Research Article

Konstantinos A. Lazopoulos* and Anastasios K. Lazopoulos

On the fractional deformation of a linearly elastic bar

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Abstract: Fractional derivatives have non-local character, although they are not mathematical derivatives, according to differential topology. New fractional derivatives satisfying the requirements of differential topology are proposed, that have non-local character. A new space, the Λ -space corresponding to the initial space is proposed, where the derivatives are local. Transferring the results to the initial space through Riemann-Liouville fractional derivatives, the non-local character of the analysis is shown up. Since fractional derivatives have been established, having the mathematical properties of the derivatives, the linearly elastic fractional deformation of an elastic bar is presented. The fractional axial stress along the distributed body force is discussed. Fractional analysis with horizon is also introduced and the deformation of an elastic bar is also presented.

Keywords: Λ -fractional derivative, Λ -fractional space, fractional stress, fractional deformation, fractional body forces, fractional strain, fractional horizon

1 Introduction

Fractional Calculus is a blooming mathematical tool for various applications in physics and engineering. Machado *et al.* [1] have presented a historical retrogression for fractional calculus originated by Leibnitz [2], Liouville [3] and established by Riemann [4]. The main advantage of fractional calculus is the non-local character of the fractional derivatives, contrary to the common derivatives, that have inherited by their definition a strong local property. Fractional calculus has been an indispensable tool in various branches in physics, mechanics, control theory, engi-

neering and economics [5–8]. Classical references are the books [9–11] including also various applications.

Although fractional calculus was intensively used for the last 50 years, it was referred only to time non-locality Klimek [12]. Lazopoulos [13] turned the attention from time to space, just to include non-local properties in space due to inhomogeneities, microcracks etc. In fact he established the fractional strain with strong non-local character, contrary to the local character of the conventional strain, Truesdell [14]. However, all the well known fractional derivatives have mainly an operative character, instead of a derivative one. All the known derivatives do not satisfy the properties of a derivative demanded by Differential Topology, just to correspond to a fractional differential. So their use was not mathematically established, but it has an ad-hoc character. Lazopoulos [15] trying to fill that gap, proposed the fractional L-derivative, that had already been presented. Nevertheless, that effort was not successful, since again the conditions demanded by differential topology were not satisfied. Lately, Lazopoulos [16] proposed the fractional Λ -derivative, that is a modification of the fractional L-derivative, along with the fractional Λ -space where the fractional Λ -derivative behaves in conventional derivative rules. Further following that procedure, the beam bending problem, Lazopoulos [22], along with the fractional Taylor Series, with the fractional variation calculus and branching analysis of an axially compressed rod, Lazopoulos [23], has been presented.

The present work is targeting to the establishment of the axial fractional strain, that acquires non-local character, contrary to the conventional strain that is local. With the establishment of the fractional Λ -derivative and fractional Λ -space, the fractional Λ -strain is proposed, that has non-local character in the initial space, whereas it preserves local character in the fractional Λ -space. A linear elastic bar with given axial displacement is studied, defining the distribution of axial stress along with the distribution of body forces. Further, motivated by peridynamic theory, the horizon of the fractional influence is introduced. Again the deformation of a linearly elastic bar is discussed with fractional horizon.

*Corresponding Author: Konstantinos A. Lazopoulos: 14 Theatrou Str., Rafina, 19009 Greece; Email: orfeakos74@gmail.com
Anastasios K. Lazopoulos: Mathematical Sciences Department, Hellenic Army Academy Vari, 16673 Greece

2 Basic properties of Fractional Calculus

Fractional Calculus has recently become a branch of pure mathematics with many applications in physics and engineering. Many definitions of fractional derivatives exist. In fact, Fractional Calculus originated by Leibniz, looking for the possibility of defining the derivative $\frac{d^n g}{dx^n}$ when $n = \frac{1}{2}$. The various types of the fractional derivatives exhibit some advantages over the others. Nevertheless they all are non local, contrary to the conventional ones.

The detailed properties of fractional derivatives may be found in Kilbas *et al.* [17], Podlubny [10], Samko *et al.* [8]. Starting from Cauchy formula for the n -fold integral of a primitive function $f(x)$

$${}_a I_x^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s) ds, \quad (1)$$

$$x > 0, \quad n \in \mathbb{N}$$

and

$${}_x I_b^n f(x) = \frac{1}{(n-1)!} \int_x^b (s-x)^{n-1} f(s) ds, \quad (2)$$

$$x > 0, \quad n \in \mathbb{N}$$

we define the left and right fractional integral of f as:

$${}_a I_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(s)}{(x-s)^{1-\gamma}} ds \quad (3)$$

$${}_x I_b^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_x^b \frac{f(s)}{(s-x)^{1-\gamma}} ds \quad (4)$$

In Eqs.(3, 4) we assume that γ is the order of fractional integrals with $m < \gamma \leq m+1$, considering $\Gamma(x) = (x-1)!$ with $\Gamma(\gamma)$ Euler's Gamma function.

Thus the left and right Riemann-Liouville (R-L) derivatives are defined by:

$${}^{RL}_a D_x^\gamma f(x) = \left(\frac{d}{dx} \right)^{m+1} ({}_a I_x^{m-\gamma} f(x)) \quad (5)$$

and

$${}^{RL}_x D_b^\gamma f(x) = \left(-\frac{d}{dx} \right)^{m+1} ({}_x I_b^{m-\gamma} f(x)) \quad (6)$$

The L-FD was first pointed out in Lazopoulos *et al.* [15] as the derivative that occurs from the Adda [8] definition of a fractional differential (See Lazopoulos *et al.* [18, 19]):

$$d^\gamma f(x) = {}^L_a D_x^\gamma f(x) d^\gamma x \quad (7)$$

where $d^\gamma f(x)$ is the fractional differential of function $f(x)$ and $d^\gamma x$ the fractional differential of x . The first definition of the L-FD was stated as the ratio of the corresponding Caputo derivatives:

$${}^L_a D_x^\gamma f(x) = \frac{{}_a^C D_x^\gamma f(x)}{{}_a^C D_x^\gamma x} \quad (8)$$

Nevertheless, in the article we revised the L-FD, using Riemann-Liouville derivatives:

$${}^L_a D_x^\gamma f(x) = \frac{{}^{RL}_a D_x^\gamma f(x)}{{}^{RL}_a D_x^\gamma x} \quad (9)$$

Eqs. (8, 9) are equivalent when $f(a)=0$ and $f(x)$ is differentiable.

3 The proposed Λ -fractional space

The main disadvantage of the existing fractional derivatives is that they fail to satisfy Leibnitz and composition rules. Therefore they cannot correspond to a differential. As a consequence, the use of the various fractional derivatives in mathematical, geometrical and physical problems is questionable. On the other hand, the proposed formulation yields derivatives corresponding to differentials, fulfilling all the necessary conditions demanded by a derivative according to Differential Topology. Hence, it could be feasible to create a fractional differential geometry theory, not in the original coordinates but in a coordinate system which will be defined below.

Let us consider the R-L derivative, of a function $f(x)$:

$$\begin{aligned} {}^{RL}_a D_x^\gamma f(x) &= \left(\frac{d}{dx} \right)^{m+1} ({}_a I_x^{m-\gamma} f(x)) \\ &= \frac{d}{dx} \left(\left(\frac{d}{dx} \right)^m ({}_a I_x^{m-1-(\gamma-1)} f(x)) \right) \end{aligned} \quad (10)$$

with $m < \gamma < m+1$. It is evident that,

$${}^{RL}_a D_x^\gamma f(x) = \frac{d}{dx} \left({}^{RL}_a D_x^{\gamma-1} f(x) \right) \quad (11)$$

Hence, Eq. (9) may define the Λ -fractional derivative with:

$${}^\Lambda_a D_x^\gamma f(x) = \frac{d \left({}^{RL}_a D_x^{\gamma-1} f(x) \right)}{d \left({}^{RL}_a D_x^{\gamma-1} x \right)} \quad (12)$$

The γ - Riemann-Liouville fractional derivative is expressed as a common derivative of the $(\gamma-1)$ -Riemann-Liouville fractional derivatives, hence as a common derivative corresponds to a differential. Although for the present purposes the R-L derivative might be satisfactory, variational problems might be raised with its use, since

Noether's variational theory might be questionable and the adoption of that derivative might not satisfy geometrical and physical demands, Atanackovic *et al.* [4]. The Λ -Fractional Derivative introduced by Lazopoulos [16], might be adopted without the problems that the R-L exhibits. The Λ -Fractional Derivative has been defined by:

$${}^{\Lambda}D_x^{\gamma}f(x) = \frac{{}^{RL}D_x^{\gamma}f(x)}{{}^{RL}D_x^{\gamma}x} = \frac{d(D^{\gamma-1}(f(x)))}{d(D^{\gamma-1}x)} \quad (13)$$

Introducing the new variable,

$$X = {}^{RL}D^{\gamma-1}x \quad (14)$$

and imposing $F(X) = D^{\gamma-1}(f(x(X)))$, the L-fractional derivative, Eq. (9) may be expressed as a common derivative of the function $F(X)$ with respect to X . Hence the analysis has been transferred to plane $(X, F(X))$ with the conventional differential analysis. Then the results may be transferred back to the initial variables. Indeed, the transferring may be performed through the relation,

$$\begin{aligned} {}^{\Lambda}I_x^{\gamma-1} \left({}^{RL}D_x^{\gamma-1}(f(x)) \right) \\ = f(x) - \sum_{j=1}^m \left[{}^{RL}D_x^{\gamma-j-1}(f(x)) \right]_{x=a} \frac{(x-a)^{\gamma-1-j}}{\Gamma(\gamma-j)} \end{aligned} \quad (15)$$

Following that procedure, various fractional mathematical analysis areas, such as Fractional Differential Geometry, Fractional Field Theory, Fractional differential equations etc., may be established. The proposed Λ -fractional derivative satisfies Leibnitz's rule for the product of the $I^{1-\alpha}f(x)$ and $I^{1-\alpha}y(x)$. Indeed,

$$\begin{aligned} \frac{d(D^{\gamma-1}f(x) \cdot D^{\gamma-1}y(x))}{d(D^{\gamma-1}x)} &= \frac{d(D^{\gamma-1}f(x))}{d(D^{\gamma-1}x)} D^{\gamma-1}y(x) \\ &+ D^{\gamma-1}f(x) \frac{d(D^{\gamma-1}y(x))}{d(D^{\gamma-1}x)} \end{aligned} \quad (16)$$

Furthermore, the fractional derivative of a composite function is defined by,

$$\frac{d(D^{\gamma-1}f(y(x)))}{d(D^{\gamma-1}x)} = \frac{d(D^{\gamma-1}f(y))}{d(D^{\gamma-1}y)} \cdot \frac{d(D^{\gamma-1}y(x))}{d(D^{\gamma-1}x)} \quad (17)$$

The linearity properties along with Leibnitz's and composition rules, Eqs. (16, 17), satisfy all the Differential Topology conditions, Chillingworth [18], for the existence of a differential and further the existence of Fractional Differential Geometry. In fact the non-local mathematical analysis is established with derivatives having non-local character.

Yet, some of the most established books in Fractional Analysis are referred [8–10], just to compare the present

version of the Fractional Analysis to the existing one. Further, the analysis is simplified when $0 < \gamma < 1$. In that case, the γ -multiple integral is defined by

$${}_a I_x^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(s)}{(x-s)^{1-\gamma}} ds \quad (18)$$

with $\Gamma(\gamma)$ Euler's Gamma function.

Further, the left Riemann-Liouville (R-L) derivatives are defined by:

$$\begin{aligned} {}^{RL}_a D_x^{\gamma} f(x) &= \frac{d}{dx} \left({}_a I_x^{1-\gamma} f(x) \right) \\ &= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \frac{f(s)}{(x-s)^{\gamma}} ds \end{aligned} \quad (19)$$

with corresponding definitions for the right fractional integrals and derivatives [9]. Moreover, the Λ -fractional derivative is defined by,

$${}^{\Lambda}_a D_x^{\gamma} f(x) = \frac{\frac{d({}_a I_x^{1-\gamma} f(x))}{dx}}{\frac{d({}_a I_x^{1-\gamma} x)}{dx}} = \frac{d({}_a I_x^{1-\gamma} f(x))}{d({}_a I_x^{1-\gamma} x)} \quad (20)$$

In addition the results from the Λ -space may be transferred to the initial space through the relation,

$${}^{RL}_a D_x^{1-\gamma} (F(X)) = {}^{RL}_a D_x^{1-\gamma} \left({}^{\Lambda}_a I_x^{1-\gamma} (f(x)) \right) = f(x) \quad (21)$$

Defining as $X = {}_a I_x^{1-\gamma} x$ and $F(X) = {}_a I_x^{1-\gamma} f(x)$, the Λ -FD is defined as a conventional derivative in the fractional space $(X, F(X))$. In fact the Fractional Differential Geometry is defined as a conventional differential geometry in the Λ -fractional space, $(X, F(X))$.

Just to clarify the ideas, let us work as an example on the function,

$$f(x) = x^2 \quad (22)$$

Then the Λ -fractional plane $(X, F(X))$ is defined by

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \quad (23)$$

$$\begin{aligned} F(X) &= {}_0 I_x^{1-\gamma} f(x(X)) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{s^2}{(x-s)^{1-\gamma}} ds \\ &= -\frac{2}{(-6+11\gamma-6\gamma^2+\gamma^3)\Gamma(1-\gamma)} x^{(3-\gamma)} \end{aligned} \quad (24)$$

Further considering from Eq. (23),

$$x = \left((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X \right)^{\frac{1}{2-\gamma}}, \quad (25)$$

Eq. (25) yields

$$F(X) = -\frac{2 \left(((2 - 3\gamma + \gamma^2) \Gamma(1 - \gamma) X)^{\frac{1}{2-\gamma}} \right)^{3-\gamma}}{\Gamma(1 - \gamma) (-6 + 11\gamma - 6\gamma^2 + \gamma^3)}. \quad (26)$$

Therefore the curve in the original plane $(x, f(x))$ shown in Figure 1 corresponds to the fractional plane (space) shown in Figure 2, for $\gamma = 0.6$. Further, the derivative

Figure 1: The original plane $(x, f(x) = x^2)$

Figure 2: The curve in the Fractional Space $(X, F(X))$ for $\gamma = 0.6$

$$\frac{dF(X)}{dX} = \frac{24(5 - \gamma)(2 - 3\gamma + \gamma^2) \Gamma(1 - \gamma) (M)^{\frac{3}{2-\gamma}}}{(2 - \gamma) \Gamma(6 - \gamma)} \quad (27)$$

with $M = (2 - 3\gamma + \gamma^2)X\Gamma(1 - \gamma)$.

For $X_0 = 0.6$ and $\gamma = 0.6$, the derivative in the fractional plane is equal to

$$D(F(X_0)) = 1.1580.$$

Since the tangent space $Y(X)$ of the curve at a point X_0 is defined by the line,

$$Y(X) = F(X_0) + \frac{d}{dX}(F(X_0))(X - X_0) \quad (28)$$

In the original plane $(x, f(x))$ the corresponding tangent space is defined by the curve that will be built as follows:

Figure 3: The curve with its tangent space in the Fractional Plane

Recalling Eq. (25), the $x_0 = 0.81$ corresponding to $X_0 = 0.60$ is defined. Then substituting in the derivative $\frac{dF(X)}{dX}$ in the fractional plane the X as a function of x , the ${}_0^A D_x^\gamma f(x)$ is defined. Therefore the corresponding function in the real space $(x, f(x))$ is defined as ${}_0^{RL} D_x^\gamma \left({}_0^A D_x^\gamma f(x) \right)$. Indeed,

$$\begin{aligned} & {}_0^{RL} D_x^\gamma \left({}_0^A D_x^\gamma f(x) \right) \\ &= \frac{1}{\Gamma(\gamma)} \frac{d}{dx} \int_0^x (x-s)^{\gamma-1} \left({}_0^A D_s^\gamma f(s) \right) ds \end{aligned} \quad (29)$$

In the present case, for the function $f(x) = x^2$,

$${}_0^{RL} D_x^\gamma \left({}_0^A D_x^\gamma x^2 \right)_{x=0.81} = 1.41 \quad (30)$$

Thus the fractional tangent space $g(x)$ in the original space $(x, f(x))$ is defined as

$$\begin{aligned} g(x) &= f(x)_{x_0} \\ &+ {}_0^{RL} D_x^\gamma \left({}_0^A D_x^\gamma f(x) \right)_{x=x_0} \left(\frac{1}{\Gamma(1-\gamma)} \frac{x^{2-\gamma}}{2-3\gamma+\gamma^2} - X_0 \right) \end{aligned} \quad (31)$$

Hence, at $X_0 = 0.6$ for $\gamma = 0.6$, $x_0 = 0.81$ the tangent space is defined by

$$g(x) = (x^2)_{x=0.81} + 1.41 \left(\frac{1.79x^{1.4}}{\Gamma(0.4)} - 0.6 \right) \quad (32)$$

for $\gamma = 0.6$ at $x = 0.81$

4 The fractional deformation of a linearly elastic bar

Let us consider a bar of length L , linearly elastic, with elastic modulus E , following the fractional analysis, see Lazopoulos [16]. If $u(x)$ and $\sigma(x)$ are the displacement and the stress at a point x of the bar in the original space, the

Figure 4: The curve $f(x)$ with its fractional tangent space $g(x)$ in the original plane

corresponding configurations in the Λ -fractional space are defined by,

$${}^l F(X) = {}_0 I_x^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{f(s)}{(x-s)^\gamma} ds \quad (33)$$

Here $f(x)$ might express the x coordinate, the displacement $u(x)$ or the stress $\sigma(x)$ functions corresponding to the ${}^l X$, the displacement ${}^l U(X)$ and the stress ${}^l \Sigma(X)$ functions in the left Λ -Fractional space. The units dimension of the function $f(x)$ in the initial space is increased during its transformation $F(X)$ in the Λ -space by $L^{1-\gamma}$ where L is the physical unit of the length in the initial space. Let us point out that:

$${}^l X = {}_0 I_x^{1-\gamma}(x) = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \quad (34)$$

Further, the displacement function $U(x)$ in the left fractional space Λ is defined by,

$${}^l U(x) = {}_0 I_x^{1-\gamma} u(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{u(s)}{(x-s)^\gamma} ds \quad (35)$$

Therefore, the fractional stress ${}^l \Sigma(X)$ may be defined through the linear elastic law,

$${}^l \Sigma({}^l X) = E \frac{d^l U({}^l X)}{d^l X} = E \frac{\frac{d^l U(x)}{dx}}{\frac{d^l X}{dx}} \quad (36)$$

Furthermore, the linear momentum equation demands for equilibrium,

$$\frac{d^l \Sigma({}^l X)}{d^l X} + {}^l P({}^l X) = 0 \quad (37)$$

where ${}^l P({}^l X)$ is the body force per unit length in the left Λ -space. Transferring the problem to the initial space, the stress and the distributed body force may be defined by the law expressed by Eq. (21),

$$f(x) = {}^{RL} D_x^{1-\gamma} (F(X)) \quad (38)$$

Having defined the stress and the distributed body force for the left fractional space, the procedure is repeated for the right Λ -space, defining the right stress and body force in the original space. Indeed, for the right part of the rod (x, L) the corresponding Λ -space is defined by,

$${}^r F(X) = {}_x I_L^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_x^L \frac{f(s)}{(s-x)^\gamma} ds \quad (39)$$

where again $f(x)$ might be the x coordinate, the displacement $u(x)$ and the stress $\sigma(x)$ functions corresponding to the ${}^r X$, ${}^r U(X)$ and ${}^r \Sigma(X)$ functions in the right Λ -Fractional space. Indeed,

$$\begin{aligned} {}^r X &= {}_x I_L^{1-\gamma}(x) = \frac{1}{\Gamma(1-\gamma)} \int_x^L \frac{s}{(s-x)^\gamma} ds \\ &= \frac{(L-x)^{2-\gamma}(L+x-L\gamma)}{\Gamma(3-\gamma)} \end{aligned} \quad (40)$$

Further, the displacement function $U(x)$ in the right fractional space Λ is defined by,

Therefore, the fractional stress ${}^r \Sigma(X)$ may be defined through the linear elastic law,

$${}^r \Sigma({}^r X) = E \frac{d^r U({}^r X)}{d^r X} = E \frac{\frac{d^r U(x)}{dx}}{\frac{d^r X}{dx}} \quad (41)$$

Furthermore, the equilibrium equation demands for equilibrium,

$$\frac{d^r \Sigma({}^r X)}{d^r X} + {}^r P({}^r X) = 0 \quad (42)$$

Transferring the stresses from the Λ -space to the initial space the Eq. (21) should be recalled.

$${}^{RL} D_x^{1-\gamma} (F(X)) = {}^{RL} D_x^{1-\gamma} \left({}^{RL} I_x^{1-\gamma} (f(x)) \right) = f(x) \quad (43)$$

Also the corresponding equation for the right Λ -space,

$${}^{RL} D_L^{1-\gamma} (F(X)) = {}^{RL} D_L^{1-\gamma} \left({}^{RL} I_L^{1-\gamma} (f(x)) \right) = f(x) \quad (44)$$

should be mentioned.

Then, the stress is defined as the mean value of the left and right stress. Just the same is valid for the body forces. The following application shows the methodology clarifying the various steps of the present procedure.

5 Application of fractional deformation of a bar

Let us consider a bar of length L and elastic modulus E , on the axis $x \in (0, L)$. The displacement field of the bar is

defined by:

$$u(x) = a(x^3 - x) \quad (45)$$

where $a \leq 1$, since the discussed problem is considered in linear elasticity where the displacement field is considered as infinitesimal.

Since the derivation is valid in the Λ -space, all the corresponding functions have to be transferred in that fractional space. For the left fractional space the coordinate, see Eq. (34),

$${}^lX = {}_0I_x^{1-\gamma}(x) = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}. \quad (46)$$

Further, the corresponding left fractional displacement function in the Λ -space is defined, see Eq. (35) by,

$$\begin{aligned} {}^lU(x) &= \frac{a}{\Gamma(1-\gamma)} \int_0^x \frac{s^3 - s}{(x-s)^\gamma} ds \\ &= \frac{x^{2-\gamma}(6x^2 - (-4+\gamma)(-3+\gamma))}{\Gamma(4-\gamma)} \end{aligned} \quad (47)$$

Since the axial stress in the left Λ -space is defined by Eq. (36)

$${}^l\Sigma({}^lX) = aE(-1 + \frac{6x^2}{6-5\gamma+\gamma^2}) \quad (48)$$

Further, the right fractional displacement function in the Λ -space, Eq. (41) is defined by,

$$\begin{aligned} {}^rU({}^rX) &= (\Gamma(5-\gamma))^{-1}(L-x)^{1-\gamma} \left\{ 6x^3 \right. \\ &\quad - x(-4+\gamma)(-3+\gamma) + L(-6x^2 \\ &\quad + (-4+\gamma)(-3+\gamma)(-1+\gamma) + 3L^2x(-2+\gamma)(-1+\gamma) \\ &\quad \left. - (-2+\gamma)(-1+\gamma) \right\} \end{aligned} \quad (49)$$

with the right stress function in the Λ -space defined by the function:

$${}^r\Sigma({}^rX)/(aE) = \frac{A}{(\gamma-3)(\gamma-2)(L\gamma-2)} \quad (50)$$

where:

$$\begin{aligned} A &= -6x^3 + 6Lx^2\gamma + L \left\{ (3 + L^2(-1+\gamma) - \gamma)(-2+\gamma)\gamma \right. \\ &\quad \left. + x(6 + (-5 + 3L^2)\gamma + (1 - 3L^2)\gamma^2) \right\}. \end{aligned}$$

The stress function in the Λ -space is the mean value of the right and left values. Therefore,

$$\Sigma(X) = \frac{1}{2} ({}^l\Sigma({}^lX) + {}^r\Sigma({}^rX)) \quad (51)$$

Further, the distributed body forces $P(X)$ in the fractional Λ -space are defined through the function

$$P(X) = -\frac{1}{2} \left(\frac{d{}^l\Sigma({}^lX)}{d{}^lX} + \frac{d{}^r\Sigma({}^rX)}{d{}^rX} \right) \quad (52)$$

Figure 5: The distribution of stresses in the Λ -space with respect to the initial x axis

Figure 6: The distribution of body forces in the Λ -space with respect to the initial x axis

For the specific values, $L = 3$, $\gamma = 0.7$, the stress function in the Λ -space with respect to the initial x axis has been shown in Figure 5.

Transferring the stresses into the initial space:

$$\sigma(x) = \frac{1}{2} \left({}_0^{RL}D_x^{1-\gamma} ({}^l\Sigma({}^lX)) + {}_x^{RL}D_L^{1-\gamma} ({}^r\Sigma({}^rX)) \right) \quad (53)$$

Recalling Eq. (11) the stress $\sigma(x)$ in the initial space might be found. Omitting the various computations performed with the help of Mathematica computing algebra pack, the stress has been computed for $L = 3$ and $\gamma = 0.7$.

Further the distributed body force in the initial space has also been computed through,

$$p(x) = \frac{1}{2} \left({}_0^{RL}D_x^{1-\gamma} ({}^lP({}^lX)) + {}_x^{RL}D_L^{1-\gamma} ({}^rP({}^rX)) \right) \quad (54)$$

Again the various computations performed with the help of Mathematica computing algebra pack, the stress has been computed for $L = 3$ and $\gamma = 0.7$.

Although the body force exhibits a singularity, the fractional stress is smooth.

Figure 7: The distribution of stresses in the initial space with respect to the initial x axis

Figure 8: Distribution of body forces in the initial space with respect to the initial x axis

6 The fractional deformation of a linearly elastic bar with horizon

Adapting the idea of horizon, introduced in peridynamic [10, 19, 21] deformation theory, the non-local influence is restricted in a neighborhood of the material point of distance δ . Therefore, the fractional influence is restricted in $x \in (x - \delta, x + \delta)$. If $u(x)$ and $\sigma(x)$ are the displacement and the stress at a point x of the bar in the original space, the corresponding configurations in the Λ -fractional space are defined by,

$${}^l F(X) = {}_{x-\delta} I_x^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_{x-\delta}^x \frac{f(s)}{(x-s)^\gamma} ds \quad (55)$$

Let us point out that,

$${}^l X = {}_{x-\delta} I_x^{1-\gamma} (x) = \frac{\delta^{1-\gamma} \left(\frac{\delta}{-2+\gamma} - \frac{x}{-1+x} \right)}{\Gamma(1-\gamma)} \quad (56)$$

Further, the displacement function $U(x)$ in the left fractional space Λ is defined by,

$${}^l U(x) = {}_{x-\delta} I_x^{1-\gamma} u(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{u(s)}{(x-s)^\gamma} ds \quad (57)$$

Therefore, the fractional stress ${}^l \Sigma(X)$ may be defined through the linear elastic law,

$${}^l \Sigma(X) = E \frac{d {}^l U(X)}{d {}^l X} = E \frac{\frac{d {}^l U(x)}{dx}}{\frac{d {}^l X}{dx}} \quad (58)$$

Furthermore, the equilibrium equation demands,

$$\frac{d {}^l \Sigma(X)}{d {}^l X} + {}^l P({}^l X) = 0 \quad (59)$$

where ${}^l P({}^l X)$ is the body force per unit length in the left Λ -space. Transferring the problem in the initial space the stress and the distributed body force may be defined by the law expressed by Eq. (21),

$$f(x) = {}^{RL} D_x^{1-\gamma} (F(X)) \quad (60)$$

Having defined the stress and the distributed body force for the left fractional space, the procedure is repeated for the right Λ -space, defining the right stress and body force in the original space. Indeed, for the right part of the rod $(x, x + \delta)$ the corresponding Λ -space is defined by,

$${}^r F(X) = {}_x I_{x+\delta}^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_x^{x+\delta} \frac{f(s)}{(s-x)^\gamma} ds \quad (61)$$

Indeed,

$$\begin{aligned} {}^r X &= {}_x I_{x+\delta}^{1-\gamma} (x) = \frac{1}{\Gamma(1-\gamma)} \int_x^{x+\delta} \frac{s}{(s-x)^\gamma} ds \\ &= \frac{\delta^{1-\gamma} (\delta + 2x - (\delta + x)\gamma)}{\Gamma(3-\gamma)} \end{aligned} \quad (62)$$

Further, the displacement function $U(x)$ in the right fractional space Λ is defined by,

$${}^r U(x) = {}_x I_{x+\delta}^{1-\gamma} u(x) = \frac{1}{\Gamma(1-\gamma)} \int_x^{x+\delta} \frac{u(s)}{(s-x)^\gamma} ds \quad (63)$$

Therefore, the fractional stress ${}^r \Sigma(X)$ may be defined through the linear elastic law,

$${}^r \Sigma(X) = E \frac{d {}^r U(X)}{d {}^r X} = E \frac{\frac{d {}^r U(x)}{dx}}{\frac{d {}^r X}{dx}} \quad (64)$$

Furthermore, the linear momentum equation demands for equilibrium,

$$\frac{d {}^r \Sigma(X)}{d {}^r X} + {}^r P({}^r X) = 0 \quad (65)$$

Transferring the stresses from the Λ -space to the initial space the eq. (21) should be recalled.

$${}^{RL}D_{x-\delta}^{1-\gamma}(F(X)) = {}^{RL}D_{x-\delta}^{1-\gamma}\left({}^{RL}I_{x-\delta}^{1-\gamma}(f(x))\right) = f(x). \quad (66a)$$

Also the corresponding equation for the right Λ -space,

$${}^{RL}D_{x+\delta}^{1-\gamma}(F(X)) = {}^{RL}D_{x+\delta}^{1-\gamma}\left({}^{RL}I_{x+\delta}^{1-\gamma}(f(x))\right) = f(x) \quad (66b)$$

should be mentioned.

Then, the stress is defined as the mean value of the left and right stress. Just the same is valid for the body forces.

7 Application of fractional deformation with horizon

Let us consider a bar of length L and elastic modulus E , on the axis x . The displacement field of the bar is defined by,

$$u(x) = a(x^3 - x) \quad (67)$$

where $a \ll 1$, since the discussed problem is considered in linear elasticity where the displacement field is considered as infinitesimal.

Since the derivation is valid in the Λ -space, all the corresponding functions have to be transferred in that fractional space. For the left fractional space the coordinate X , see Eq. (56),

$${}^lX = {}_{x-\delta}I_x^{1-\gamma}(x) = \frac{\delta^{1-\gamma}\left(\frac{\delta}{-2+\gamma} - \frac{x}{-1+x}\right)}{\Gamma(1-\gamma)} \quad (68)$$

Further, the corresponding left fractional displacement function in the Λ -space with horizon δ is defined, see Eq. (57) by,

$$\begin{aligned} {}^lU(x) &= \frac{a}{\Gamma(1-\gamma)} \int_0^x \frac{s^3 - s}{(x-s)^\gamma} ds \\ &= \frac{a\delta^{1-\gamma}\left(\frac{\delta^3}{\gamma-4} - \frac{3\delta^2}{\gamma-3}x + \frac{\delta(-1+3x^2)}{\gamma-2} + \frac{x-x^3}{\gamma-1}\right)}{\Gamma(4-\gamma)} \end{aligned} \quad (69)$$

Since the axial stress in the left Λ -space is defined by Eq. (58)

$${}^l\Sigma({}^lX) = aE\left(\frac{3\delta^2}{\gamma-3} - \frac{6\delta x}{\gamma-2} + \frac{1-3x^2}{\gamma-1}\right)(\gamma-1) \quad (70)$$

Further, the right fractional displacement function in the Λ -space, Eq. (63) is defined by,

$${}^rU({}^rX) = \frac{\delta^{1-\gamma}}{\Gamma(1-\gamma)} \cdot \left(\frac{\delta^3}{4-\gamma} + \frac{3\delta^2 x}{3-\gamma} + \frac{\delta-3\delta x^2}{-2+\gamma} + \frac{x-x^3}{-1+\gamma}\right) \quad (71)$$

with the right stress function in the Λ -space defined by the function,

$$\frac{{}^r\Sigma({}^rX)}{(aE)} = -\left(-\frac{3\delta^2}{-3+\gamma} + \frac{6\delta x}{-2+\gamma} + \frac{1-3x^2}{-1+\gamma}\right)(-1+\gamma) \quad (72)$$

The stress function in the Λ -space is the mean value of the right and left values. Therefore,

$$\Sigma(X) = \frac{1}{2}\left({}^l\Sigma({}^lX) + {}^r\Sigma({}^rX)\right) \quad (73)$$

Further, the distributed body forces $P(X)$ in the fractional Λ -space are defined through the function

$$P(X) = -\frac{1}{2}\left(\frac{d{}^l\Sigma({}^lX)}{d{}^lX} + \frac{d{}^r\Sigma({}^rX)}{d{}^rX}\right) \quad (74)$$

For the specific values, $\delta = 0.2$, $\gamma = 0.7$, the stress function in the Λ -space with respect to the initial x axis has been shown in Figure 9.

Figure 9: The distribution of stresses in the fractional with horizon Λ -space with respect to the initial x axis

Figure 10: The distribution of body forces in the fractional with horizon Λ -space with respect to the initial x axis

Transferring the stresses into the initial space, we have:

$${}^{RL}D_{x+\delta}^{1-\gamma}(F(X)) = {}^{RL}D_{x+\delta}^{1-\gamma}\left({}^{RL}I_{x+\delta}^{1-\gamma}(f(x))\right) = f(x) \quad (75)$$

Recalling Eq. (11) the stress $\sigma(x)$ in the initial space might be found. Omitting the various computations performed with the help of Mathematica computing algebra pack, the stress has been computed for $\delta = 0.2$ and $\gamma = 0.7$.

Figure 11: The distribution of stresses in the initial space with respect to the initial x axis

Further the distributed body force in the initial space has also been computed through,

$$p(x) = \frac{1}{2} \left({}^{RL}D_x^{1-\gamma} \left({}^lP \left({}^lX \right) \right) + {}^{RL}D_L^{1-\gamma} \left({}^rP \left({}^rX \right) \right) \right) \quad (76)$$

Again the various computations performed with the help of Mathematica computing algebra pack, the stress has been computed for $\delta = 3$ and $\gamma = 0.7$.

Figure 12: Distribution of body forces in the initial space with respect to the initial x axis

8 The 3D fractional equilibrium problem

Let us point out that geometry exists only on the fractional Λ -space. Therefore all the conventional mechanics rules

are valid in that space. The various results may be transferred as simple functions in the initial space. Let us consider the 3D displacement vector $\mathbf{u}(\mathbf{x})$ in the initial space. Then the strain function in the fractional Λ -space may be defined by Eq. (33)

$$\mathbf{U}(\mathbf{x}) = {}_0I_x^{1-\gamma} \mathbf{u}(\mathbf{x}) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{\mathbf{u}(s)}{(\mathbf{x}-s)^\gamma} ds \quad (77)$$

Further, the corresponding $\mathbf{X}$ vector in the fractional Λ -space is expressed by:

$$\mathbf{X} = {}_0I_x^{1-\gamma} \mathbf{x} = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{\mathbf{x}(s)}{(\mathbf{x}-s)^\gamma} ds \quad (78)$$

Inserting $\mathbf{X}$ from the Eq. (77) into Eq. (78), the fractional Λ -space displacement vector $\mathbf{U}(\mathbf{X})$ is defined. Hence the fractional linear strain function is given by:

$$E(X) = \frac{1}{2} \left(\frac{\partial U(X)}{\partial X} + \left(\frac{\partial U(X)}{\partial X} \right)^T \right) \quad (79)$$

Further the geometrical shape of the body $b(\mathbf{x})$ may be transferred to the fractional Λ -space as $B(\mathbf{X})$. In addition any function, like stresses, forces etc may be transferred to the fractional Λ -space. In the Λ -fractional space the derivatives are local and everything works following the conventional procedure. The linear elastic Hook's law is valid with,

$$\Sigma(X)_{ij} = L_{ijk} E_{km} \quad (80)$$

Solving that problem in the fractional Λ -space, the various results $\mathbf{Q}(\mathbf{X})$ may be pulled back like functions to the original space, using the equation,

$$q(x) = {}^{RL}D_{x(X)}^{1-\gamma} (Q(X)) \quad (81)$$

9 Conclusion and further research

Introducing the Λ -fractional derivative satisfying the properties of a derivative in the Λ -fractional space, demanded by differential topology, the deformation of an elastic bar governed by fractional analysis has been presented. The fractional stresses along with the fractional body forces have been defined for a given displacement field of the bar. The same problem has been discussed for fractional deformation of the bar with horizon. Again the fractional stresses along with fractional body forces have been defined for the same displacement field.

The next step is the discussion of fractional continuum mechanics and fractional dynamics demanding a sound definition and development of fractional differential geometry.

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