Short Communication

George Petsos and Elias C. Aifantis*

Some remarks on deformation, localization, and front propagation

Keywords: gradient theory; propagation front; strain localization.

*Corresponding author: Elias C. Aifantis, Laboratory of Mechanics and Materials, Aristotle University, Thessaloniki 54124, Greece, e-mail: mom@mom.gen.auth.gr

Elias C. Aifantis: College of Engineering, Michigan Tech, Houghton, MI 49931, USA; and King Abdulaziz University, Jeddah 21589, Saudi Arabia

George Petsos: Laboratory of Mechanics and Materials, Aristotle University, Thessaloniki 54124, Greece

This note provides some analytical results concerning the localization of deformation in one-dimensional solids. It supplements earlier ones obtained by the second author and his coworkers in the past [1–4]. These previous results were mainly concerned with stationary shear bands emerging when the material enters the unstable (negative slope) regime of its (nonmonotonous) stress-strain relation. Such nonmonotonous constitutive equations were generalized with the Laplacian of strain to stabilize the solution of the nonlinear equation obtained by further using the equation of stress equilibrium. The present results are derived within a more general framework based on the minimization of a gradient-dependent nonconvex free energy, but also taking into account viscosity and inertia. At the same time, they may also be viewed as being restricted by the assumption of the existence of free energy in far-from-thermodynamic-equilibrium conditions and the validity of the variational principles used. This assumption was not used in the mechanical theory of fluid interfaces [5, 6], which was the motivation for the approach adopted in [1–4].

We begin by defining kinetic (T), free (F), and dissipation (R) energies per unit length as follows

$$T = \frac{1}{2} \rho \int_{-L/2}^{L/2} \left(\frac{\partial u}{\partial t} \right)^2 dx; \quad F = \int_{-L/2}^{L/2} \{ f_L[\gamma] + f_{NL} \} dx; \\ R = \frac{1}{2} \lambda \int_{-L/2}^{L/2} \left(\frac{\partial \gamma}{\partial t} \right)^2 dx, \quad (1)$$

where L denotes the specimen thickness in the x -direction ($-L/2 < x < L/2$) loaded with a shear stress τ_0 in the

perpendicular y -direction. The field variables $u(x, t)$ and $\gamma(x, t) = \partial u / \partial x$ denote displacement and strain, respectively, whereas ρ and t denote mass density and time, respectively. The nonmonotonous function $\kappa(\gamma)$ denotes the homogeneous part of the stress, whereas the homogeneous part of the local free energy density functional $f_L[\gamma]$ and its inhomogeneous gradient-dependent (nonlocal) counterpart f_{NL} are given by

$$f_L[\gamma] = \int_0^\gamma \kappa(\gamma) d\gamma - \tau_0 \gamma, \quad f_{NL} = l^2 \left(\frac{\partial \gamma}{\partial x} \right)^2. \quad (2)$$

The quantity l is the gradient coefficient identified with an internal length, and λ denotes the viscosity coefficient (the Rayleigh dissipation constant).

Following [7, 8], we postulate the variational principle $\delta L / \delta u = \delta R / \delta \dot{u}$, where $L = T - F$ is the Lagrangian of the system. As a result, we obtain the differential equation of dynamic equilibrium in its usual form, i.e., $\rho \partial^2 u / \partial t^2 = \partial \tau / \partial x$ and the corresponding constitutive equation, i.e., $\tau(x, t) = \kappa(\gamma) - 2l^2 \partial^2 \gamma / \partial x^2 + \lambda \partial \gamma / \partial t$. By combining the last two equations, we arrive at the following governing partial differential equation for the displacement

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{d\kappa}{d\gamma} \frac{\partial^2 u}{\partial x^2} - 2l^2 \frac{\partial^4 u}{\partial x^4} + \lambda \frac{\partial^3 u}{\partial t \partial x^2}, \quad (3)$$

and a similar one for the strain derived by differentiating Eq. (3) with respect to the spatial coordinate x , i.e.,

$$\rho \frac{\partial^2 \gamma}{\partial t^2} = \frac{d\kappa}{d\gamma} \frac{\partial^2 \gamma}{\partial x^2} + \frac{d^2 \kappa}{d\gamma^2} \left(\frac{\partial \gamma}{\partial x} \right)^2 - 2l^2 \frac{\partial^4 \gamma}{\partial x^4} + \lambda \frac{\partial^3 \gamma}{\partial t \partial x^2}. \quad (4)$$

Equations (3) and (4) are nonlinear wave equation with dissipation for the displacement u strain γ . The linear sound velocity v_s is given by $v_s = \pm \sqrt{\kappa'(0) / \rho}$, where $\kappa'(0)$ denotes the Young modulus.

To obtain the corresponding boundary conditions, we use the energy relation [7–9] $\partial E / \partial t = -2R$, where $E = T + F$ is the mechanical energy of the system. The result is

$$\frac{\partial u}{\partial t} \bigg|_{x=-L/2} \left[\tau \left(\frac{L}{2}, t \right) - \tau_0 \right] = \frac{\partial u}{\partial t} \bigg|_{x=L/2} \left[\tau \left(\frac{L}{2}, t \right) - \tau_0 \right] = 0, \quad (5)$$

and

$$2l^2 \frac{\partial^2 u}{\partial t \partial x} \bigg|_{x=L/2} \frac{\partial \gamma}{\partial x} \bigg|_{x=L/2} = 2l^2 \frac{\partial^2 u}{\partial t \partial x} \bigg|_{x=L/2} \frac{\partial \gamma}{\partial x} \bigg|_{x=L/2} = 0. \quad (6)$$

By confining attention to a soft device, it follows from the above that the appropriate boundary conditions are

$$\tau \left(\pm \frac{L}{2}, t \right) = \tau_0; \quad \frac{\partial \gamma}{\partial x} \bigg|_{x=\pm L/2} = 0. \quad (7)$$

By differentiating Eq. (2)₁ with respect to γ , we find that

$$\frac{df_L}{d\gamma} = \kappa(\gamma) - \tau_0. \quad (8)$$

On searching for traveling wave solutions of the form $\xi = x + vt$, where v denotes the speed of the front, we can easily deduce that the stress $\tau(\gamma)$ is given by

$$\tau(\gamma) = \tau_0 + \rho v^2 (\gamma - \gamma_1), \quad (9)$$

where γ_1 is the strain at the boundaries where τ_0 is applied. From Eq. (9), we can see that for the dynamic case $d\tau/d\gamma = \rho v^2 > 0$. Moreover, for stationary states ($v=0$), we have $\tau(\gamma) = \tau_0$ and $d\tau/d\gamma = 0$. Using the relation $\tau(x, t) = \kappa(\gamma) - 2l^2 \partial^2 \gamma / \partial x^2 + \lambda \partial \gamma / \partial t$ and Eq. (9), it follows that the strain γ is obtained from the solution of the nonlinear differential equation

$$-2l^2 \gamma'' + \lambda v \gamma' = \tau_0 - \kappa(\gamma) + \rho v^2 (\gamma - \gamma_1), \quad (10)$$

with prime denoting differentiation with respect to ξ .

To proceed further, we assume a specific form of $\kappa(\gamma)$, which is appropriate for polymers. This form reads $\kappa(\gamma) = A\gamma^3 + B\gamma^2 + \Gamma\gamma$ ($A > 0$, $B < 0$, $\Gamma > 0$), and the material parameters A , B , Γ must satisfy the condition $3 < (B^2/A\Gamma) < 4$ for nonmonotonicity [10, 11]. The corresponding form for $f_L[\gamma]$ is given by $f_L[\gamma] = (1/4)A\gamma^4 + (1/3)B\gamma^3 + (1/2)\Gamma\gamma^2 - \tau_0\gamma$, and the graphs for both $\kappa(\gamma)$ and $f_L[\gamma]$ are given in Figure 1.

Strain localization, i.e., a stationary reversal solution [5, 6] is possible when $\tau_0 < \tau_{\max}$, where $\tau_{\max} = [2B^3 - 9AB\Gamma + 2(B^2 - 3A\Gamma)^{3/2}]/(27A^2)$ is the maximum value of $\kappa(\gamma)$. Also, for $\tau_0 = \tau_d = (2B^3 - 9AB\Gamma)/(27A^2)$, we have $\int_{\gamma_1}^{\gamma_3} [\kappa(\gamma) - \tau_d] d\gamma = 0$, and the stationary solution is a transition. For $\tau_0 > \tau_{\max}$, the energy functional $f_L[\gamma]$ has only one global minimum at $\gamma = \gamma_3$, whereas for $\tau_0 = \tau_d$, we have $f_L[\gamma_1] = f_L[\gamma_3]$.

From the governing equation, i.e., Eq. (10), and the boundary conditions, i.e., Eq. (7) for an infinite system, it is easily seen [5, 6, 10–12] that the following equal “equal area or Maxwell’s rule” conditions should hold true for a solution to exist:

– For stationary states ($v=0$), we have

$$\int_{\gamma_1}^{\gamma_{\max}^{ss}} [\kappa(\gamma) - \tau_0] d\gamma = 0, \quad (11)$$

where $\gamma_{\max}^{ss}$ is the maximum value of the strain of the stationary (steady state, or ss) reversal solution.

– For dynamic states without dissipation ($\lambda=0$), we have

$$\int_{\gamma_1}^{\gamma_{\max}^d} [\kappa(\gamma) - \tau(\gamma)] d\gamma = 0, \quad (12)$$

where $\gamma_{\max}^d$ is the maximum value of the strain of the dynamic reversal solution. Also, we have $\gamma_{\max}^{ss} > \gamma_{\max}^d$.

It is noted that Eq. (11) suggests that the straight line τ_0 cuts equal areas above and below with the graph $\kappa(\gamma)$ as defined by γ_1 and $\gamma_{\max}^{ss}$. This is not shown in Figure 1A to avoid complication. The same holds for the dynamic case without dissipation, Eq. (12), where the straight line τ_0 (parallel to the γ -axis) is replaced by the straight line $\tau(\gamma)$ (with slope ρv^2 with respect to the γ -axis), while $\gamma_{\max}^{ss}$ is replaced by $\gamma_{\max}^d$. In this connection, it is pointed out that dynamic transition solutions without dissipation are possible. This may be inferred from Figure 1A and an equal area construction determined by $\tau(\gamma_1) = \tau_0$ and $\tau(\bar{\gamma}_3)$.

Figure 1 (A) Double-loop constitutive relation for $\kappa(\gamma)$ and stress $\tau(\gamma)$ for the dynamic case. (B) Double-well local free energy density functional $f_L[\gamma]$.

instead of $\tau(\gamma_3) = \tau_0$ in the stationary case. In the aforementioned equal area rule construction for the dynamic case, the velocity of the propagation front is uniquely determined by the (different) values of the two stresses at the specimen ends. These remarks are in accordance with the conclusions of [10–12], where it is also shown that in the dynamic case with dissipation, only transition solutions exist [12].

We consider the following analytical solutions of Eq. (10).

(i) *Stationary solutions without dissipation* ($\lambda = 0$): They are obtained by following the method of Aifantis and Serrin [5, 6] for “reversal” solutions in fluid microstructures (see also [1–4, 12]) for the equation

$$2l^2 \frac{d^2 \gamma}{dx^2} = \kappa(\gamma) - \tau_0. \quad (13)$$

They read

$$\gamma(x) = \frac{M(\gamma_1) + \Delta Z - \sqrt{Z^2 - \Lambda(1 - \Delta^2) + M(\gamma_1)[M(\gamma_1) + 2\Delta Z]}}{1 - \Delta^2} \quad (14)$$

where

$$\begin{aligned} M(\gamma_1) &= -\gamma_1 \cdot \frac{2B}{3A}; \quad K(\gamma_1) = -2\gamma_1^2 \cdot \frac{4B}{3A} \gamma_1 + \frac{4B^2}{9A^2} \cdot \frac{2\Gamma}{A}; \quad \Lambda = [M(\gamma_1)]^2 \cdot K(\gamma_1); \quad g(\gamma_1) = \gamma_1 \cdot M(\gamma_1) \\ f(\gamma_1) &= \sqrt{[\gamma_1 \cdot M(\gamma_1)]^2 \cdot K(\gamma_1)}; \quad Q(\gamma_1) = \frac{2l}{\sqrt{A}}; \quad H(\gamma_{\max}^{ss}) = Q(\gamma_1) \ln \left[\frac{f(\gamma_1)}{2\sqrt{K(\gamma_1)}} \right]; \quad \Delta = \frac{1}{2} e^{\gamma} \cdot \frac{g(\gamma_1)}{f(\gamma_1)} \\ Z &= -\frac{1}{2} \gamma_1 e^{\gamma} + \frac{2K(\gamma_1) + g(\gamma_1)M(\gamma_1)}{f(\gamma_1)}; \quad \gamma = \frac{\pm(x - x_{\max}) - H(\gamma_{\max}^{ss})}{Q(\gamma_1)}, \end{aligned} \quad (15)$$

with (+) for $x > x_{\max}$ and (−) for $x < x_{\max}$. The graphical representation of the solution is depicted in Figure 2 for a nucleus of strain at the “center” or the “edge” of the specimen. The values of the model parameters are $A=1$, $B=-1$, $\Gamma=0.27$, $\tau_0=0.018$, $\gamma_1=0.1$, $2l^2=10^{-6}$, with $x_{\max}=0$. The value

of the strain $\gamma_1=0.1$ is determined by numerically solving the equation $A\gamma_1^3 + B\gamma_1^2 + \Gamma\gamma_1 = \tau_0$. From the equal area rule condition of Eq. (11), it follows that $f_L[\gamma_1] = f_L[\gamma_{\max}^{ss}]$ (see Figure 1B).

(ii) *Dynamic solutions without dissipation* ($\lambda = 0$):

In this case, the governing equation reads

$$-2l^2 \gamma'' = \tau_0 - \kappa(\gamma) + \rho v^2 (\gamma - \gamma_1). \quad (16)$$

The solutions are found as in the previous paragraph by extending the Aifantis-Serrin method for reversals [5, 6] according to [10–12]. This can be done by setting $x \rightarrow \xi$ and $\Gamma \rightarrow \Gamma - \rho v^2$ in the form of the stationary reversal solution of (i). From the equal area rule condition, i.e. Eq. (12), we find that the propagation velocity of the front is determined by the relation

$$v^2 = \frac{2}{\Gamma} v_s^2 \frac{\int_{\gamma_1}^{\gamma_{\max}^d} [\kappa(\gamma) - \tau_0] d\gamma}{(\gamma_{\max}^d - \gamma_1)^2}, \quad (17)$$

where $v_s = \pm \sqrt{\kappa'(0)/\rho} = \pm \sqrt{\Gamma/\rho}$ is the sound velocity and $\gamma_{\max}^d$ is the maximum value of the strain. Alternatively, Eq. (17) can be written as

$$v^2 = \frac{2}{\Gamma} v_s^2 \frac{f_L[\gamma_{\max}^d] - f_L[\gamma_1]}{(\gamma_{\max}^d - \gamma_1)^2}. \quad (18)$$

Also, from Eq. (12), it follows that $\bar{f}_L[\gamma_1] = \bar{f}_L[\gamma_{\max}^d]$, where

Figure 2 (A) Profile of the strain nucleus at the center of the specimen. (B) Profile of the strain nucleus at the edge of the specimen.

Figure 3 (A) Kink profiles at two different times. The propagation is to the right. (B) Strain nucleus at the edge of the specimen and the kink profile. The maximum value of the strain nucleus is equal to $\gamma_{\max}^{ss}$.

$$\bar{f}_L[\gamma] = \int_0^\gamma [\kappa(\gamma) - \tau(\gamma) d\gamma]. \quad (19) \quad \text{where } \beta = \sqrt{d} \frac{b-a_1}{\sqrt{2}} > 0 \text{ with } (a_1, b, c) \text{ being the roots } (b > c > a_1) \text{ of the following cubic equation}$$

The relation between the functionals $f_L[\gamma]$ and $\bar{f}_L[\gamma]$ is

$$\bar{f}_L[\gamma] = f_L[\gamma] - \rho v^2 \gamma \left(\frac{1}{2} \gamma - \gamma_1 \right), \quad (20)$$

and a similar graph for $\bar{f}_L[\gamma]$ as for $f_L[\gamma]$ in Figure 1B holds. For stationary states where $v=0$, we have $\bar{f}_L[\gamma] = f_L[\gamma]$. Transition solutions for the dynamic case have been discussed in [12], and this (with or without dissipation) will be further elaborated upon in a future article.

(iii) *Quasi-static solutions with dissipation* ($\lambda \neq 0$): They are found from the governing equation

$$-2l^2 \gamma'' + \lambda v \gamma' = \tau_0 - \kappa(\gamma). \quad (21)$$

This is the equation of an anharmonic oscillator with linear friction. It is known [13, 14] that it possesses a kink-type solution of the form

$$\gamma(\xi) = a_1 + \frac{b-a_1}{1+e^{\pm \beta \xi}}, \quad (22)$$

$$d(\gamma-a_1)(\gamma-b)(\gamma-c) = \frac{1}{2l^2} (A\gamma^3 + B\gamma^2 + \Gamma\gamma - \tau_0). \quad (23)$$

Also, the velocity of the propagating front is

$$v = \mp \frac{2l^2}{\lambda} \sqrt{d} \frac{a_1 + b - 2c}{\sqrt{2}}. \quad (24)$$

Related results for the evolution and propagation of a fluctuating strain nucleus at the boundary are given in Figure 3. The graphs of the kink profiles for $t=0$ and $t=0.05$ are shown in Figure 3A, whereas the graph of the strain nucleus at the boundary and the propagating kink is shown in Figure 3B. The values of the model parameters are: $A=1$, $B=-1$, $\Gamma=0.27$, $\tau_0=0.018$, $2l^2=10^{-6}$, $\lambda=7 \times 10^{-5}$ and $v \approx -1$. The propagation velocity has a minus sign because the kink is traveling to the right. We have $\lim_{x \rightarrow -\infty} \gamma(x, t) = \gamma_3 = 0.6$ and $\lim_{x \rightarrow +\infty} \gamma(x, t) = \gamma_1 = 0.1$.

References

- [1] Aifantis EC. *Int. J. Plasticity* 1987, 3, 211–247.
- [2] Zbib HM, Aifantis EC. *Res Mech.* 1988, 23, 261–277.
- [3] Zbib HM, Aifantis EC. *Res Mech.* 1988, 23, 279–292.
- [4] Zbib HM, Aifantis EC. *Res Mech.* 1988, 23, 293–305.
- [5] Aifantis EC, Serrin JB. *J. Colloid Interface Sci.* 1983, 96, 517–529.
- [6] Aifantis EC, Serrin JB. *J. Colloid Interface Sci.* 1983, 96, 530–547.
- [7] van Zyl BP, Gooding RJ. *Phys. Rev. B* 1996, 54, 15700–15707.
- [8] Bales GS, Gooding RJ. *Phys. Rev. Lett.* 1991, 67, 3412–3415.
- [9] Landau LD, Lifshitz EM. *Theory of Elasticity*. Oxford: Pergamon, 1986.
- [10] Tsagarakis I, Chasiotis I, Aifantis EC. *Proc. 2nd Greek Conf. Chem. Eng.* Assael M, et al., Eds. Tziola Publishing: Thessaloniki, Greece, 1999, pp. 187–203.
- [11] Tsagarakis I. *The Phenomenon of Necking in Polymers*, diploma thesis. Aifantis EC, Advisor. Aristotle University: Thessaloniki, Greece, 1996.
- [12] Coleman BD. *Comp. Math. Appl.* 1985, 11, 35–65.
- [13] Collins MA, Blumen A, Currie JF, Ross J. *Phys. Rev. B* 1979, 19, 3630–3644.
- [14] Dixon JW, Tuszynski JA, Otwinowski M. *Phys. Rev. A* 1991, 44, 3484–3491.