odd. Then, when $\varepsilon = (-1)^{e-1}$, there is a unique non-singular form f (up to linear equivalence) preserved by $F(\chi)$, and when $\varepsilon = (-1)^e$, there is a unique non-singular form f (up to linear equivalence) preserved by $F(\chi)^{\oplus 2}$, but no non-singular form is preserved by $F(\chi)$.*
 - (b) *If $p = 2$, we use [2, Theorem 3.1] to see that if $g \cong F(\chi)^{\oplus c}$, i.e., g acts χ -isotypically (see the beginning of Section 3), then any non-singular form f (and the associated quadratic form Q) restricts to all but at most constantly many blocks $F(\chi)$ to the indecomposable $W(e)$ from this theorem (which acts as $F(\chi)^{\oplus 2}$).*

An indecomposable block of an element g from a classical group of Lie type stabilizing the sesquilinear form f with natural module V is a subspace $W \neq 0$ on which f is non-singular (i.e., $W \cap W^\perp = 0$) and which is minimal with this

property. Out of the theory of indecomposables presented in [4, Subsection 3.4.2, § 1], one can also deduce the following fact which we will use in the text.

Fact 2. *Let $H = \text{GI}(V, f)$ (resp. $H = \text{GI}(V, Q)$) in the orthogonal case for $p = 2$) be a classical group of Lie type stabilizing the non-singular form f (and Q). Let $g \in H$ and $h: U \rightarrow W$ be a partial isometry between the subspaces $U, W \leq V$ commuting with g . Then h extends to a full isometry $h' \in H$ which commutes with g .*

The next result, which can also be deduced from [4, Subsection 3.4.2, § 1], describes the shape of f on a “semisimple” indecomposable block.

Fact 3. *Let H be as in Fact 2, and let $g \in H$ be a semisimple element, i.e., for each Frobenius block $F(\chi)$ in the generalized Jordan normal form of g (see above), we have that $\chi = i^1$ is irreducible (or alternatively the order $\text{ord}(h)$ of h is coprime to the characteristic p of the group H). Then the indecomposable blocks of g look as follows. Assume $W \leq V$ is an indecomposable block of g such that $g|_W$ has a Jordan block $F(\chi)$ acting on the subspace $U \leq W$ of dimension $\deg(\chi)$ for $\chi \in k[X]$ irreducible. Write $K_\chi := k[X]/(\chi)$, and let $\lambda_\chi := \bar{X} \in K_\chi$ be a root of χ in K_χ . Let $\tau: K_{\chi^*} \rightarrow K_\chi$ be the field isomorphism given by $\tau|_k = \sigma$ and $\lambda_{\chi^*} \mapsto \lambda_\chi^{-1}$. If χ is self-dual, i.e., $\chi = \chi^*$, we clearly have that $\tau \in \text{Aut}(K_\chi/k_\sigma)$ is an involution (or the identity in the bilinear case when $\chi = X \pm 1$).*

(i) Assume $\chi \neq X \pm 1$ in the bilinear case.

- (a) Assume χ is not self-dual. There exists $U' \leq W$ such that $W = U \oplus U'$ and g acts as $F(\chi) \oplus F(\chi^*)$ on $U \oplus U'$. So, as $k[g]$ -modules, $U \cong K_\chi$ and $U' \cong K_{\chi^*}$, where g acts on $K_\chi \times K_{\chi^*} \cong U \oplus U' = W$ as multiplication by $(\lambda_\chi, \lambda_{\chi^*})$. The pairing f on W is given by

$$U \times U' \cong K_\chi \times K_{\chi^*} \ni (u, v) \mapsto \text{tr}_{K_\chi/k}(uv^\tau),$$

$f|_U = 0$, $f|_{U'} = 0$, and extension to a sesquilinear form of sign ε (this is clearly g -invariant). In the orthogonal case for $p = 2$, the associated quadratic form Q is uniquely determined by f and g . It is given by $W = U \oplus U' \ni u \oplus v \mapsto f(u, v)$.

- (b) Assume now that χ is self-dual. Then $U = W$ and g acts as $F(\chi)$ on U . Hence $U \cong K_\chi$ as a $k[g]$ -module, where g acts on $K_\chi \cong U$ as multiplication by λ_χ . In this case, the form f is given by

$$U^2 \cong K_\chi^2 \ni (u, v) \mapsto \text{tr}_{K_\chi/k}(\delta uv^\tau),$$

where $\delta \in K_\chi$ is such that $\delta^\tau = \varepsilon\delta$. In the orthogonal case for $p = 2$, the quadratic form Q is given by $U \ni v \mapsto \text{tr}_{(K_\chi)_\tau/k}(vv^\tau)$. Here $(K_\chi)_\tau$ denotes the fixed field of the involution τ .

- (ii) Assume now that $\chi = X \pm 1$ and we are in the bilinear case. Then we have $g|_W = \mp \text{id}_W$.
- (a) If we are in the symplectic case, then $W \cong k^2 = \mathbb{F}_q^2$ and $f|_W$ is the unique (up to linear equivalence) non-singular alternating form on the vector space W .
- (b) Assume we are in the orthogonal case and $p \neq 2$. Then $W \cong k = \mathbb{F}_q$ and f is given by $(u, v) \mapsto \beta uv$, where $\beta \in \mathbb{F}_q^\times$ is either 1 (the “standard square”) or a “standard non-square”.
- (c) Assume we are in the orthogonal case and $p = 2$. Then $W \cong k^2 = \mathbb{F}_q^2$ and f is the unique (up to linear equivalence) non-singular alternating form on it and Q is a quadratic form of plus type or minus type, i.e., either Q is given by $k^2 \ni (u, v) \mapsto uv$ (plus type) or

$$k^2 \cong \mathbb{F}_{q^2} \ni v \mapsto N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(v) = v^{q+1},$$

where $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}: \mathbb{F}_{q^2} \rightarrow \mathbb{F}_q$ is the field norm.

We will also make use of the relations presented in the next remark.

Remark 1. When D is the block $W \cong k^2$ of case (ii)(b) of Fact 3 where $\beta = 1$ and E is the same block for a non-square $\beta \in \mathbb{F}_q^\times$, then $D \oplus D \cong E \oplus E$. Similarly, when D is the block of minus type and E the block of plus type on $W \cong k^2$ in Fact 3 (ii)(c), then $D \oplus D \cong E \oplus E$.

3 Description of conjugacy classes in $S_{\mathcal{U}}$, $\text{GL}_{\mathcal{U}}(q)$, and $\text{PGL}_{\mathcal{U}}(q)$

In this section, we give a description of the conjugacy classes of groups of type $S_{\mathcal{U}}$ or $\text{PGL}_{\mathcal{U}}(q)$. We will make use of this in the subsequent sections.

We start with some definitions. Write $S_n = \text{Sym}(\underline{n})$ for the symmetric group acting on the set $\underline{n} := \{1, \dots, n\}$. Let $C_k(\sigma) \subseteq S_n$ (for a permutation $\sigma \in S_n$ and $k \in \mathbb{Z}_+$) denote the set of all k -cycles of σ , and let $c_k(\sigma) := |C_k(\sigma)|$ denote the number of k -cycles of σ . Moreover, let $\Omega_k(\sigma)$ be the set on which all k -cycles of σ act. Set $n_k(\sigma) := |\Omega_k(\sigma)| = kc_k(\sigma)$ to be the cardinality of $\Omega_k(\sigma)$. Call the permutation $\sigma \in S_n$ *k-isotypic* when $n_k(\sigma) = n$, i.e., σ has only cycles of length k . Call σ *isotypic* if it is k -isotypic for one number $k \in \mathbb{Z}_+$. Similarly, for $g \in \mathbf{M}_n(k)$ and a monic primary polynomial $\chi \in k[X]$, let $c_\chi(g)$ be the number

of Frobenius blocks $F(\chi)$ in the generalized Jordan normal form of g (see Section 2). Let $V_\chi(g)$ be the subspace on which g acts as $F(\chi)^{\oplus c_\chi(g)}$ with respect to this normal form, and set $n_\chi(g) := \dim(V_\chi(g))$. Clearly, $V_\chi(g)$ is not uniquely determined as it depends on the decomposition of the vector space k^n associated to the chosen generalized Jordan normal form of g . However, by writing $V_\chi(g)$ we mean that these subspaces of k^n are the decomposition of k^n coming from such a normal form of g . Then, of course, $n_\chi(g) = \deg(\chi)c_\chi(g)$. For a monic polynomial $\chi \in k[X]$, say that g is χ -isotypic if $g \cong F(\chi)^{\oplus c}$ for some $c \in \mathbb{Z}_+$.

At first, we consider $\mathrm{GL}_{\mathcal{U}}(q)$ instead of $\mathrm{PGL}_{\mathcal{U}}(q)$. Subsequently, all polynomials from $k[X]$ that occur in the text are meant to be *monic* polynomials.

Conjugacy classes in $\mathrm{S}_{\mathcal{U}}$ and $\mathrm{GL}_{\mathcal{U}}(q)$. For an integer $k \in \mathbb{Z}_+$ and a polynomial $\xi \in k[X]$, define $r_k(\sigma) := |\mathrm{fix}(\sigma^k)|/n$ resp. $r_\xi(g) := \dim(\ker(\xi(g)))/n$ for $\sigma \in \mathrm{S}_n$ resp. $g \in \mathbf{M}_n(q)$. Here $\mathrm{fix}(\sigma) := \{x \in \underline{n} \mid x.\sigma = x\}$ is the set of *fixed points* of the permutation σ . Extend this definition to $\mathrm{S}_{\mathcal{U}}$ and $\mathbf{M}_{\mathcal{U}}(q)$ by setting

$$r_k(\sigma) := \lim_{\mathcal{U}} r_k(\sigma_i) \quad \text{and} \quad r_\xi(g) := \lim_{\mathcal{U}} r_\xi(g_i)$$

for $\sigma = (\overline{\sigma_i})_{i \in I}$ resp. $g = (\overline{g_i})_{i \in I}$. Both expressions are well-defined since, for $\sigma = (\overline{\sigma_i})_{i \in I} = (\overline{\tau_i})_{i \in I} \in \mathrm{S}_{\mathcal{U}}$, one has

$$\begin{aligned} \frac{1}{n_i} |\mathrm{fix}(\sigma_i^k) - \mathrm{fix}(\tau_i^k)| &\leq d_{\mathrm{H}}(\sigma_i^k, \tau_i^k) \\ &\leq d_{\mathrm{H}}(\sigma_i^k, \sigma_i^{k-1} \tau_i) + \cdots + d_{\mathrm{H}}(\sigma_i \tau_i^{k-1}, \tau_i^k) \\ &= k d_{\mathrm{H}}(\sigma_i, \tau_i) \rightarrow_{\mathcal{U}} 0. \end{aligned} \tag{3.1}$$

Similarly, if $g = (\overline{g_i})_{i \in I} = (\overline{h_i})_{i \in I} \in \mathrm{GL}_{\mathcal{U}}(q)$ and

$$\xi = a_0 + a_1 X + \cdots + a_{k-1} X^{k-1} + X^k \in k[X],$$

we have

$$\begin{aligned} \frac{1}{n_i} |\dim(\ker \xi(g_i)) - \dim(\ker \xi(h_i))| \\ \leq d_{\mathrm{rk}}(\xi(g_i), \xi(h_i)) &\leq \sum_{j=0}^k d_{\mathrm{rk}}(a_j g_i^j, a_j h_i^j) \leq \sum_{j=0}^k d_{\mathrm{rk}}(g_i^j, h_i^j) \\ &\leq \left(\sum_{j=0}^k j \right) d_{\mathrm{rk}}(g_i, h_i) = \binom{k+1}{2} d_{\mathrm{rk}}(g_i, h_i), \end{aligned} \tag{3.2}$$

and the latter tends to zero along $\mathcal{U}$. Here we used the same trick as in estimate (3.1) above to bound $d_{\mathrm{rk}}(g_i^j, h_i^j)$ by $j d_{\mathrm{rk}}(g_i, h_i)$ ($j = 0, \dots, k$) in estimate (3.2). Write $r(\sigma) := (r_k(\sigma))_{k \in \mathbb{Z}_+}$ and $r(g) := (r_\xi(g))_{\xi \in k[X]}$. Now define

$q_k(\sigma)$ for $k \in \mathbb{Z}_+$ via the equality

$$\sum_{d|k} q_d(\sigma) = r_k(\sigma) \quad \text{for all } k \in \mathbb{Z}_+.$$

Write $q(\sigma) := (q_k(\sigma))_{k \in \mathbb{Z}_+}$. Applying Möbius inversion, we obtain

$$q_k(\sigma) = \sum_{d|k} \mu(k/d) r_d(\sigma).$$

Alternatively, one can think of $q_k(\sigma)$ as the $\mathcal{U}$ -limit of the normalized cardinality of the support of all k -cycles in σ_i ($i \in I$), i.e.,

$$q_k(\sigma) = \lim_{\mathcal{U}} n_k(\sigma_i)/n_i = k \lim_{\mathcal{U}} c_k(\sigma_i)/n_i.$$

Similarly, for $\chi = i^e \in k[X]$ primary, define $q_\chi(g)$ via the equality

$$r_\xi(g) = \sum_{\chi \text{ primary}} \frac{\deg(\gcd\{\chi, \xi\})}{\deg(\chi)} q_\chi(g)$$

for all polynomials $\xi \in k[X]$. Here $\gcd S$ is the greatest common divisor of the elements from S . Write $q(g) := (q_\chi(g))_{\chi \text{ primary}}$. Alternatively, one can think of $q_\chi(g)$ as the $\mathcal{U}$ -limit of the normalized dimensions of the (not unique) subspaces $V_\chi(g_i)$ ($i \in I$), i.e.,

$$q_\chi(g) = \lim_{\mathcal{U}} n_\chi(g_i)/n_i = k_\chi \lim_{\mathcal{U}} c_\chi(g_i)/n_i,$$

where $k_\chi = \deg(\chi) = e \deg(i)$. This is because, when g acts as $F(\chi)$ and $\xi \in k[X]$, then $\dim(\ker(\xi(g))) = \deg(\gcd\{\chi, \xi\})$.

We claim that the conjugacy classes in $S_{\mathcal{U}}$ resp. $\mathbf{M}_{\mathcal{U}}(q)$ are in one-to-one correspondence with all tuples $(q_k(\sigma))_{k \in \mathbb{Z}_+}$ resp. $(q_\chi(g))_{\chi \text{ primary}}$, where the only condition on the sequences are that

$$\sum_{k \in \mathbb{Z}_+} q_k(\sigma) \leq 1 \quad \text{resp.} \quad \sum_{\chi \text{ primary}} q_\chi(g) \leq 1.$$

Here we let $\text{GL}_{\mathcal{U}}(q)$ act on $\mathbf{M}_{\mathcal{U}}(q)$ by conjugation. The element g lies in $\text{GL}_{\mathcal{U}}(q)$ iff $q_\chi(g) = 0$ for $\chi = X^e$ ($e \geq 1$). Indeed, one sees easily that $r(\sigma)$ resp. $r(g)$ is conjugacy invariant, and so is $q(\sigma)$ resp. $q(g)$ for $\sigma \in S_{\mathcal{U}}$ and $g \in \mathbf{M}_{\mathcal{U}}(q)$.

To see the converse, let

$$\begin{aligned} \sigma &= (\overline{\sigma_i})_{i \in I}, \tau = (\overline{\tau_i})_{i \in I} \in S_{\mathcal{U}}, \\ g &= (\overline{g_i})_{i \in I}, h = (\overline{h_i})_{i \in I} \in \mathbf{M}_{\mathcal{U}}(q) \end{aligned}$$

be elements such that $q(\sigma) = q(\tau)$ and $q(g) = q(h)$, respectively.

Find a sequence $(N_i)_{i \in I}$ tending to infinity along $\mathcal{U}$ such that

$$\sum_{k=1}^{N_i} |q_k(\sigma) - q_k(\sigma_i)|, \sum_{k=1}^{N_i} |q_k(\tau) - q_k(\tau_i)| \rightarrow_{\mathcal{U}} 0$$

resp. such that

$$\sum_{\substack{\chi \text{ primary} \\ \deg(\chi) \leq N_i}} |q_\chi(g) - q_\chi(g_i)|, \sum_{\substack{\chi \text{ primary} \\ \deg(\chi) \leq N_i}} |q_\chi(h) - q_\chi(h_i)| \rightarrow_{\mathcal{U}} 0.$$

Then, by the triangle inequality,

$$\frac{1}{n_i} \sum_{k=1}^{N_i} |n_k(\sigma_i) - n_k(\tau_i)| \rightarrow_{\mathcal{U}} 0$$

resp.

$$\frac{1}{n_i} \sum_{\substack{\chi \text{ primary} \\ \deg(\chi) \leq N_i}} |\dim(V_\chi(g_i)) - \dim(V_\chi(h_i))| \rightarrow_{\mathcal{U}} 0.$$

Hence we can conjugate a big part of $\bigsqcup_{k=1}^{N_i} \Omega_k(\sigma_i)$ equivariantly to a big part of $\bigsqcup_{k=1}^{N_i} \Omega_k(\tau_i)$ resp. an almost full-dimensional part of

$$\bigoplus_{\substack{\chi \text{ primary} \\ \deg(\chi) \leq N_i}} V_\chi(g_i) \quad \text{to} \quad \bigoplus_{\substack{\chi \text{ primary} \\ \deg(\chi) \leq N_i}} V_\chi(h_i)$$

equivariantly (with no error in the limit; here again $V_\chi(g_i)$ resp. $V_\chi(h_i)$ are not unique). The remaining part of σ_i and τ_i resp. g_i and h_i can be modified into one big cycle resp. one big Frobenius block with no change of σ and τ resp. g and h , since $N_i \rightarrow_{\mathcal{U}} \infty$. Then we conjugate this cycle of σ_i resp. Frobenius block of g_i onto the one of τ_i resp. h_i .

The case of $\mathbf{PM}_{\mathcal{U}}(q)$. Let the group $k^\times = \mathbb{F}_q^\times$ act on all (monic) polynomials $\xi \in k[X]$ by $\xi.z := z^{-k_\xi} \xi(zX)$, where $k_\xi := \deg(\xi)$. Extend this action to all tuples $q = (q_\chi)_{\chi \text{ primary}}$ with $\sum_{\chi \text{ primary}} q_\chi \leq 1$ via

$$q.z = (q_{\chi.z})_{\chi \text{ primary}},$$

and denote by $\bar{q}$ the orbit $\text{orb}_{k^\times}(q)$ of q under this action of $k^\times$.

Suppose that $\bar{G} = \text{PGL}_{\mathcal{U}}(q)$. We claim that the conjugacy classes of elements $\bar{g} \in \mathbf{PM}_{\mathcal{U}}(q)$ are classified by the bijection $\bar{g} \mapsto \overline{(q_\chi(g))}_{\chi \text{ primary}}$, where g is any lift of $\bar{g}$ in $\mathbf{M}_{\mathcal{U}}(q)$ (here we exclude the tuple q where $q_X = 1$ and $q_\chi = 0$ otherwise).

Indeed, the map is well-defined since any other h such that $\bar{h} = \bar{g} \in \mathbf{PM}_{\mathcal{U}}(q)$ is of the form zg for some $z \in k^{\times}$ (as $k = \mathbb{F}_q$ is finite) so that

$$(q_{\chi}(h))_{\chi \text{ primary}} = (q_{\chi}(zg))_{\chi \text{ primary}} = (q_{\chi.z}(g))_{\chi \text{ primary}} = q(g).z.$$

Also q is constant on conjugacy classes of $\mathbf{M}_{\mathcal{U}}(q)$ (under the action of $\mathbf{GL}_{\mathcal{U}}(q)$). Conversely, if $q_{\chi}(h) = q_{\chi.z}(g) = q_{\chi}(zg)$ for some fixed $z \in k^{\times}$ and all $\chi \in k[X]$ primary, then, from the above, we derive that the elements g and $z^{-1}h$ of $\mathbf{M}_{\mathcal{U}}(q)$ are conjugate so that $\bar{g}$ and $\bar{h}$ are conjugate in $\mathbf{PM}_{\mathcal{U}}(q)$. This proves the claim.

Remark 2. For G of type $\mathbf{Sp}_{\mathcal{U}}(q)$, $\mathbf{GO}_{\mathcal{U}}(q)$, or $\mathbf{GU}_{\mathcal{U}}(q)$, the conjugacy classes of elements $g \in G$ for which $\sum_{\chi \text{ primary}} q_{\chi}(g) = 1$ are still characterized by the tuples $(q_{\chi}(g))_{\chi \text{ primary}}$. The only necessary additional restriction on these tuples is that $q_{\chi}(g) = q_{\chi^*}(g)$ for all $\chi \in k[X]$ primary (see the beginning of Section 2 for the definition of χ^*).

Indeed, assume $g = (\bar{g}_i)_{i \in I}$, $h = (\bar{h}_i)_{i \in I} \in G$, $q(g) = q(h)$, $q_{\chi}(g) = q_{\chi^*}(g)$ for all $\chi \in k[X]$ primary, and

$$\sum_{\chi \text{ primary}} q_{\chi}(g) = \sum_{\chi \text{ primary}} q_{\chi}(h) = 1.$$

Then g is conjugate to h .

This holds since on all but constantly many Frobenius blocks $F(\chi)$ of g_i resp. h_i ($i \in I$), where χ is a self-dual primary polynomial, and on all but constantly many blocks $F(\chi) \oplus F(\chi^*) \cong F(\chi\chi^*)$, where χ is not self-dual and primary, according to Fact 1, the form f_i (and in characteristic $p = 2$ the quadratic form Q_i) is uniquely determined up to linear equivalence so that we can map these blocks of g_i to such blocks of h_i and extend this partial map by Witt's lemma.

Conversely, if we have a tuple $(q_{\chi})_{\chi \text{ primary}}$ such that $\sum_{\chi \text{ primary}} q_{\chi} = 1$ and $q_{\chi} = q_{\chi^*}$, we can deduce again from Fact 1 that there exists an element $g \in G$ such that $q_{\chi}(g) = q_{\chi}$ for all $\chi \in k[X]$ primary.

Recall that $\bar{G} = G/Z$, where $Z = \mathbf{Z}(G)$. For a tuple $q = (q_{\chi})_{\chi \text{ primary}}$, let $\bar{q}$ denote its orbit $\text{orb}_Z(q)$ under the action of $Z \leq k^{\times}$. Then, for the elements $\bar{g} \in \bar{G}$ (i.e., $\bar{G}$ is $\mathbf{PSp}_{\mathcal{U}}(q)$, $\mathbf{PGO}_{\mathcal{U}}(q)$, or $\mathbf{PGU}_{\mathcal{U}}(q)$) such that $\sum_{\chi \text{ primary}} q_{\chi}(g) = 1$ for one lift $g \in G$ of $\bar{g}$, the same characterization as for $\mathbf{PGL}_{\mathcal{U}}(q)$ above holds by the same argument. Again we need to restrict the tuples $q = (q_{\chi})_{\chi \text{ primary}}$ so that $q_{\chi} = q_{\chi^*}$ for all χ primary.

However, we conjecture that the above characterization for G of type $\mathbf{Sp}_{\mathcal{U}}(q)$, $\mathbf{GO}_{\mathcal{U}}(q)$, or $\mathbf{GU}_{\mathcal{U}}(q)$ is false if

$$\sum_{\chi \text{ primary}} q_{\chi}(g) < 1 \quad \text{for an element } g \in G.$$

Remark 3. It is easy to see that $(\chi.z)^* = \chi^*.z$ for $z \in Z \leq k^\times$. Indeed, since $z^\sigma = z^{-1}$ for $z \in Z$, we have

$$\begin{aligned} (\chi.z)^* &= (z^{-k} \chi(zX))^* = (z^k)^\sigma a_0^{-\sigma} X^k (z^{-k})^\sigma \chi^\sigma (z^\sigma X^{-1}) \\ &= a_0^{-\sigma} X^k \chi^\sigma ((zX)^{-1}) = z^{-k} \chi^*(zX) = \chi^*.z, \end{aligned}$$

where $\chi = a_0 + a_1 X + \dots + a_{k-1} X^{k-1} + X^k$ (recall that $\sigma \in \text{Aut}(k)$ is defined in Section 2).

4 Characterization of torsion elements in $\mathbf{S}_{\mathcal{U}}$, $\mathbf{GL}_{\mathcal{U}}(q)$, and $\mathbf{PGL}_{\mathcal{U}}(q)$

In this section, we characterize torsion elements in metric ultraproducts of the above type. At first, note that an invertible element in $\mathbf{M}_{\mathcal{U}}(q)$, i.e., an element of $\mathbf{GL}_{\mathcal{U}}(q)$, is algebraic over $k = \mathbb{F}_q$ if and only if it is torsion. Indeed, if g is torsion, then $g^o - 1 = 0$ for some integer $o \geq 1$. Conversely, if g is algebraic and invertible, let $\chi \in k[X]$ be its minimal polynomial. Setting $o := |(k[X]/(\chi))^\times| < \infty$, one sees that $g^o = 1$ as g is invertible.

Here comes the promised characterization of torsion elements.

Lemma 1. *An element $\sigma \in \mathbf{S}_{\mathcal{U}}$ resp. $g \in \mathbf{GL}_{\mathcal{U}}(q)$ is torsion if and only if there is $N \in \mathbb{Z}_+$ such that*

$$\sum_{k=1}^N q_k(\sigma) = 1 \quad \text{resp.} \quad \sum_{\substack{\chi \text{ primary} \\ \deg(\chi) \leq N}} q_\chi(g) = 1.$$

An element $\bar{g} \in \mathbf{PGL}_{\mathcal{U}}(q)$ is torsion if and only if any lift $g \in \mathbf{GL}_{\mathcal{U}}(q)$ is torsion.

Proof. Write $\text{lcm } S$ for the least common multiple of the elements of S . Indeed, if the above two conditions are fulfilled, then writing $o := \text{lcm}\{1, \dots, N\}$ resp. $o := \text{lcm}\{|(k[X]/(\chi))^\times| \mid \chi \text{ primary, } \deg(\chi) \leq N\}$, we have $\ell_{\mathbf{H}}(\sigma_i^o) \rightarrow_{\mathcal{U}} 0$ resp. $\ell_{\text{rk}}(g_i^o) \rightarrow_{\mathcal{U}} 0$, meaning that $\sigma^o = 1$ resp. $g^o = 1$.

Conversely, if we assume $\sigma^o = 1$ resp. $g^o = 1$, we get that $\ell_{\mathbf{H}}(\sigma_i^o) \rightarrow_{\mathcal{U}} 0$ resp. $\ell_{\text{rk}}(g_i^o) \rightarrow_{\mathcal{U}} 0$, meaning that, asymptotically, the d -cycles in σ_i for $d \mid o$ support the whole set resp. all Frobenius blocks $F(\chi)$ for $\chi \mid X^o - 1$ primary support the whole vector space, so taking $N := o$ above, we get the converse direction.

The last statement follows since the kernel of the surjective homomorphism $\mathbf{GL}_{\mathcal{U}}(q) \rightarrow \mathbf{PGL}_{\mathcal{U}}(q)$ equals $k^\times = \mathbb{F}_q^\times$, which is finite. Hence, if $g \in \mathbf{GL}_{\mathcal{U}}(q)$ represents $\bar{g} \in \mathbf{PGL}_{\mathcal{U}}(q)$ and the latter is of order $o < \infty$, then we have that $\text{ord}(g) \mid o(q-1) < \infty$. $\square$

5 Faithful action of $S_{\mathcal{U}}$ and $\text{PGL}_{\mathcal{U}}(q)$ on the Loeb space and the associated continuous geometry

In this section, we show that the groups $S_{\mathcal{U}}$ and $\text{PGL}_{\mathcal{U}}(q)$ faithfully act on natural associated objects. For this purpose, we need the so-called *Loeb space*

$$L(n_i)_{i \in I} := (\mathcal{S}, \mu)$$

resp. its vector space analog, the *continuous geometry*

$$V(n_i)_{i \in I} := (\mathcal{V}, \dim),$$

which are associated naturally to the metric ultraproduct $S_{\mathcal{U}}$ resp. $\text{PGL}_{\mathcal{U}}(q)$.

Here $\mathcal{S}$ resp. $\mathcal{V}$ equals $\prod_{i \in I} \mathcal{P}(n_i)$ resp. $\prod_{i \in I} \text{Sub}(k^{n_i})$ modulo the equivalence relation $(S_i)_{i \in I} \sim (T_i)_{i \in I}$ resp. $(U_i)_{i \in I} \sim (V_i)_{i \in I}$ iff $\mu_i(S_i \Delta T_i) \rightarrow_{\mathcal{U}} 0$ resp. $\dim_i(U_i + V_i) - \dim_i(U_i \cap V_i) \rightarrow_{\mathcal{U}} 0$, where μ_i resp. $\dim_i$ is the normalized counting measure resp. dimension on n_i resp. k^{n_i} (and $A \Delta B$ denotes the symmetric difference of the sets A and B). Then one defines μ resp. $\dim$ by

$$\mu(S) = \mu(\overline{(S_i)_{i \in I}}) := \lim_{\mathcal{U}} \mu_i(S_i),$$

$$\dim(V) = \dim(\overline{(V_i)_{i \in I}}) := \lim_{\mathcal{U}} \dim_i(V_i).$$

It is easy to check that both are well-defined in this way. Also the operations $\cup, \cap$ resp. $+, \cap$ are inherited to $\mathcal{S}$ resp. $\mathcal{V}$ in a natural way, e.g.,

$$\overline{(S_i)_{i \in I}} \cap \overline{(T_i)_{i \in I}} := \overline{(S_i \cap T_i)_{i \in I}}.$$

Write $S \subseteq T$ resp. $U \leq V$ iff $\mu(S \cap T) = \mu(S)$ resp. $\dim(U \cap V) = \dim(U)$. Call a permutation of $\mathcal{S}$ resp. $\mathcal{V}$ an automorphism iff it preserves μ resp. $\dim$ and the relation $\subseteq$ resp. $\leq$.

Then one observes that $S_{\mathcal{U}}$ resp. $\text{PGL}_{\mathcal{U}}(q)$ is faithfully represented as group of automorphisms of $(\mathcal{S}, \mu)$ resp. $(\mathcal{V}, \dim)$.

At first, consider the case $\overline{G} = S_{\mathcal{U}}$. Indeed, assume for fixed $\sigma = (\sigma_i)_{i \in I} \in S_{\mathcal{U}}$ that $S \cdot \sigma = S$ for all $S \in \mathcal{S}$. Then take $S = \overline{(S_i)_{i \in I}}$, where $S_i \subseteq n_i$ is taken in the following way. For each k -cycle $c \subseteq n_i$ ($k > 1$; here seen as a set), we pick $s_c \in c$ and define S_i by $S_i \cap c = \{s_c, s_c \cdot \sigma_i^2, \dots, s_c \cdot \sigma_i^{2(\lfloor k/2 \rfloor - 1)}\}$ and $S_i \cap \Omega_1(\sigma_i) = \emptyset$. Then $S_i \Delta S_i \cdot \sigma_i = \emptyset$ and $\mu_i(S_i) \geq 1/3 |\text{supp}(\sigma_i)|$. This means that S is fixed by σ if and only if $\text{supp}(\sigma) := (\text{supp}(\sigma_i))_{i \in I} = (\emptyset)$ has measure zero. But this means $\sigma = \text{id}$ in the metric ultraproduct $S_{\mathcal{U}}$.

Now we consider the case $\overline{G} = \text{PGL}_{\mathcal{U}}(q)$. Here, similarly, assume for fixed $g = (g_i)_{i \in I} \in \text{GL}_{\mathcal{U}}(q)$ that $V \cdot g = V$ for all $V \in \mathcal{V}$. Then take $V = \overline{(V_i)_{i \in I}}$ in

the following way. The linear transformation g_i is a direct sum of Frobenius blocks $F(\chi)$, where $\chi \in k[X]$ runs through all (monic) primary polynomials. For each such block $b \leq k_i^{n_i}$ of dimension $k_b > 1$ (here seen as a subspace) of g_i select a cyclic vector v_b . Then define V_i by

$$V_i = \bigoplus_{b, k_b > 1} \langle v_b, v_b \cdot g_i^2, \dots, v_b \cdot g_i^{2(\lfloor k_b/2 \rfloor - 1)} \rangle.$$

Then one observes that $V_i \cap V_i \cdot g_i = 0$ so that

$$\dim_i(V_i + V_i \cdot g_i) - \dim_i(V_i \cap V_i \cdot g_i) = 2 \dim_i(V_i).$$

This shows that $q_\chi(g) = 0$ for all $\chi \in k[X]$ primary of degree $k_\chi > 1$. But one observes that, if $q_{(X-\lambda)}(g), q_{(X-\mu)}(g) \geq \varepsilon > 0$ for $\lambda \neq \mu$ elements of k , we can use the following construction. Let

$$e_{i1}, \dots, e_{ik_i} \in V_{X-\lambda}(g_i) \quad \text{and} \quad f_{i1}, \dots, f_{ik_i} \in V_{X-\mu}(g_i)$$

such that $\lim_{\mathcal{U}} k_i/n_i = \varepsilon$. Define $V_i := \langle e_{ij} + f_{ij} \mid j = 1, \dots, k_i \rangle$ ($i \in I$). Assume $v \in V_i \cap V_i \cdot g_i$. Then there exist numbers $\alpha_1, \dots, \alpha_{k_i}, \beta_1, \dots, \beta_{k_i} \in k$ such that

$$v = \sum_{j=1}^{k_i} \alpha_j (e_{ij} + f_{ij}) = \sum_{j=1}^{k_i} \beta_j (\lambda e_{ij} + \mu f_{ij}).$$

This gives that

$$\sum_{j=1}^{k_i} (\alpha_j - \beta_j \lambda) e_{ij} = \sum_{j=1}^{k_i} (\beta_j \mu - \alpha_j) f_{ij}$$

so that by disjointness of $V_{X-\lambda}(g_i)$ and $V_{X-\mu}(g_i)$ both sides are zero and so, since the e_{ij}, f_{ij} ($j = 1, \dots, k_i$) are linearly independent, we get that

$$\alpha_j - \beta_j \lambda = \beta_j \mu - \alpha_j = 0$$

so that, since $\lambda \neq \mu$, we obtain $\alpha_j = \beta_j = 0$. Hence $v = 0$ and $V_i \cap V_i \cdot g_i = 0$. But $\lim_{\mathcal{U}} \dim(V_i)/n_i \geq \varepsilon$, yielding the same contradiction as above. Therefore, we must have $g = \lambda \text{id}$ (for $\lambda \in \mathbb{F}_q^\times$, as $k = \mathbb{F}_q$ is finite) in the metric ultraproduct $\text{GL}_{\mathcal{U}}(q)$, i.e., $\text{PGL}_{\mathcal{U}}(q)$ is faithfully represented.

Remark 4. The above statement about $\text{PGL}_{\mathcal{U}}(q)$ holds for any such metric ultraproduct of groups $\text{PGL}_{n_i}(k_i)$ where the fields k_i are not restricted with the same proof. Here the kernel of the action $\text{GL}_{\mathcal{U}} \rightarrow \text{Aut}(\mathcal{V}, \dim)$ is given by $\prod_{\mathcal{U}} k_i^\times$ (the algebraic ultraproduct of these groups).

Remark 5. Hence, if the sequence of subsets $(S_i)_{i \in I}$ resp. subspaces $(V_i)_{i \in I}$ is almost stabilized by each element of a subgroup H of $\overline{G} = \mathbf{S}_{\mathcal{U}}$ resp. $\overline{G} = \mathbf{PGL}_{\mathcal{U}}(q)$ (or of $G = \mathbf{GL}_{\mathcal{U}}(q)$), we can restrict H to $S := \overline{(S_i)_{i \in I}}$ resp. $V := \overline{(V_i)_{i \in I}}$.

Remark 6. For an element $\sigma = \overline{(\sigma_i)_{i \in I}}$, set

$$\Omega_k(\sigma) := \overline{(\Omega_k(\sigma_i))_{i \in I}} \in \mathcal{S} \quad \text{for } k \in \mathbb{Z}_+.$$

Similarly, for a *semisimple* element $g = \overline{(g_i)_{i \in I}} \in \mathbf{GL}_{\mathcal{U}}(q)$ (see the next section for the definition of semisimple elements in such an ultraproduct), set

$$V_{\chi}(g) := \overline{(V_{\chi}(g_i))_{i \in I}} \in \mathcal{V} \quad \text{for } \chi \in k[X] \text{ irreducible.}$$

Note that these definitions are independent of the chosen representatives (for the uniqueness of $V_{\chi}(g)$, we need that g is semisimple since then $V_{\chi}(g_i) = \ker(\chi(g_i))$ is unique for a suitable representative $(g_i)_{i \in I}$ of g and $\chi \in k[X]$ irreducible).

Remark 7. Call a sequence of subsets $(B_i)_{i \in I} \subseteq k^{n_i}$ a basis of $V \in \mathcal{V}$ if there is a representative $(V_i)_{i \in I}$ of V such that B_i is a basis of V_i ($i \in I$).

Remark 8. Call $V \in \mathcal{V}$ totally singular if it has a representative $(V_i)_{i \in I}$ such that each V_i is totally singular (i.e., $f_i|_{V_i} = 0$ for $i \in I$).

6 Centralizers in $\mathbf{S}_{\mathcal{U}}$, $\mathbf{GL}_{\mathcal{U}}(q)$, $\mathbf{Sp}_{\mathcal{U}}(q)$, $\mathbf{GO}_{\mathcal{U}}(q)$, and $\mathbf{GU}_{\mathcal{U}}(q)$

In this section, we provide tools (Lemmas 2 and 3) to compute centralizers of certain elements from the metric ultraproducts $\mathbf{S}_{\mathcal{U}}$ and $\mathbf{GL}_{\mathcal{U}}(q)$. We will use this in Section 7 to compute centralizers of elements in $\mathbf{PGL}_{\mathcal{U}}(q)$. We write $\mathbf{C}(g)$ for the *centralizer* of the group element g .

Centralizers of elements in $G = \mathbf{S}_{\mathcal{U}}, \mathbf{GL}_{\mathcal{U}}(q)$. Note that, for $\sigma = \overline{(\sigma_i)_{i \in I}} \in \mathbf{S}_{\mathcal{U}}$ resp. $g = \overline{(g_i)_{i \in I}} \in \mathbf{GL}_{\mathcal{U}}(q)$, we have $\prod_{\mathcal{U}} \mathbf{C}(\sigma_i) \leq \mathbf{C}(\sigma)$ resp. $\prod_{\mathcal{U}} \mathbf{C}(g_i) \leq \mathbf{C}(g)$ (subsequently, by this notation we mean the metric ultraproduct of subgroups of the H_i ($i \in I$)). In the following lemma, we characterize when the above inclusion is actually an equality in the case of $\mathbf{S}_{\mathcal{U}}$.

Lemma 2. *An element $\sigma \in \mathbf{S}_{\mathcal{U}}$ satisfies $\sum_{k \in \mathbb{Z}_+} q_k(\sigma) = 1$ if and only if, for each choice of a representative $(\sigma_i)_{i \in I}$ of σ , the centralizer $\mathbf{C}(\sigma)$ equals $\prod_{\mathcal{U}} \mathbf{C}(\sigma_i)$.*

Before proving Lemma 2, we turn to $\mathbf{GL}_{\mathcal{U}}(q)$. An element $g \in \mathbf{GL}_{\mathcal{U}}(q)$ is called *semisimple* if it has a representative $(g_i)_{i \in I}$ such that each $g_i \in \mathbf{GL}_{n_i}(q)$ is semisimple, i.e., of order prime to q .

Lemma 3. *A semisimple element $g \in \mathrm{GL}_{\mathcal{U}}(q)$ satisfies $\sum_{\chi \text{ primary}} q_{\chi}(g) = 1$ if and only if for each choice of a representative $(g_i)_{i \in I}$ of g where each g_i is semisimple ($i \in I$), the centralizer $\mathbf{C}(g)$ equals $\prod_{\mathcal{U}} \mathbf{C}(g_i)$.*

To prove Lemmas 2 and 3, we need the following auxiliary result.

Lemma 4. *The following are true.*

- (i) *Assume $\sigma \in \mathrm{Sym}(\underline{n})$ is of order k and $S \subseteq \underline{n}$ has normalized counting measure $\mu(S)$. Then S contains a σ -invariant subset T of measure*

$$\mu(T) \geq 1 - k(1 - \mu(S)).$$

- (ii) *Assume $g \in \mathrm{GL}(V)$ for a k -vector space V and that the minimal polynomial of g over k has degree k . Assume $U \leq V$. Then there exists a g -invariant subspace of U of codimension at most $k \operatorname{codim}(U)$.*

Proof. (i) Observe that the biggest σ -invariant subset of S is equal to

$$T = \bigcap_{i \in \mathbb{Z}} S \cdot \sigma^i.$$

But since $\sigma^k = \mathrm{id}$ by assumption, we see that actually $T = \bigcap_{i=0}^{k-1} S \cdot \sigma^i$. Hence, since $\mu(S \cdot \sigma^i) = \mu(S)$ for all $i \in \mathbb{Z}$, we have that $\mu(T) \geq 1 - k(1 - \mu(S))$.

(ii) Similarly to the above, the biggest g -invariant subspace contained in U is $W = \bigcap_{i \in \mathbb{Z}} U \cdot g^i$. Now $v \in \bigcap_{i=0}^{k-1} U \cdot g^i$ means that $v, \dots, v \cdot g^{-(k-1)} \in U$. But then

$$v \cdot g^{-k} = -\frac{1}{a_0}(a_1 v \cdot g^{-(k-1)} + \dots + a_{k-1} v \cdot g^{-1} + v) \in U,$$

where $\chi = a_0 + a_1 X + \dots + a_{k-1} X^{k-1} + X^k$ is the minimal polynomial of g . Note that $a_0 = (-1)^k \det(g) \neq 0$. This shows that actually $W = \bigcap_{i=0}^{k-1} U \cdot g^i$ so that $\operatorname{codim}(W) \leq k \operatorname{codim}(U)$. $\square$

Remark 9. The bounds in Lemma 4 are sharp. For example, take σ of type (k^{c_k}) , and set $n = c_k k$. Take S of size $n - s$ such that, for precisely $s \leq c_k$ k -cycles of σ , S contains $k - 1$ elements of each of them and all elements of the remaining k -cycles. Then the set T constructed in Lemma 4 has size $n - ks$. In (ii), we can use a similar construction.

Now we are able to prove Lemmas 2 and 3.

Proof of Lemmas 2 and 3. At first, we prove Lemma 2. Assume that $\sigma = \overline{(\sigma_i)}_{i \in I}$, $\tau = \overline{(\tau_i)}_{i \in I} \in S_{\mathcal{U}}$ commute, and assume that $\sum_{k=1}^{\infty} q_k(\sigma) = 1$. Find a sequence $(N_i)_{i \in I}$ tending to infinity along $\mathcal{U}$ such that

$$\lim_{\mathcal{U}} \sum_{k=1}^{N_i} q_k(\sigma_i) = 1 \quad \text{and} \quad \binom{N_i + 1}{2} \ell_H([\sigma_i, \tau_i]) \rightarrow_{\mathcal{U}} 0.$$

Recall that $C_k(\sigma_i)$ denotes the set of k -cycles of the permutation σ_i . Call a k -cycle of σ_i *bad* if it is not mapped σ_i -equivariantly to another k -cycle of σ_i by τ_i . Collect the set of bad k -cycles of σ_i in $C'_k(\sigma_i)$. For each bad k -cycle of σ_i , we get at least one non-fixed point of $[\sigma_i, \tau_i]$ so that $|C'_k(\sigma_i)|/n_i \leq \ell_H([\sigma_i, \tau_i])$ for all $k \in \mathbb{Z}_+$. Hence, if we change τ_i such that all bad k -cycles of σ_i are mapped accurately for $k \leq N_i$ and all k -cycles for $k > N_i$ are mapped identically, we get a permutation τ'_i such that

$$\begin{aligned} d_H(\tau_i, \tau'_i) &\leq \frac{1}{n_i} \sum_{k=1}^{N_i} k |C'_k(\sigma_i)| + \sum_{k=N_i+1}^{\infty} q_k(\sigma_i) \\ &\leq \left(\sum_{k=1}^{N_i} k \right) \ell_H([\sigma_i, \tau_i]) + \sum_{k=N_i+1}^{\infty} q_k(\sigma_i) \\ &= \binom{N_i + 1}{2} \ell_H([\sigma_i, \tau_i]) + \sum_{k=N_i+1}^{\infty} q_k(\sigma_i). \end{aligned}$$

By the assumption $\sum_{k=1}^{\infty} q_k(\sigma) = 1$, the last term in the above estimate tends to zero along $\mathcal{U}$. Hence $\tau = \overline{(\tau_i)}_{i \in I} = \overline{(\tau'_i)}_{i \in I}$ and $[\sigma_i, \tau'_i] = 1$.

Conversely, assume that $\sum_{k=1}^{\infty} q_k(\sigma) < 1$. Choose the sequence $(N_i)_{i \in I}$ such that $\lim_{\mathcal{U}} \sum_{k=1}^{N_i} q_k(\sigma_i) = \sum_{k=1}^{\infty} q_k(\sigma)$ and $\lim_{\mathcal{U}} N_i/n_i = 0$.

For each $i \in I$, change σ_i to σ'_i such that the k -cycles of σ'_i are the same as in σ_i for $1 \leq k \leq N_i$ and the other k -cycles of σ'_i ($k > N_i$, if they exist) are grouped into one big K_i -cycle so that $d_H(\sigma_i, \sigma'_i)$ is minimal possible. It is easy to see that then still $d_H(\sigma_i, \sigma'_i) \rightarrow_{\mathcal{U}} 0$ as $N_i \rightarrow_{\mathcal{U}} \infty$. Now σ'_i eventually has precisely one K_i -cycle for $K_i > N_i$. Obtain σ''_i by dividing this K_i -cycle (if it exists) into two $\lfloor K_i/2 \rfloor$ -cycles and at most one fixed point so that $d_H(\sigma'_i, \sigma''_i) \leq 3/n_i$ is minimal. Note that $K_i/n_i = 1 - \sum_{k=1}^{N_i} q_k(\sigma_i) \rightarrow_{\mathcal{U}} \varepsilon > 0$ so that $\lfloor K_i/2 \rfloor > N_i$ along $\mathcal{U}$, as $\lim_{\mathcal{U}} N_i/n_i \rightarrow_{\mathcal{U}} 0$ by assumption.

Now consider the restriction of the centralizers $\mathbf{C}(\sigma'_i)$ and $\mathbf{C}(\sigma''_i)$ to the support of the unique K_i -cycle of σ'_i (which certainly both fix setwise by the previous inequality). The first group is isomorphic to the cyclic group C_{K_i} , whereas the

second is isomorphic to $C_{\lfloor K_i/2 \rfloor} \wr C_2$. Taking the metric ultraproducts of these groups restricted to this support (in the sense of Remark 5), we get an abelian group in the first case, and a non-abelian group in the second case. Hence, in at least one case, $\prod_{\mathcal{U}} C(\sigma'_i) \neq C(\sigma)$ or $\prod_{\mathcal{U}} C(\sigma''_i) \neq C(\sigma)$.

Now we prove Lemma 3. Assume that $g = (g_i)_{i \in I}$, $h = (\overline{h_i})_{i \in I} \in \text{GL}_{\mathcal{U}}(q)$ commute, i.e., $[g, h] = \text{id}$, that g and each g_i ($i \in I$) is semisimple, and assume that

$$\sum_{\chi \text{ irreducible}}^{\infty} q_{\chi}(g) = 1.$$

Recall that semisimplicity means that, for each Frobenius block $F(\chi)$ in the generalized Jordan normal form of g_i , $\chi = t^1$ is irreducible. Choose the sequence $(N_i)_{i \in I}$ such that

$$\lim_{\mathcal{U}} \sum_{\substack{\chi \text{ irreducible} \\ \deg(\chi) \leq N_i}} q_{\chi}(g_i) = 1 \quad \text{and} \quad \left(\sum_{\substack{\chi \text{ irreducible} \\ \deg(\chi) \leq N_i}} \deg(\chi) \right) \ell_{\text{rk}}([g_i, h_i]) \rightarrow_{\mathcal{U}} 0.$$

Define $U_i := \ker([g_i, h_i] - \text{id})$. Fix an irreducible polynomial $\chi \in k[X]$ and apply Lemma 4(ii) inside $V := V_{\chi}(g_i)$ to the subspace $U := U_i \cap V_{\chi}(g_i)$ to get a g_i -invariant subspace $W = W_{i\chi} \leq U$ such that $\text{codim}_V(W_{i\chi}) \leq k_{\chi} \text{codim}(U_i)$, where $k_{\chi} = \deg(\chi)$. Note here that $V_{\chi}(g_i) = \ker(\chi(g_i))$ is unique since g_i is semisimple. This large-dimensional subspace $W_{i\chi}$ is mapped accurately by h_i , as g_i commutes with h_i on it. Define h'_i to be equal to h_i on each $W_{i\chi}$, and complete it on each $V_{i\chi}$ to a map commuting with g_i for $\deg(\chi) \leq N_i$ (here we use semisimplicity of g_i). On $V_{\chi}(g_i)$ with $\deg(\chi) > N_i$, set h'_i to be the identity. As in the proof for $S_{\mathcal{U}}$ above, it follows that $d_{\text{rk}}(h_i, h'_i) \rightarrow_{\mathcal{U}} 0$ and $[g_i, h'_i] = 1$.

Conversely, assume that $\sum_{\chi \text{ irreducible}}^{\infty} q_{\chi}(g) < 1$. Choose the sequence $(N_i)_{i \in I}$ such that

$$\lim_{\mathcal{U}} \sum_{\substack{\chi \text{ irreducible} \\ \deg(\chi) \leq N_i}}^{N_i} q_{\chi}(g_i) = \sum_{\chi \text{ irreducible}}^{\infty} q_{\chi}(g) \quad \text{and} \quad \lim_{\mathcal{U}} N_i/n_i = 0.$$

For each $i \in I$, change g_i into g'_i such that all Frobenius blocks $F(\chi)$ for χ irreducible of degree at most N_i are left unchanged and all bigger Frobenius blocks (if there is any such block) are grouped into one big Frobenius block $F(\varphi)$ of size K_i (for φ irreducible). Define g''_i in the same way, but split the Frobenius block $F(\varphi)$ (if it exists) into two or three blocks, two of which are $F(\phi)$ for ϕ irreducible of degree $\lfloor K_i/2 \rfloor$ and, if K_i is odd, one block of size one, which is the identity. Then, as above, the centralizer of g'_i restricted to the large Frobenius block $F(\varphi)$ of it,

equals $\mathbf{C}(g'_i) \cong (k[X]/(\varphi))^\times$, whereas the centralizer $\mathbf{C}(g''_i)$ restricted to the same subspace is non-abelian (again in the sense of Remark 5). Also one sees that their metric ultraproducts are non-isomorphic, similarly to the case of permutations. The proof is complete. $\square$

Remark 10. If G is one of $\mathrm{Sp}_{\mathcal{U}}(q)$, $\mathrm{GO}_{\mathcal{U}}(q)$, or $\mathrm{GU}_{\mathcal{U}}(q)$ and a semisimple $g \in G$ is represented by $(g_i)_{i \in I}$ and $\sum_{\chi \text{ irreducible}} q_{\chi}(g) = 1$, one can adapt the above argument for $\mathrm{GL}_{\mathcal{U}}(q)$ to see that still $\mathbf{C}(g) = \prod_{\mathcal{U}} \mathbf{C}(g_i)$ when all g_i are semisimple.

Indeed, from Fact 3, it follows that, in the space $W_{i\chi} + W_{i\chi^*}$ (where $W_{i\chi}, W_{i\chi^*}$ are constructed as above), we can still find a big, i.e., almost full-dimensional, g_i -invariant non-singular subspace $W'_{i\chi, \chi^*}$. This is because each block $F(\chi)$ must be paired with a block $F(\chi^*)$ (which can possibly be the same as the first one) to get an indecomposable. Now $W_{i\chi}$ contains almost all blocks $F(\chi)$ of g_i , and $W_{i\chi^*}$ contains almost all blocks $F(\chi^*)$ of g_i , so many of them must be paired together.

By Fact 2, we can then complete the partial map

$$h_i|_{W'_{i\chi, \chi^*}} : W'_{i\chi, \chi^*} \rightarrow W'_{i\chi, \chi^*} \cdot h_i$$

to a map $h'_i \in \mathrm{GL}(V_{\chi}(g_i) + V_{\chi^*}(g_i))$ stabilizing f_i (and Q_i in the orthogonal case for $p = 2$). So we have found an element $h'_i \in H_i$ close to h_i which commutes with g_i . This completes the proof.

As a consequence of Lemma 1 together with Lemmas 2 and 3, and Remark 10, we get the following corollary.

Corollary 1. *If an element $\sigma \in \mathbf{S}_{\mathcal{U}}$ resp. a semisimple element*

$$g \in \mathrm{GL}_{\mathcal{U}}(q), \mathrm{Sp}_{\mathcal{U}}(q), \mathrm{GO}_{\mathcal{U}}(q), \text{ or } \mathrm{GU}_{\mathcal{U}}(q)$$

is torsion, then $\mathbf{C}(\sigma)$ resp. $\mathbf{C}(g)$ is equal to $\prod_{\mathcal{U}} \mathbf{C}(\sigma_i)$ resp. $\prod_{\mathcal{U}} \mathbf{C}(g_i)$ for each representative $(\sigma_i)_{i \in I}$ resp. $(g_i)_{i \in I}$ of σ resp. g , where we require all g_i ($i \in I$) to be semisimple.

7 Centralizers in $\mathrm{PGL}_{\mathcal{U}}(q)$, $\mathrm{PSp}_{\mathcal{U}}(q)$, $\mathrm{PGO}_{\mathcal{U}}(q)$, and $\mathrm{PGU}_{\mathcal{U}}(q)$

Now we can deduce the structure of centralizers of semisimple elements from $\mathrm{PGL}_{\mathcal{U}}(q)$. By this we mean elements that lift to semisimple elements in $\mathrm{GL}_{\mathcal{U}}(q)$. Let $g = (\overline{g_i})_{i \in I} \in \mathrm{GL}_{\mathcal{U}}(q)$ be a semisimple element which maps to

$$\bar{g} \in \mathrm{PGL}_{\mathcal{U}}(q) = \mathrm{GL}_{\mathcal{U}}(q)/k^\times.$$

All g_i are assumed to be semisimple ($i \in I$).

Assume that $h = \overline{(h_i)}_{i \in I} \in \mathrm{GL}_{\mathcal{U}}(q)$ is such that $[g, h] = \mu \mathrm{id}$ for $\mu \in k^\times$. Then $g^h = \mu g$ so that $q(g) = q(g^h) = q(\mu g) = q(g) \cdot \mu$, i.e., $\mu \in \mathrm{stab}_{k^\times}(q(g))$. Let $v \in \mathrm{stab}_{k^\times}(q(g)) \leq k^\times$ be a generator of this cyclic group.

It is now easy to see that the *conformal centralizer*

$$\mathbf{C}_{\mathrm{conf}}(g) := \{h \in \mathrm{GL}_{\mathcal{U}}(q) \mid \text{there is } \mu \in k^\times \text{ such that } g^h = \mu g\}$$

is an extension $\mathbf{C}(g) \cdot \mathrm{stab}_{k^\times}(q(g)) = \mathbf{C}(g) \cdot \langle v \rangle$ of $\mathbf{C}(g)$ by $\mathrm{stab}_{k^\times}(q(g))$. Hence $\mathbf{C}(\bar{g}) = (\mathbf{C}(g) \cdot \langle v \rangle) / k^\times$.

Remark 11. The analog statement of Lemma 3 is false in $\mathrm{PGL}_{\mathcal{U}}(q)$. Indeed, take a semisimple element $\bar{g} \in \mathrm{PGL}_{\mathcal{U}}(q)$ such that, for a lift $g \in \mathrm{GL}_{\mathcal{U}}(q)$, the group $\mathrm{stab}_{k^\times}(q(g))$ is non-trivial. Choose a representative $(g_i)_{i \in I}$ of $g \in \mathrm{GL}_{\mathcal{U}}(q)$ such that $q_\chi(g_i) \neq q_\xi(g_i)$ for all $\chi, \xi \in k[X]$ distinct irreducible and g_i is semisimple ($i \in I$). Then $\mathbf{C}(g_i)$ stabilizes each subspace $V_\chi(g_i) = \ker(\chi(g_i)) \leq k^{n_i}$. But this means that, if $h \in C := \prod_{\mathcal{U}} \mathbf{C}_{\mathrm{conf}}(g_i)$, we have that $g^h = g$ so that $C/k^\times$ is properly contained in $\mathbf{C}(\bar{g})$ (namely, $\mathbf{C}(\bar{g})/(C/k^\times) \cong \mathrm{stab}_{k^\times}(q(g))$, which is non-trivial).

Remark 12. For the groups $\mathrm{PSp}_{\mathcal{U}}(q)$, $\mathrm{PGO}_{\mathcal{U}}(q)$, and $\mathrm{PGU}_{\mathcal{U}}(q)$, the same structure for $\mathbf{C}(\bar{g})$ holds true, where $\mathrm{Sp}_{\mathcal{U}}(q)$, $\mathrm{GO}_{\mathcal{U}}(q)$ resp. $\mathrm{GU}_{\mathcal{U}}(q)$ play the role of $\mathrm{GL}_{\mathcal{U}}(q)$. The possible scalars $\mu \in k^\times$ (from Z) are restricted to $\mu \in \{\pm 1\}$ in the bilinear case, and to $\mu^{q+1} = 1$ in the unitary case.

8 Double centralizers of torsion elements

In this section, we compute the double centralizers of semisimple torsion elements of the groups $\bar{G}$ of type $\mathrm{S}_{\mathcal{U}}$, $\mathrm{PGL}_{\mathcal{U}}(q)$, $\mathrm{PSp}_{\mathcal{U}}(q)$, $\mathrm{PGO}_{\mathcal{U}}(q)$, and $\mathrm{PGU}_{\mathcal{U}}(q)$. Note that, for $g \in G$ a group element, $\mathbf{C}(\mathbf{C}(g)) = \mathbf{Z}(\mathbf{C}(g))$ since $g \in \mathbf{C}(g)$ so that $\mathbf{C}(\mathbf{C}(g)) \leq \mathbf{C}(g)$. Set $\mathbf{C}^2(g) := \mathbf{C}(\mathbf{C}(g))$ and $\mathbf{C}_{\mathrm{conf}}^2(g) := \mathbf{C}_{\mathrm{conf}}(\mathbf{C}_{\mathrm{conf}}(g))$ to be the *double centralizer* resp. *double conformal centralizer* of g . Here

$$\mathbf{C}_{\mathrm{conf}}(g) := \{h \in G \mid [g, h] \in \mathbf{Z}(G)\}.$$

8.1 The case $\mathrm{S}_{\mathcal{U}}$

Let $\sigma = \overline{(\sigma_i)}_{i \in I} \in \mathrm{S}_{\mathcal{U}} = \bar{G}$ be torsion of order o . Then we have $\sum_{k|o} q_k(\sigma) = 1$ by Lemma 1. By Corollary 1, we have that $\mathbf{C}(\sigma) = \prod_{\mathcal{U}} \mathbf{C}(\sigma_i)$. But $\mathbf{C}(\sigma_i)$ has a subgroup

$$\prod_{k|o} \mathbf{C}_k \wr \mathrm{Sym}(c_k(\sigma_i))$$

which is dense in it along $\mathcal{U}$ so that $C := \mathbf{C}(\sigma) = \prod_{\mathcal{U}} \prod_{k|o} \mathbf{C}_k \wr \mathrm{Sym}(c_k(\sigma_i))$.

At first, for simplicity, assume that σ_i is isotypic of type $(k^{c_{ik}})$ (so $n_i = c_{ik}k$). Assume that $\tau = (\overline{\tau_i})_{i \in I} \in \mathbf{Z}(C)$ and $\tau_i = (a_{ij}).\varphi_i \in C_k \wr \text{Sym}(c_{ik})$. Assume that $\lim_{\mathcal{U}} |\text{supp}(\varphi_i)|/c_{ik} = \varepsilon > 0$. Then we can conjugate φ_i by

$$\phi_i \in \text{Sym}(c_{ik}) \leq C_k \wr \text{Sym}(c_{ik}) = \mathbf{C}(\sigma_i)$$

such that

$$\lim_{\mathcal{U}} d_H(\varphi_i \phi_i, \phi_i \varphi_i) \geq \varepsilon > 0.$$

But this leads to the contradiction

$$\lim_{\mathcal{U}} d_H(\tau_i \phi_i, \phi_i \tau_i) \geq \varepsilon > 0.$$

Hence we may assume that $\varphi_i = \text{id}$, applying a small change to τ_i along $\mathcal{U}$ if necessary ($i \in I$). Now assume that $\lim_{\mathcal{U}} |\{j \mid a_{ij} = c\}|/c_{ik} = \varepsilon \in (0, 1)$. Then we find permutations $\phi_i \in \text{Sym}(c_{ik}) \leq C_k \wr \text{Sym}(c_{ik}) = \mathbf{C}(\sigma_i)$ such that

$$d_H(\tau_i, \tau_i^{\phi_i}) = |\{j \mid a_{ij} \neq a_{ij.\phi_i}\}|/c_{ik} \geq \min\{\varepsilon, 1 - \varepsilon\} > 0.$$

Hence we can assume that all a_{ij} are equal. This shows that, in this case, $\mathbf{Z}(\mathbf{C}(\sigma))$ is the metric ultraproduct $\prod_{\mathcal{U}} C_k \cong C_k$, where C_k in the i th component is generated by the element σ_i itself ($i \in I$).

In the general case, we obtain that

$$\mathbf{Z}(\mathbf{C}(\sigma)) = \prod_{k|o, q_k(\sigma) > 0} \prod_{\mathcal{U}} C_k \cong \prod_{k|o, q_k(\sigma) > 0} C_k.$$

This holds because $\sigma \in \mathbf{C}(\sigma)$ so that, when $\tau \in \mathbf{Z}(\mathbf{C}(\sigma))$, it must commute with σ . But this implies that $\lim_{\mathcal{U}} |\Omega_k(\sigma_i) \triangle \Omega_k(\sigma_i).\tau_i| = 0$ so that τ must stabilize the isotypic components of σ (in the sense of Remark 5), and we can apply the above argument.

8.2 The case $\text{PGL}_{\mathcal{U}}(q)$, $\text{PSp}_{\mathcal{U}}(q)$, $\text{PGO}_{\mathcal{U}}(q)$, and $\text{PGU}_{\mathcal{U}}(q)$

Recall from Section 2 that $k = \mathbb{F}_q$ when G is $\text{GL}_{\mathcal{U}}(q)$, $\text{Sp}_{\mathcal{U}}(q)$, or $\text{GO}_{\mathcal{U}}(q)$, and $k = \mathbb{F}_{q^2}$ when $G = \text{GU}_{\mathcal{U}}(q)$. Set $d = 1$ in the first three cases and $d = 2$ when G is unitary over $\mathbb{F}_{q^2}$.

Recall that $Z = k^{\times}$ when $G = \text{GL}_{\mathcal{U}}(q)$, $Z = \{\pm 1\} \subseteq k^{\times}$ when $G = \text{Sp}_{\mathcal{U}}(q)$ or $G = \text{GO}_{\mathcal{U}}(q)$, and $Z = \{z \in k^{\times} \mid z^{q+1} = 1\} \subseteq k^{\times} = \mathbb{F}_{q^2}^{\times}$ when $G = \text{GU}_{\mathcal{U}}(q)$. Also recall that, if G is not of shape $\text{GL}_{\mathcal{U}}(q)$, we have $z^{\sigma} = z^{-1}$ for $z \in Z$, where $\sigma: k \rightarrow k$ is the identity in the bilinear case, and the q -Frobenius endomorphism $x \mapsto x^q$ when $G = \text{GU}_{\mathcal{U}}(q)$. Let $g = (\overline{g_i})_{i \in I} \in G \leq \text{GL}_{\mathcal{U}}(k)$ be semisimple,

with g_i ($i \in I$) semisimple such that $\bar{g} \in \bar{G} \leq \mathrm{PGL}_{\mathcal{U}}(k) = \mathrm{GL}_{\mathcal{U}}(k)/k^\times$ is torsion of order dividing o , i.e., there is $\mu \in k^\times$ such that $g^o = \mu \mathrm{id}$. This implies $\mu \in Z$. Then

$$\sum_{\substack{\chi \text{ irreducible} \\ X^o \equiv \mu(\chi)}} q_\chi(g) = 1$$

by Lemma 1. Set

$$P := \{\chi \in k[X] \mid \chi \text{ (monic) irreducible, } \chi \mid X^o - \mu\},$$

$T := \mathrm{stab}_Z(q(g))$, $K_\chi := k[X]/(\chi)$ for $\chi \in k[X]$ irreducible (as in Fact 3), and $c_{i\chi} := c_\chi(g_i)$ ($i \in I$). Hence, similarly to the above, we have

$$\mathbf{C}(g) = \prod_{\mathcal{U}} \prod_{\substack{\chi \text{ irreducible} \\ X^o \equiv \mu(\chi) \\ q_\chi(g) > 0}} \mathbf{M}_{c_{i\chi}}(K_\chi),$$

the centralizer being computed in $\mathbf{M}_{\mathcal{U}}(k)$. Now, by Section 7, we “know” the structure of $\mathbf{C}_{\mathrm{conf}}(g) \leq G$. For $\chi \in k[X]$ irreducible, consider the g -invariant subspace $V := V_{\bar{\chi}}(g) := \bigoplus_{\xi \in \bar{\chi}} V_\xi(g) \in \mathcal{V}$, where $\bar{\chi} := \mathrm{orb}_T(\chi)$ is the orbit of χ under T (see Remark 6 for the definition of $V_\xi(g) \in \mathcal{V}$). Set $l_\chi := |\bar{\chi}|$, $m_\chi := |T|/l_\chi$. Note that $m_\chi = |\mathrm{stab}_T(\chi)|$, and so

$$m_\chi = \max\{m \text{ divides } |T| \mid \text{there exists } \chi' \text{ such that } \chi = \chi'(X^m)\}.$$

The restriction of the action of $\mathbf{C}_{\mathrm{conf}}(g)/Z$ to $V_{\bar{\chi}}(g)$ is given by

$$\left(\left(\prod_{\mathcal{U}} \prod_{\xi \in \bar{\chi}} \mathbf{C}(g|_{V_\xi(g)}) \right) \cdot T \right) / Z.$$

We will explain this below. In most cases, the above extension is even split.

Definition of the action of T . Let $t \in Z$ be a generator of the above cyclic group T . Then the map φ_t , which is constructed in the following, generates the above extension.

At first, note that we can see each space $V_\xi(g_i)$ as a vector space over the field $K_\xi = k[X]/(\xi)$ ($\xi \in k[X]$ irreducible) as g acts on it as multiplication by $\lambda_\xi = \bar{X}$ acts on K_ξ (cf. Fact 3). With this convention, find K_ξ -bases $(B_{\xi,i})_{i \in I}$ of each $V_\xi(g)$ ($\xi \in \bar{\chi}$ for all representatives χ of orbits of the action of T on the irreducible polynomials; see Remark 7) and compatible bijections $\alpha_{\xi_1, \xi_2, i}: B_{\xi_1, i} \rightarrow B_{\xi_2, i}$ (for $i \in I$, i.e., $\alpha_{\xi_2, \xi_3, i} \circ \alpha_{\xi_1, \xi_2, i} = \alpha_{\xi_1, \xi_3, i}$ for all $\xi_1, \xi_2, \xi_3 \in \bar{\chi}$, all χ , and all $i \in I$).

If G comes from groups preserving a form, we still find bijections $\bullet^*: B_{\xi,i} \rightarrow B_{\xi^*,i}$ such that $b^{**} = b$, the pairing f_i restricted to

$$K_{\xi}b \times K_{\xi^*}b^* = k[g]b \times k[g]b^* \rightarrow k$$

is non-singular, the pairing f_i restricted to $K_{\xi}b \times K_{\xi'}b'$ is zero for all $b \in B_{\xi,i}$, $b' \in B_{\xi',i}$, $b' \neq b^*$, and such that $\bullet^*$ commutes with the maps $\alpha_{\xi_1, \xi_2, i}$ ($i \in I$). Such bases exist by an application of Fact 3. The last condition can be fulfilled since $(\xi.t)^* = \xi^*.t$ for all $\xi \in k[X]$ by Remark 3.

Now we define maps $\varphi_{t,i}$ on k^{n_i} by defining their restrictions

$$\varphi_{t,i}|_{\langle B_{\xi,i} \rangle_{K_{\xi}}} : \langle B_{\xi,i} \rangle_{K_{\xi}} \rightarrow \langle B_{\xi.t,i} \rangle_{K_{\xi.t}}.$$

We do this by defining a k -linear map $\varphi_{\xi,t}: K_{\xi} \rightarrow K_{\xi.t}$ (which is not necessarily a homomorphism) and extending it to a map $\varphi_{t,i}$ by linearity,

$$\varphi_{t,i} \left(\sum_{b \in B_{\xi,i}} \lambda_b b \right) = \sum_{b \in B_{\xi,i}} \varphi_{\xi,t}(\lambda_b) \alpha_{\xi, \xi.t, i}(b).$$

Let $\phi_{\xi,t}: K_{\xi} \rightarrow K_{\xi.t}$ denote the k -homomorphism such that $\lambda_{\xi} \mapsto t \lambda_{\xi.t}$. Here is our definition of $\varphi_{\xi,t}$.

If not $p > 2$, $\varepsilon = -1$, $\chi = \chi^*$, $t = -1$, and $\chi.t = \chi$, set $\varphi_{\xi,t} := \phi_{\xi,t}$. In this case, $\varphi_t^{|T|} = \text{id}$, and the above extension is split.

In the opposite case,

$$\xi = \xi^* = \xi.t = \xi.(-1) \quad \text{for all } \xi \in \bar{\chi}.$$

Let $\eta \in K_{\xi}$ be a generator of the cyclic group $K_{\xi}^{\times}$ and set $\zeta := \eta^{(q^{k_{\chi}/2}-1)/2}$, where $k_{\chi} = \deg(\chi) = \deg(\xi)$, so that $\zeta^{q^{k_{\chi}/2}+1} = -1$. Then set $\varphi_{\xi,t}: K_{\xi} \rightarrow K_{\xi.t}$ to be the map $x \mapsto \zeta \phi_{\xi,t}(x)$. In this case, $\varphi_{\xi,t}^2(x) = -x$, so the above extension needs not to be split.

Doing this for all representatives χ of orbits of the action of T on the irreducible polynomials $\chi \in k[X]$ with $\chi \mid X^o - \mu$, this defines, up to a small error in the rank metric, a map $\varphi_{t,i}: k^{n_i} \rightarrow k^{n_i}$, so set φ_t to be $(\varphi_{t,i})_{i \in I}$.

The map g t -commutes with φ_t . Recall from Fact 3 that g_i ($i \in I$) acts on K_{ξ} as multiplication by λ_{ξ} (and so on $K_{\xi.t}$ as multiplication by $\lambda_{\xi.t}$). Hence, by the above definition of φ_t , we need to verify that

$$\varphi_{\xi,t}(\lambda_{\xi}x) = t \lambda_{\xi.t} \varphi_{\xi,t}(x) \tag{8.1}$$

for all $x \in K_{\xi}$, $\xi \in \bar{\chi}$.

In the first of the above cases, this is true since then $\varphi_{\xi,t} = \phi_{\xi,t}$ is a field isomorphism (and hence multiplicative) and $\phi_{\xi,t}(\lambda_{\xi}) = t\lambda_{\xi,t}$ by definition. In the second case, the left-hand side of equation (8.1) evaluates to

$$\varphi_{\xi,t}(\lambda_{\xi}x) = \zeta\phi_{\xi,t}(\lambda_{\xi}x) = \zeta\phi_{\xi,t}(\lambda_{\xi})\phi_{\xi,t}(x) = \zeta t\lambda_{\xi,t}\phi_{\xi,t}(x),$$

whereas the right-hand side is $t\lambda_{\xi,t}\varphi_{\xi,t}(x) = t\lambda_{\xi,t}\zeta\phi_{\xi,t}(x)$, so they are equal.

The action of φ_t preserves the forms f_i (and Q_i in the orthogonal case when $p = 2$, $i \in I$). Assume G is not $\mathrm{GL}_{\mathcal{U}}(q)$. Then one sees that φ_t preserves the forms f_i ($i \in I$). We verify this in the different cases of Fact 3. Let $b \in B_{\xi,i}$ (for $\xi \in \bar{\chi}$), and consider the restriction of f_i to $U \times U^* := K_{\xi}b \times K_{\xi}^*b^*$. Recall from Fact 3 that, in this situation, $\tau: K_{\xi}^* \rightarrow K_{\xi}$ is the homomorphism such that $\tau: \lambda_{\xi}^* \mapsto \lambda_{\xi}^{-1}$ and $\tau|_h = \sigma$. Let $\tau': K_{\xi^*,t} \rightarrow K_{\xi,t}$ be the corresponding homomorphism such that $\lambda_{\xi^*,t} \mapsto \lambda_{\xi,t}^{-1}$ and $\tau'|_k = \sigma$.

In case (i) (a) of Fact 3, we have

$$\begin{aligned} f_i(u.t, v.t) &= \mathrm{tr}_{K_{\xi,t}/k}(\varphi_{t,i}(u)\varphi_{t,i}(v)^{\tau'}) \\ &= \mathrm{tr}_{K_{\xi,t}/k}(\phi_{\xi,t}(u)\phi_{\xi^*,t}(v)^{\tau'}) \\ &= \mathrm{tr}_{K_{\xi,t}/k}(\phi_{\xi,t}(uv^{\tau})) \\ &= \mathrm{tr}_{K_{\xi}/k}(uv^{\tau}) = f_i(u, v). \end{aligned}$$

Here we use in the third line that $\tau' \circ \phi_{\xi^*,t} = \phi_{\xi,t} \circ \tau$ since both induce σ on the ground field k and

$$\begin{aligned} \phi_{\xi^*,t}(\lambda_{\xi^*})^{\tau'} &= (t\lambda_{\xi^*,t})^{\tau'} = t^{\sigma}\lambda_{\xi,t}^{-1} = (t\lambda_{\xi,t})^{-1} \\ &= \phi_{\xi,t}(\lambda_{\xi})^{-1} = \phi_{\xi,t}(\lambda_{\xi}^{-1}) = \phi_{\xi,t}(\lambda_{\xi}^{\tau}). \end{aligned}$$

In the orthogonal case for $p = 2$, the forms Q_i are preserved by a similar argument.

In case (i) (b) of Fact 3, let $\delta \in K_{\xi}$ be the element used there, and let $\delta' \in K_{\xi,t}$ be the analogous element in the $\xi.t$ -block. We may assume $\delta = \delta' = 1$ when $\varepsilon = 1$. Then the same computation as in the previous case verifies that φ_t preserves the forms f_i (and Q_i). Now assume that $\varepsilon = -1$. If $t = 1$, there is nothing to show, so $t = -1$. When $\chi \neq \chi.t$, then $\xi \neq \xi.t$ for $\xi \in \bar{\chi}$, so we can assume that $\delta' = \phi_{\xi,t}(\delta)$. Then

$$\begin{aligned} f_i(u.t, v.t) &= \mathrm{tr}_{K_{\xi,t}/k}(\delta'\varphi_{t,i}(u)\varphi_{t,i}(v)^{\tau'}) \\ &= \mathrm{tr}_{K_{\xi,t}/k}(\delta'\phi_{\xi,t}(u)\phi_{\xi,t}(v)^{\tau'}) \\ &= \mathrm{tr}_{K_{\xi,t}/k}(\delta'\phi_{\xi,t}(uv^{\tau})) \end{aligned}$$

$$\begin{aligned}
&= \text{tr}_{K_{\xi,t}/k}(\phi_{\xi,t}(\delta)\phi_{\xi,t}(uv^\tau)) \\
&= \text{tr}_{K_{\xi}/k}(\delta uv^\tau) = f_i(u, v).
\end{aligned}$$

So assume $\chi = \chi.t$ so that $\xi = \xi.t$ (for $\xi \in \bar{\chi}$), $\tau = \tau'$, and $\delta' = \delta$. Then

$$\begin{aligned}
f_i(u.t, v.t) &= \text{tr}_{K_{\xi,t}/k}(\delta' \varphi_{t,i}(u) \varphi_{t,i}(v)^{\tau'}) \\
&= \text{tr}_{K_{\xi}/k}(\delta \zeta \phi_{\xi,t}(u) \zeta^{\tau'} \phi_{\xi,t}(v)^{\tau'}) \\
&= \text{tr}_{K_{\xi}/k}(\zeta^{q^{k\chi/2}+1} \delta \phi_{\xi,t}(uv^\tau)) \\
&= \text{tr}_{K_{\xi}/k}(\phi_{\xi,t}(\delta uv^\tau)) \\
&= \text{tr}_{K_{\xi}/k}(\delta uv^\tau) = f_i(u, v).
\end{aligned}$$

Here we use in the third line that $\zeta^{q^{k\chi/2}+1} = -1$ and $\phi_{\xi,t}(\delta) = -\delta$ since $\phi_{\xi,t} = \tau$ is the unique involution $x \mapsto x^{q^{k\chi/2}}$ in this case.

Assume now we are in case (ii). Then we may again assume that $t = -1$ and $p > 2$ since $|T| \leq |Z| \leq 2$, and also $\chi = X \pm 1 \neq \chi.t = X \mp 1$. Then we have $\varphi_{\xi,t} = \phi_{\xi,t} = \text{id}_k$ so that f_i is preserved according to Fact 3. So we have shown that φ_t preserves the forms f_i (and Q_i).

Now let us fix $h \in \mathbf{C}_{\text{conf}}^2(g)$. We want to understand the shape of h .

Step 1: h stabilizes each $V_\chi(g)$ ($\chi \in k[X]$ irreducible). Assume that

$$h \in \mathbf{C}_{\text{conf}}^2(g|_V) \leq \mathbf{C}_{\text{conf}}(g|_V)$$

does not stabilize each subspace $V_\xi(g)$ of V ($\xi \in \bar{\chi}$). Write $\bar{\chi} = \{\xi_1, \dots, \xi_l\}$ and assume that $V_{\xi_1}(g).h = V_{\xi_2}(g)$. Take

$$f = (M_1, M_2, *, \dots, *) \in \mathbf{C}(g|_V) \leq \mathbf{C}_{\text{conf}}(g|_V),$$

where the j th component of f acts on $V_{\xi_j}(g)$ ($j = 1, \dots, l$); then

$$f^h = h^{-1} f h = (*, M_1^h, *, \dots, *).$$

Now there are three cases according to the classification presented in Fact 3. If $G = \text{GL}_{\mathcal{U}}(q)$, we can take $M_2 = 1_{V_{\xi_2}(g)}$ and M_1 far away from $k^\times \text{id}_{V_{\xi_1}(g)}$. Then $[f, h] = (*, M_1^h, *, \dots, *)$ is far away from $k^\times \text{id}_V$. If G is one of $\text{Sp}_{\mathcal{U}}(q)$, $\text{GO}_{\mathcal{U}}(q)$, or $\text{GU}_{\mathcal{U}}(q)$, ξ_1 is not self-dual and $\xi_2 \neq \xi_1^*$, we can do the same as before. When $\xi_1^* = \xi_2$ in this case, we must have $M_2 = (M_1^{-\sigma})^\top$ so that

$$[f, h] = (*, M_1^{\sigma^\top} M_1^h, *, \dots, *).$$

Again we can choose $M_1 \in \prod_{\mathcal{U}} \text{GL}_{c_{i\xi_1}}(K_{\xi_1})$ such that $(M_1^\sigma)^\top M_1^h$ is far away from Z . In the last case, $\xi_1 = \xi_1^*$ is self-dual. Then again M_1 and M_2 are independent of each other, and we can choose $M_2 = 1_{V_{\xi_2}(g)}$. The only restriction

on M_1 is that it lies in $\prod_{\mathcal{U}} \mathrm{GU}_{c_{i\xi_1}}(K_{\xi_1})$ if $\xi_1 \neq X \pm 1$ or G is $\mathrm{GU}_{\mathcal{U}}(q)$ (see Fact 3 (i) (b)) resp. $M_1 \in \prod_{\mathcal{U}} X_{c_{i\xi_1}}(k)$ in the opposite case when $\xi_1 = X \pm 1$, where $G = X_{\mathcal{U}}(q)$ ($X = \mathrm{Sp}$ or GO ; see Fact 3 (ii)), so again we can choose M_1 such that $[f, h] = (*, M_1^h, *, \dots, *)$ is far away from Z . In all cases, we get a contradiction. This shows that $h \in \mathbf{C}_{\mathrm{conf}}^2(g)$ fixes each $V_{\chi}(g) \in \mathcal{V}$ ($\chi \in k[X]$ irreducible).

Assume now that $h|_{V_{\chi}(g)} = M \cdot \alpha$, where α corresponds to an element of

$$T^{l_{\chi}} = \{s^{l_{\chi}} \mid s \in T\}$$

which induces a non-trivial field automorphism on K_{χ} and M is an ultraproduct of matrices over K_{χ} . This is the case since $h \in \mathbf{C}_{\mathrm{conf}}(g)$ and by the above definition of φ_t .

Step 2: the automorphism α equals the identity $\mathrm{id}_{K_{\chi}}$. Then, for $\lambda \in K_{\chi}$, we have $(\lambda \mathrm{id})^h = \lambda^{\alpha} \mathrm{id} = (\lambda^{\alpha} \lambda^{-1}) \lambda \mathrm{id}$. This implies that, for all $\lambda \in K_{\chi}^{\times}$ stabilizing the forms f_i (or Q_i ; $i \in I$) on $V_{\chi}(g)$, we have $\lambda^{\alpha} \lambda^{-1} \in Z \leq k^{\times}$. When $G = \mathrm{GL}_{\mathcal{U}}(q)$ or χ is not self-dual, there is no restriction on λ (of course, if G is one of $\mathrm{Sp}_{\mathcal{U}}(q)$, $\mathrm{GO}_{\mathcal{U}}(q)$, or $\mathrm{GU}_{\mathcal{U}}(q)$, then if h acts as M on $V_{\chi}(g)$, it must act as $(M^{-\sigma})^{\top}$ on $V_{\chi^*}(g)$). Hence, in this case, for each $\lambda^{\times} \in K_{\chi}$, there exists $\kappa_{\lambda} \in k^{\times}$ such that $\lambda^{\alpha} \lambda^{-1} = \kappa_{\lambda}$. However, then every vector $\lambda \in K_{\chi}$ is an eigenvector of the k -linear map α , which forces $\alpha = \mathrm{id}_{K_{\chi}}$, since $1 \in K_{\chi}$ is fixed, a contradiction.

In the opposite case, G is one of $\mathrm{Sp}_{\mathcal{U}}(q)$, $\mathrm{GO}_{\mathcal{U}}(q)$, or $\mathrm{GU}_{\mathcal{U}}(q)$ and χ is self-dual. Then we are in case (i) (b) and (ii) of Fact 3. Recall that $\tau: K_{\chi} \rightarrow K_{\chi}$ is defined by $\tau|_k = \sigma$ and $\tau: \lambda_{\chi} \mapsto \lambda_{\chi}^{-1}$, where $\lambda_{\chi} \in K_{\chi}$ is the root of χ . Then $\tau^2 = \mathrm{id}_{K_{\chi}}$ and $\tau = \mathrm{id}_{K_{\chi}}$ if and only if we are in case (ii) of Fact 3. Here, if we are in case (i) (b) of Fact 3, $\mathbf{C}(g)|_{V_{\chi}(g)}$ is an ultraproduct of unitary groups over the field K_{χ} equipped with the involution τ . In case (ii) of that fact, $\mathbf{C}(g)|_{V_{\chi}(g)}$ is an ultraproduct of symplectic resp. orthogonal groups over $K_{\chi} = k = \mathbb{F}_q$. We proceed as follows. Find totally singular K_{χ} -subspaces

$$U = \overline{(U_i)_{i \in I}}, U' = \overline{(U'_i)_{i \in I}}, U'' = \overline{(U''_i)_{i \in I}} \in \mathcal{V} \quad \text{of } V_{\chi}(g)$$

(in the sense of Remark 8) such that $U \oplus U' = U \oplus U'' = V_{\chi}(g)$, $U' \cap U'' = 0$, and $\dim(U) = \dim(U') = \dim(U'') = \dim(V_{\chi}(g))/2$. We may assume that

$$\dim_{K_{\chi}}(U_i) = \dim_{K_{\chi}}(U'_i) = \dim_{K_{\chi}}(U''_i)$$

and that the restrictions $f_i|_{U_i \times U'_i}$ and $f_i|_{U_i \times U''_i}$ are non-degenerate ($i \in I$, as we may by modifying U_i , U'_i , and U''_i a little if necessary). Then define $f' = \overline{(f'_i)_{i \in I}}$, $f'' = \overline{(f''_i)_{i \in I}} \in \mathbf{C}(g) \leq G$ such that f'_i and f''_i act $F(\varphi)$ -isotypically on U_i and such that f'_i resp. f''_i act $F(\varphi^*)$ -isotypically on U'_i resp. U''_i ($i \in I$) for a fixed

irreducible polynomial $\varphi \in K_\chi[X]$ which is not self-dual with respect to τ . Then $f'^h|_{V_\chi(g)} = z' f'|_{V_\chi(g)}$ and $f''^h|_{V_\chi(g)} = z'' f''|_{V_\chi(g)}$ for $z', z'' \in Z$. Note that

$$\begin{aligned} q_{\varphi.z'^{-1}}(z' f'|_{V_\chi(g)}) &= q_\varphi(f'|_{V_\chi(g)}) = 1/2, \\ q_{\varphi.z''^{-1}}(z'' f''|_{V_\chi(g)}) &= q_\varphi(f''|_{V_\chi(g)}) = 1/2, \end{aligned}$$

and $\varphi.z'^{-1}$ and $\varphi.z''^{-1}$ are both also not self-dual since $\varphi \in K_\chi[X]$ is not self-dual and $z'^{-1}, z''^{-1} \in Z$ so that $z'^{-\tau} = z'^{-\sigma} = z'$ and $z''^{-\tau} = z''^{-\sigma} = z''$, whence, e.g., $(\varphi.z'^{-1})^* = \varphi^*.z'^{-1} \neq \varphi.z'^{-1}$. Then h must stabilize the decompositions $V_\chi(g) = U \oplus U' = U \oplus U''$, so it must stabilize U . But on the h -invariant totally isotropic subspace U , we can do the same argument as above for $G = \text{GL}_{\mathcal{U}}(q)$ to see that $\alpha = \text{id}_{K_\chi}$. Hence we have obtained that $h|_{V_\chi(g)} = M \in \prod_{\mathcal{U}} \mathbf{M}_{c_{i_\chi}}(K_\chi)$ so that $h \in \mathbf{C}(g)$.

Step 3: we have that $h|_{V_\chi(g)} = M = \lambda \text{id}$ for $\lambda \in K_\chi$. According to Fact 3, we can find V_i ($i \in I$) such that $(V_i)_{i \in I} = V_\chi(g)$ such that either all V_i are totally singular (case (i) (a) of that fact, i.e., if χ is not self-dual; this includes the case $G = \text{GL}_{\mathcal{U}}(q)$) or H_i preserves a unitary form (case (i) (b) of Fact 3) or a symplectic or orthogonal form (and a corresponding quadratic form when $p = 2$, case (ii) of Fact 3) over K_χ on V_i ($i \in I$). Note that it follows from the classification in [4, Subsection 3.4.2, § 1] that orthogonally indecomposable blocks involving a Frobenius block of size ≥ 2 are non-central in the ambient projective linear classical group. This shows that $q_\xi(M) = 0$ for all $\xi \in K_\chi[X]$ of degree ≥ 2 . Assume now that there exist distinct $\lambda, \mu \in K_\chi^\times$ such that $q_{X-\lambda}(M), q_{X-\mu}(M) \geq \varepsilon > 0$. If $G = \text{GL}_{\mathcal{U}}(q)$ or we are in case (i) (a) of Fact 3,

$$F(X - \lambda) \oplus F(X - \mu) = \text{diag}(\lambda, \mu) \in \text{GL}_2(K_\chi)$$

is mapped to a non-central element in $\text{PGL}_2(K_\chi)$ so that, by the assumption, since we have “many” of these blocks, $h|_{V_\chi(g)}$ would not commute modulo scalars with all of $\mathbf{C}(g)|_{V_\chi(g)} \cong \prod_{\mathcal{U}} \mathbf{M}_{c_{i_\chi}}(K_\chi)$. In case (i) (b) of Fact 3, we use the same argument for a block of shape $\text{diag}(\lambda, \lambda^{-\tau}, \mu, \mu^{-\tau})$ acting on a four-dimensional (K_χ, τ) -unitary space. In case (ii) of Fact 3, we use the same argument with a block $\text{diag}(\lambda, \lambda^{-1}, \mu, \mu^{-1})$ acting on a four-dimensional symplectic or orthogonal space. In total, we get that $M = \lambda \text{id}$ for $\lambda \in K_\chi$. If we are in case (i) (b) of Fact 3, we have the additional assumption that $N_\tau(\lambda) = 1$, where $N_\tau: K_\chi \rightarrow (K_\chi)_\tau$ is the field norm $x \mapsto x x^\tau$. In case (ii) of Fact 3, we must have $\lambda^2 = 1$.

Step 4: the precise shape of $C := \mathbf{C}_{\text{conf}}^2(g)$. We know now $h|_{V_\chi(g)} = \lambda_\chi(h) \text{id}$ for each irreducible $\chi \in k[X]$, and so h commutes with all of $\mathbf{C}(g)$. In order that $h \in \mathbf{C}_{\text{conf}}^2(g)$, we still need to check that $[h, \varphi_t] = z \text{id}$, where $t \in T$ is the generator fixed above and $z \in Z$. Let $\chi \in k[X]$ run through a system of representatives

of the orbits of the action of T and $\bullet^*$ on the irreducible polynomials (the action of $\bullet^*$ is only used when G is not $\mathrm{GL}\mathcal{U}(q)$). This means $zh = h^{\varphi_t}$, so since $h|_{V_\chi(g)} = \lambda_\chi(h) \mathrm{id}$, we must have $h|_{V_{\chi,t}(g)} = z^{-1} \varphi_{\chi,t}(\lambda_\chi(h)) \mathrm{id}_{V_{\chi,t}}$ so that h is determined on all of $V = V_{\bar{\chi}}(g)$ by $\lambda_\chi(h)$. In this situation, the only condition that needs to be satisfied is that

$$h|_{V_\chi(g)} = h|_{V_{\chi,t^{l_\chi}}(g)} = \lambda_\chi(h) \mathrm{id} = z^{-l_\chi} \varphi_{\chi,t}^{l_\chi}(\lambda_\chi(h)) \mathrm{id}.$$

Note that

$$\varphi_{\chi,t}^{l_\chi}: K_\chi \cong \mathbb{F}_{q^{dk_\chi}} \rightarrow K_\chi \cong \mathbb{F}_{q^{dk_\chi}} \quad (d = 1, 2)$$

is given by $x \mapsto \kappa_\chi x^{q^{dk_\chi/m_\chi}}$, where $\kappa_\chi = 1$ unless $p > 2$, $\varepsilon = -1$, $t = -1$, and $\chi.t = \chi$, in which case $\kappa_\chi = \zeta$ is the scalar chosen above (which depends on χ), so that the previous condition becomes

$$z^{l_\chi} = \kappa_\chi (\lambda_\chi(h))^{q^{dk_\chi/m_\chi} - 1}. \quad (8.2)$$

Hence we can write C as follows. When $G = \mathrm{GL}\mathcal{U}(q)$, we have

$$C = \left\{ h = \bigoplus_{\substack{\chi \text{ irreducible} \\ X^o \equiv \mu(\chi) \\ q_\chi(g) > 0}} \lambda_\chi(h) \mathrm{id}_{V_\chi(g)} \mid \begin{array}{l} \text{there exists } z \in Z \text{ such that} \\ \lambda_{\chi,t}(h) = z^{-1} \varphi_{\chi,t}(\lambda_\chi(h)) \text{ for all } \chi \end{array} \right\}. \quad (8.3)$$

Here the condition from equation (8.3) is equivalent to equation (8.2) for χ running through a system of representatives of the action of T on the set

$$P := \{ \chi \in k[X] \text{ irreducible} \mid \chi \text{ divides } X^o - \mu \text{ and } q_\chi(g) > 0 \}.$$

For G one of $\mathrm{Sp}\mathcal{U}(q)$, $\mathrm{GO}\mathcal{U}(q)$, or $\mathrm{GU}\mathcal{U}(q)$, we have

$$C = \left\{ \bigoplus_{\substack{\chi \text{ irreducible} \\ X^o \equiv \mu(\chi) \\ q_\chi(g) > 0}} \lambda_\chi(h) \mathrm{id}_{V_\chi(g)} \mid R \right\}, \quad (8.4)$$

where condition R is that there exists $z \in Z$ such that $\lambda_{\chi,t}(h) = z^{-1} \varphi_{\chi,t}(\lambda_\chi(h))$ for all $\chi \in P$ (as in the previous case) and $\lambda_\chi(h) \lambda_{\chi^*}(h)^\tau = 1$ for all $\chi \in P$, where $\tau: K_{\chi^*} \rightarrow K_\chi$ is defined as in Fact 3. If G is one of $\mathrm{Sp}\mathcal{U}(q)$ or $\mathrm{GO}\mathcal{U}(q)$ and $\chi = \chi^* \neq X \pm 1$ is self-dual, this means k_χ is even and $\lambda_\chi(h)^{q^{k_\chi/2} + 1} = 1$. Also, in this case, if $\chi = X \pm 1$, it means $\lambda_\chi(h)^2 = 1$. If $G = \mathrm{GU}\mathcal{U}(q)$ and $\chi = \chi^*$, this means that k_χ is odd (since τ needs to induce $\sigma: x \mapsto x^q$ on $k = \mathbb{F}_{q^2}$) and $\lambda_\chi(h)^{q^{k_\chi} + 1} = 1$.

9 Distinction of metric ultraproducts

Now we want to distinguish all (simple) metric ultraproducts $\overline{G} = \overline{X}_{\mathcal{U}}(q)$ for distinct pairs (X, q) , where $X \in \{\text{GL}, \text{Sp}, \text{GO}, \text{GU}\}$ and q is a prime power (all but $\text{PSp}_{\mathcal{U}_1}(q)$ and $\text{PGO}_{\mathcal{U}_2}(q)$ as mentioned in Theorem 1). For a group H , define the quantity

$$e_H(o) := \max_{h \in H: h^o = 1_H} \exp(\mathbf{C}^2(h)).$$

Clearly, when $H \cong L$, we have

$$e_H(o) = e_L(o) \quad \text{for all values } o \in \mathbb{Z}_+.$$

Our strategy is to compute $e_H(o)$ for the groups $H = \overline{G}$, where $\overline{G} = \overline{X}_{\mathcal{U}}(q)$ as above, for certain values of o to distinguish these groups (with the only exception $\text{PSp}_{\mathcal{U}_1}(q) \cong \text{PGO}_{\mathcal{U}_2}(q)$?).

9.1 Computation of $e_{\overline{G}}(o)$ when $\gcd\{o, p\} = \gcd\{o, |Z|\} = 1$

If o is coprime to $|Z|$ (and by semisimplicity of $g \in G$ coprime to p), from the previous considerations, we can compute $e_{\overline{G}}(o)$. Note that, in this situation, when $g^o = \mu \in Z$, we can replace g by $g' = \lambda g \in G$ such that $g'^o = 1$, choosing $\lambda \in Z$ such that $\lambda^o = \mu^{-1}$, since the homomorphism $Z \rightarrow Z, x \mapsto x^o$ is then bijective. So we may assume that $g^o = 1$. Then

$$P \subseteq Q := \{\chi \in k[X] \text{ irreducible} \mid \chi \text{ divides } X^o - 1\}.$$

The case $G = \text{GL}_{\mathcal{U}}(q)$. From equation (8.3), we see that the bigger the group T is, for an element $h \in \mathbf{C}_{\text{conf}}^2(g)$, the more restrictions are imposed to the scalars $\lambda_{\chi}(h) \in K_{\chi}^{\times}$ ($\chi \in P$). Also the bigger the set P is, the “bigger” is the group $\mathbf{C}_{\text{conf}}^2(g)$, i.e., there are more components. Hence, to optimize the exponent of $\mathbf{C}^2(\overline{g}) = \mathbf{C}_{\text{conf}}^2(g)/Z$, we choose g such that

$$P = Q \quad \text{and} \quad 0 < q_{X-1}(g) \neq q_{\chi}(g) > 0 \quad \text{for all } \chi \in P \setminus \{X-1\}.$$

Namely, then $T = \text{stab}_Z(q(g))$ must fix the polynomial $X-1$ so that we must have $T = \mathbf{1}$. Set

$$f_q(o) := \min\{q^e - 1 \mid o \text{ divides } q^e - 1\}.$$

Equation (8.3) then gives

$$e_{\overline{G}}(o) = \exp(\mathbf{C}^2(\overline{g})) = \exp(\mathbf{C}_{\text{conf}}^2(g)/Z) = \begin{cases} 1 & \text{if } o = 1, \\ f_q(o) & \text{if } o > 1. \end{cases} \quad (9.1)$$

Let us demonstrate equation (9.1). The first equality in it holds by the previous argument. When $o = 1$, we have $\bar{g} = 1\bar{G}$, and so

$$\mathbf{C}^2(\bar{g}) = \mathbf{Z}(\bar{G}) = \mathbf{1}$$

so that $e\bar{G}(1) = 1$. Now assume $o > 1$. For each $\chi \in P$, if $\lambda \in \bar{k}^\times$ is a root of χ , the condition $\chi \mid X^o - 1$ is equivalent to $\lambda^o = 1$. Also $K_\chi = k[\lambda]$. Let $\mu \in \bar{k}^\times$ be an element of order o with minimal polynomial $\xi \in k[X]$. Then, if λ is a root of $\chi \in P$, we must have $\lambda^o = 1$ and thus $\lambda = \mu^f$ for some $f \in \mathbb{N}$. Hence

$$K_\chi = k[\lambda] = k[\mu^f] \subseteq k[\mu] = K_\xi$$

so that, in equation (8.3), $\text{ord}(\lambda_\chi(h))$ divides $|K_\chi^\times|$ which divides $|K_\xi^\times| = f_q(o)$. This shows that

$$\begin{aligned} \exp(\mathbf{C}^2(\bar{g})) \mid \exp(\mathbf{C}_{\text{conf}}^2(g)) \mid \text{lcm}\{|K_\chi^\times| \mid \chi \text{ divides } X^o - 1\} \\ = |K_\xi^\times| = |k[\mu]^\times| = f_q(o). \end{aligned}$$

To show that $\exp(\mathbf{C}^2(\bar{g})) = f_q(o)$, take $h \in \mathbf{C}_{\text{conf}}^2(g)$ such that $\lambda_{X-1}(h) = 1$ and $\lambda_\xi(h)$ has order $f_q(o) = |K_\xi^\times|$ in $\bar{k}^\times$. Then, when $h^l = 1\bar{G}$, we must have $h^l \in Z$. But $\lambda_{X-1}(h)^l = 1$ so that, since $q_{X-1}(g) > 0$, it follows that $h^l = 1_G$. Then $\lambda_\xi(h)^l = 1$ so that $\exp(\mathbf{C}^2(\bar{g})) \geq l \geq \text{ord}(\lambda_\xi(h)) = f_q(o)$. This completes the proof.

The case $G = \text{Sp}_{\mathcal{U}}(q)$ or $\text{GO}_{\mathcal{U}}(q)$. As in the linear case, equation (8.4) shows that the optimal exponent of $\mathbf{C}^2(\bar{g})$ is obtained when

$$P = Q \quad \text{and} \quad 0 < q_{X-1}(g) \neq q_X(g) > 0 \quad \text{for all } \chi \in P \setminus \{X-1\}$$

so that $T = \mathbf{1}$. Set

$$f'_q(o) := \begin{cases} q^{e/2} + 1 & \text{if } f_q(o) = q^e - 1, \text{ } e \text{ is even, and } o \mid q^{e/2} + 1, \\ f_q(o) & \text{otherwise.} \end{cases} \quad (9.2)$$

Equation (8.4) then gives

$$e\bar{G}(o) = \exp(\mathbf{C}^2(\bar{g})) = \exp(\mathbf{C}_{\text{conf}}^2(g)/Z) = \begin{cases} 1 & \text{if } o = 1, \\ 2 & \text{if } o = 2, \\ f'_q(o) & \text{if } o > 2. \end{cases} \quad (9.3)$$

We demonstrate equation (9.3). If $o = 1$, we obtain, as in the linear case, that $\mathbf{C}^2(\bar{g}) = \mathbf{1}$, and so $e\bar{G}(1) = 1$. If $o = 2$, $g^2 = 1$, and so $P = \{X-1, X+1\}$. From equation (8.4), we see that, if $h \in \mathbf{C}_{\text{conf}}^2(g)$, we have

$$\lambda_{X-1}(h)^2 = \lambda_{X+1}(h)^2 = 1$$

so that $h^2 = 1$. Also, defining h by $\lambda_{X-1}(h) := 1$ and $\lambda_{X+1}(h) = -1$, we obtain $h \notin Z$, so $\text{ord}_{\bar{G}}(\bar{h}) = e_{\bar{G}}(2) = 2(1 \neq -1$ since the case $p = 2$ does not occur due to the condition $\gcd\{o, p\} = 1$). Assume now that $o > 2$. As in the linear case, for each $\chi \in P$, if $\lambda \in \bar{k}$ is a root of χ , the condition that $\chi \mid X^o - 1$ is equivalent to $\lambda^o = 1$. Choose $\mu \in \bar{k}^\times$ of order o , and let $\xi \in P$ be its minimal polynomial. Then, as previously, if λ is a root of $\chi \in P$, we have $\lambda = \mu^f$ for some $f \in \mathbb{N}$. There are two cases.

In the first case, ξ is not self-dual. This means that μ and μ^{-1} are not conjugate in K_ξ/k . If they were conjugate, say by an automorphism α , i.e., $\mu^\alpha = \mu^{-1}$, then $\alpha = \tau \in \text{Gal}(K_\xi/k)$ needs to be the unique involution (since $\mu \neq \mu^{-1}$ as $o > 2$) given by $x \mapsto x^{q^{k_\xi/2}}$; in particular, $e = k_\xi$ would need to be even. Hence this case is equivalent to either $e = k_\xi$ being odd or $\mu\mu^\alpha = \mu^{q^{k_\xi/2}+1} \neq 1$, i.e., $o \nmid q^{e/2} + 1 = q^{k_\xi/2} + 1$. This is precisely the opposite of the first case in equation (9.2). Here, for an element $h \in \mathbf{C}_{\text{conf}}^2(g)$, we can choose $\lambda_\xi(h) \in K_\xi^\times = k[\mu]^\times$ arbitrarily ($\lambda_{\xi^*}(h)$ is then determined by $\lambda_\xi(h)$). Arguing as in the linear case, we obtain $\exp(\mathbf{C}^2(\bar{g})) = f_q(o)$. Indeed, for $h \in \mathbf{C}_{\text{conf}}^2(g)$, as above, $\text{ord}(\lambda_\chi(h))$ divides $f_q(o)$, and defining h such that $\lambda_{X-1}(h) = 1$ and $\lambda_\xi(h)$ has order $f_q(o)$, we see that $\text{ord}_{\bar{G}}(\bar{h}) = e_{\bar{G}}(o) = f_q(o)$.

In the opposite case, ξ is self-dual and $\xi \neq X \pm 1$ as $o > 2$. Then $e = k_\xi$ needs to be even and

$$\mu\mu^\tau = \mu^{q^{k_\xi/2}+1} = \lambda_\xi(h)\lambda_\xi(h)^\tau = \lambda_\xi(h)^{q^{k_\xi/2}+1} = 1,$$

where τ is the involution $x \mapsto x^{q^{k_\xi/2}}$ of $K_\xi \cong \mathbb{F}_{q^{k_\xi}}$ from Fact 3 (i) (b). This means that $o \mid q^{e/2} + 1$ and we are in the first case of equation (9.2). Note that, for each $\chi \in P$, the map τ restricts to an automorphism of each $K_\chi \subseteq K_\xi$ of order dividing two (as all the fields are finite). Then $\tau|_{K_\chi} = \text{id}$ if and only if $k_\chi \mid k_\xi/2$, and $\tau|_{K_\chi}$ is the unique involution of K_χ if k_ξ/k_χ is odd. Now, if $\lambda \in \bar{k}^\times$ is a root of χ , then in the first case $\lambda^2 = 1$, and in the second case $\lambda^{q^{k_\chi/2}+1} = 1$, so all $\chi \in P$ are self-dual. Hence, if $h \in \mathbf{C}_{\text{conf}}^2(g)$, for each $\chi \in P$, one of $\lambda_\chi(h)^2 = 1$ or $\lambda_\chi(h)^{q^{k_\chi/2}+1} = 1$ must hold. But $2 \mid q^{k_\xi/2} + 1$ if $p > 2$, and in the second case, $q^{k_\chi/2} + 1 \mid q^{k_\xi/2} + 1$ since k_ξ/k_χ is then odd. Hence

$$\exp(\mathbf{C}_{\text{conf}}^2(g)) \mid q^{e/2} + 1 = q^{k_\xi/2} + 1 = f'_q(o).$$

Defining $h \in \mathbf{C}_{\text{conf}}^2(g)$ such that $\lambda_{X-1}(h) = 1$ and $\lambda_\xi(h)$ has order $f'_q(o)$, we see that $\text{ord}_{\bar{G}}(\bar{h}) = f'_q(o)$.

The case $G = \text{GU}_{\mathcal{U}}(q)$. Here, as well, equation (8.4) shows that the optimal exponent of $\mathbf{C}^2(\bar{g})$ is obtained when $P = Q$ and $0 < q_{X-1}(g) \neq q_\chi(g) > 0$ for all $\chi \in P \setminus \{X - 1\}$ so that $T = \mathbf{1}$. Set

$$f''_q(o) := \begin{cases} q^e + 1 & \text{if } f_{q^2}(o) = q^{2e} - 1, e \text{ is odd, and } o \mid q^e + 1, \\ f_{q^2}(o) & \text{otherwise.} \end{cases}$$

Equation (8.4) gives

$$e_{\overline{G}}(o) = \exp(\mathbf{C}^2(\overline{g})) = \exp(\mathbf{C}_{\text{conf}}^2(g)/Z) = \begin{cases} 1 & \text{if } o = 1, \\ f_q''(o) & \text{if } o > 1. \end{cases} \quad (9.4)$$

Again $e_{\overline{G}}(1) = 1$ is clear. If $o > 1$, take $\mu \in \overline{k}^\times$ of order o with minimal polynomial ξ . Then the argument proceeds as in the bilinear case. But the condition that ξ is self-dual is here equivalent to μ being conjugate to μ^{-1} in K_ξ by an automorphism τ such that $\tau|_k = \sigma$; $x \mapsto x^q$. This forces $e = k_\xi$ to be odd and $\mu^{q^{d k_\xi/2} + 1} = 1$, i.e., $o \mid q^e + 1 = q^{k_\xi} + 1$.

9.2 Proof of Theorem 1

Set $G_j := X_j \mathfrak{u}_j(q_j)$, $Z_j := \mathbf{Z}(G_j)$, and $\overline{G}_j := G_j/Z_j$ ($j = 1, 2$). Let p_j be the characteristic of the field $\mathbb{F}_{q_j}$ ($j = 1, 2$). Assume that $\overline{G} \cong \overline{G}_1 \cong \overline{G}_2$. We start by showing that $p_1 = p_2$.

Determining the characteristic p . Choose o large enough and coprime to p_1 , p_2 , $|Z_1|$, $|Z_2|$. Then from equations (9.1), (9.3), and (9.4), we see that $e_{\overline{G}}(o)$ is of the form $q_1^{e_1} \pm 1$ and $q_2^{e_2} \pm 1$. If we have $e_{\overline{G}}(o) = q_1^{e_1} - 1 = q_2^{e_2} - 1$ or $e_{\overline{G}}(o) = q_1^{e_1} + 1 = q_2^{e_2} + 1$, then $q_1^{e_1} = q_2^{e_2}$ so that $p_1 = p_2$ by the uniqueness of the prime factorization. So we may assume that $e_{\overline{G}}(o) = q^{e_1} - 1 = q_2^{e_2} + 1$ for infinitely many o , and so for infinitely many pairs $(e_1, e_2) \in \mathbb{Z}_+^2$. If $p_1 \neq p_2$, we get a contradiction to [1, Corollary 1.8]. Hence $p_1 = p_2 =: p$.

Determining q^d . We can now assume $q_j = p^{e_j}$ ($j = 1, 2$). Choose $j \in \{1, 2\}$, and set $X := X_j$, $q := q_j$, and $d := d_j$. Consider the quantity

$$f := \gcd\{e_{\overline{G}}(o) \mid o \in O\},$$

where

$$O := \{o \in \mathbb{Z}_+ \mid 2 < o \text{ coprime to } p, |Z_1|, |Z_2|; e_{\overline{G}}(o) \equiv -1 \text{ modulo } p^3\}$$

From equations (9.1), (9.3), and (9.4), it follows that, for every element $o \in O$, the number $e_{\overline{G}}(o)$ is either of the form $q^{de} - 1$ or $q^e + 1$. But the second case is excluded by the condition that $e_{\overline{G}}(o) \equiv -1$ modulo p^3 . Hence $q^d - 1$ divides $e_{\overline{G}}(o) = q^{de} - 1$, and so $q^d - 1$ divides f .

For a prime r , set $t_r := \frac{q^r - 1}{q - 1}$. Then, for distinct primes r and s , we have

$$\begin{aligned} \gcd\{t_r, t_s\} &= \gcd\left\{\frac{q^r - 1}{q - 1}, \frac{q^s - 1}{q - 1}\right\} \\ &= \frac{1}{q - 1} \gcd\{q^r - 1, q^s - 1\} = \frac{q^{\gcd\{r, s\}} - 1}{q - 1} = 1. \end{aligned}$$

Hence the numbers t_r (r prime), being pairwise coprime, have arbitrarily large prime divisors. Take for $r > 2$ a prime such that t_r has a prime divisor $o > p$, $|Z_1|$, $|Z_2|$, $q^d - 1$. Then o is coprime to p , $|Z_1|$, and $|Z_2|$ so that, by equations (9.1), (9.3), and (9.4), we have $e_{\overline{G}}(o) \mid f_{q^d}(o) \mid q^{dr} - 1$, as $o \mid q^r - 1 \mid q^{dr} - 1$. Hence the number $f_{q^d}(o)$ must be one of $q^{dr} - 1$ or $q^d - 1$, the latter being excluded by the condition $o > q^d - 1$, so $f_{q^d}(o) = q^{dr} - 1$. If $X = \text{GL}$, $X = \text{Sp}$, or $X = \text{GO}$, since r is odd and $d = 1$, equations (9.1) and (9.3) show that we must have $e_{\overline{G}}(o) = f_{q^d}(o) = q^{dr} - 1 = q^r - 1 \equiv -1$ modulo p^3 . Hence, in this case, we have $o \in O$. If $X = \text{GU}$, it could be that $e_{\overline{G}}(o) = q^r + 1$ when $o \mid q^r + 1$. However,

$$\gcd\{q^r + 1, t_r\} \mid \gcd\{q^r + 1, q^r - 1\} \mid 2,$$

and t_r is always odd so that $\gcd\{q^r + 1, t_r\} = 1$, and hence also $\gcd\{q^r + 1, o\} = 1$ as $o \mid t_r$. This shows that here also $e_{\overline{G}}(o) = f_{q^d}(o) = q^{dr} - 1 = q^{2r} - 1 \equiv -1$ modulo p^3 . Therefore, again $o \in O$.

Applying this argument for two different primes r , say r_1 and r_2 , which produces two different primes o , say o_1 and o_2 , we get

$$\begin{aligned} f &= \gcd\{e_{\overline{G}}(o) \mid o \in O\} \mid \gcd\{e_{\overline{G}}(o_1), e_{\overline{G}}(o_2)\} \\ &= \gcd\{q^{dr_1} - 1, q^{dr_2} - 1\} = q^{\gcd\{dr_1, dr_2\}} - 1 = q^d - 1. \end{aligned}$$

Altogether, we have shown that $f = q^d - 1$. Plugging in $j = 1, 2$, we obtain $q_1^{d_1} - 1 = q_2^{d_2} - 1$ implying that $q_1^{d_1} = q_2^{d_2}$.

Now exclude all remaining possible isomorphisms but $\text{PSp}_{u_1}(q) \cong \text{PGO}_{u_2}(q)$.

Proof that $\text{PGL}_{u_1}(q) \not\cong \text{PSp}_{u_2}(q)$ and $\text{PGL}_{u_1}(q) \not\cong \text{PGO}_{u_2}(q)$. Let

$$G_1 = \text{GL}_{u_1}(q) \quad \text{and} \quad G_2 = X_{u_2}(q), \quad \text{where } X = \text{Sp} \text{ or } \text{GO}.$$

Set

$$o := \begin{cases} \frac{q^2+1}{2} & \text{if } p > 2, \\ q^2 + 1 & \text{if } p = 2. \end{cases} \quad (9.5)$$

Note that $o > 2$ is coprime to p , $|Z_1| = q - 1$, and $|Z_2| = |\{\pm 1\}|$. Hence, by equation (9.3), we have $e_{\overline{G}}(o) = e_{\overline{G}_2}(o) = q^2 + 1$. Indeed, $o \mid q^2 + 1 \mid q^4 - 1$. But $o \nmid q^f - 1$ for f properly dividing 4 since then $o \mid q^2 - 1$, but it is easy to see that $\gcd\{o, q^2 - 1\} = 1$. This shows

$$f_q(o) = q^4 - 1 \quad \text{and} \quad e_{\overline{G}}(o) = e_{\overline{G}_2}(o) = f'_q(o) = q^2 + 1.$$

But then, by $\overline{G}_1 \cong \overline{G}_2$, we obtain

$$e_{\overline{G}}(o) = e_{\overline{G}_1}(o) = f_q(o) = q^4 - 1 > q^2 + 1 = e_{\overline{G}_2}(o) = e_{\overline{G}}(o),$$

a contradiction.

Proof that $\mathrm{PSp}_{\mathcal{U}_1}(q^2) \not\cong \mathrm{PGU}_{\mathcal{U}_2}(q)$ and $\mathrm{PGO}_{\mathcal{U}_1}(q^2) \not\cong \mathrm{PGU}_{\mathcal{U}_2}(q)$. Let

$$G_1 = X_{\mathcal{U}_1}(q^2), \quad \text{where } X = \mathrm{Sp} \text{ or } \mathrm{GO}, \quad \text{and} \quad G_2 = \mathrm{GU}_{\mathcal{U}_2}(q).$$

Define o as in equation (9.5). Note that $o > 2$ is coprime to p , $|Z_1| = |\{\pm 1\}|$, and $|Z_2| = q + 1$. Then, by equation (9.3), $e_{\overline{G}}(o) = e_{\overline{G}_1}(o) = f'_{q^2}(o) = q^2 + 1$ (as above). But by equation (9.4), we obtain that +

$$e_{\overline{G}} = e_{\overline{G}_2}(o) = f''_q(o) = q^4 - 1 > q^2 + 1 = e_{\overline{G}_1}(o) = e_{\overline{G}}(o)$$

since $e = 2$ is even, a contradiction.

Proof that $\mathrm{PGL}_{\mathcal{U}_1}(q^2) \not\cong \mathrm{PGU}_{\mathcal{U}_2}(q)$. Let $G_1 = \mathrm{GL}_{\mathcal{U}_1}(q^2)$, $G_2 = \mathrm{GU}_{\mathcal{U}_2}(q)$. Set

$$o := \begin{cases} \frac{q^5+1}{5(q+1)} & \text{if } q \equiv -1 \text{ modulo } 5, \\ \frac{q^5+1}{q+1} & \text{otherwise.} \end{cases}$$

Note that o is coprime to p , $|Z_1| = q^2 - 1$, and $|Z_2| = q + 1 \mid q^2 - 1$. Indeed,

$$\gcd\{o, q + 1\} \mid \gcd\left\{\frac{q^5 + 1}{q + 1}, q + 1\right\} = \gcd\{5, q + 1\} \mid 5.$$

But $5 \nmid o$ so that $\gcd\{o, q + 1\} = 1$. Similarly,

$$\gcd\{o, q - 1\} \mid \gcd\{q^5 + 1, q - 1\} \mid \gcd\{2, q - 1\} \mid 2.$$

But o is always odd, so $\gcd\{o, q - 1\} = 1$. We have $o \mid q^{10} - 1$ so that, from equation (9.1), we obtain that $e_{\overline{G}}(o) = e_{\overline{G}_1}(o) = f_{q^2}(o)$ is either $q^{10} - 1$ or $q^2 - 1$. But clearly $q^2 - 1 < o$ so that we must have $f_{q^2}(o) = q^{10} - 1$. But equation (9.4) gives that

$$e_{\overline{G}}(o) = e_{\overline{G}_2}(o) = f''_q(o) = q^5 + 1 < q^{10} - 1 = f_q(o) = e_{\overline{G}_1}(o),$$

a contradiction.

Remark 13. If $q_i \rightarrow_{\mathcal{U}} \infty$, then double centralizers of semisimple torsion elements are infinite groups.

Remark 14. If q is even, then $\mathrm{PSp}_{\mathcal{U}_1}(q) \cong \mathrm{PGO}_{\mathcal{U}_2}(q)$ is possible due to the isomorphism $\mathrm{Sp}_{2m}(q) \cong \mathrm{GO}_{2m+1}(q)$. Also it seems hard to distinguish a group $\mathrm{PSp}_{\mathcal{U}_1}(q)$ from a group $\mathrm{PGO}_{\mathcal{U}_2}(q)$ for q odd.

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