Research Article

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Capitulation of the 2-ideal classes of type $(2, 2, 2)$ of some quartic cyclic number fields

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Abstract: Let $p \equiv 3 \pmod{4}$ and $l \equiv 5 \pmod{8}$ be different primes such that $\left(\frac{p}{l}\right) = 1$ and $\left(\frac{2}{p}\right) = \left(\frac{p}{l}\right)_4$. Put $k = \mathbb{Q}(\sqrt{l})$, and denote by ϵ its fundamental unit. Set $K = k(\sqrt{-2p\epsilon\sqrt{l}})$, and let $K_2^{(1)}$ be its Hilbert 2-class field, and let $K_2^{(2)}$ be its second Hilbert 2-class field. The field K is a cyclic quartic number field, and its 2-class group is of type $(2, 2, 2)$. Our goal is to prove that the length of the 2-class field tower of K is 2, to determine the structure of the 2-group $G = \text{Gal}(K_2^{(2)}/K)$, and thus to study the capitulation of the 2-ideal classes of K in all its unramified abelian extensions within $K_2^{(1)}$. Additionally, these extensions are constructed, and their abelian-type invariants are given.

Keywords: 2-groups, capitulation, Hilbert class field

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1 Introduction

Let K be an algebraic number field. For a prime number p , let $K_p^{(0)} = K$ and $K_p^{(i)}$ denote the Hilbert p -class field of $K_p^{(i-1)}$ for $i \geq 1$. Then the sequence $(K_p^{(i)})_{i \in \mathbb{N}}$ is called the p -class field tower of K . Put $\tilde{K}_p = \bigcup_{i=0}^{\infty} K_p^{(i)}$. It is well known that the p -group $G = \text{Gal}(\tilde{K}_p/K)$ can be finite as it can be infinite, thanks to Golod and Shafarevich [10], but we do not know enough criteria to decide whether G is finite or not. In general, this group is far from understood. In addition, the structure of G incorporates a lot of arithmetic information about the field K and its unramified p -extensions. It is quite difficult to determine this structure, and we do not have efficient methods for describing it. An important method of describing a p -group is via a presentation by generators and relations. In this paper, we are interested in giving the structure of $G = \text{Gal}(\tilde{K}_2/K)$, via generators and relations, for some quartic cyclic number field K , such that G/G' is of type $(2, 2, 2)$, where G' is the derived subgroup of G .

Let $p \equiv 3 \pmod{4}$ and $l \equiv 5 \pmod{8}$ be different primes. Put $k = \mathbb{Q}(\sqrt{l})$, and denote by ϵ its fundamental unit. Let $K = k(\sqrt{-2p\epsilon\sqrt{l}})$ and $K_2^{(1)}$ be its Hilbert 2-class field; denote by $K_2^{(2)}$ the Hilbert 2-class field of $K_2^{(1)}$ and by $K^{(*)}$ the genus field of K that is the maximal absolute abelian subfield of $K_2^{(1)}/K$. If $\left(\frac{p}{l}\right) = -1$, then $\mathbf{C}_{K,2}$, the 2-class group of K , is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (see [9]) and $K^{(*)} = K_2^{(1)}$; this case is studied in [3]. But if $\left(\frac{p}{l}\right) = 1$, then $K^{(*)} \subsetneq K_2^{(1)}$ and the rank of $\mathbf{C}_{K,2}$ is greater than 3, and the situation becomes more interesting. Assuming that $\mathbf{C}_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we aim, in this paper, the following four objectives.

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- Construct the fourteen unramified subextensions of $K_2^{(1)}/K$.
- Prove that the class field tower of K stops at the second stage, and thus $G = \text{Gal}(\tilde{K}_2/K) = \text{Gal}(K_2^{(2)}/K)$.
- Give the structure of the 2-group G via generators and relations.
- Concretely answer, via the transfer kernels for the finite 2-group G and their relation to capitulation kernels, the capitulation problem of the 2-ideal classes of K in the fourteen unramified subextensions of $K_2^{(1)}/K$.

Note that, in [2], we treated the same topics for the field $K = k(\sqrt{-p\epsilon\sqrt{l}})$, whose 2-class group is isomorphic to $G/G' \simeq (2, 2, 2)$, but G , in this case, admits three maximal normal abelian subgroups of index 2, and its order is exactly 16. Whereas, in the present case, the 2-group G admits, whenever $p \equiv 3 \pmod{8}$, only one maximal normal abelian subgroup of index 2, and its order is equal to 2^m for some integer $m > 4$. On the other hand, the same subjects are treated in the papers [4–6] by considering base fields of the form $k = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ for some square-free integer d , and in [15] by considering some imaginary quadratic number field as a base field.

Notations

Let m be a square free integer, and let k be a number field. Throughout this paper, we adopt the following notations:

FSU	a fundamental system of units of k
$h(m), h(k)$	the class number of $\mathbb{Q}(\sqrt{m})$ (resp. k)
$h_2(m), h_2(k)$	the 2-class number of $\mathbb{Q}(\sqrt{m})$ (resp. k)
$\mathbf{C}_{k,2}$	the 2-part of the class group of k
k^+	the maximal real subfield of k if it is a CM-field
E_k	the unit group of k
W_k	the group of roots of unity contained in k
w_k	the order of W_k
$Q_k = [E_k : W_k E_{k^+}]$	Hasse's unit index if k is a CM-field
$N_{L/k}$	the relative norm for an extension L/k
$\kappa_{L/k}$	the capitulation kernel of L/k
$\mathfrak{p}_k$	a prime ideal of k
$(\frac{x,y}{\mathfrak{p}_k}), (\frac{x}{\mathfrak{p}_k})$	the Hilbert symbol (resp. the quadratic residue symbol) for the prime $\mathfrak{p}_k$ applied to (x, y) (resp. x)
ξ_m	a primitive m -th root of unity
$\mathbb{Q}(\xi_m)$	the m -th cyclotomic number field

2 Main results

In all this paper, we consider the following conditions and notations: p and l are different primes satisfying

$$p \equiv 3 \pmod{4}, \quad l \equiv 5 \pmod{8}, \quad \left(\frac{p}{l}\right) = 1 \quad \text{and} \quad \left(\frac{2}{p}\right) = \left(\frac{p}{l}\right)_4. \quad (2.1)$$

Let $k = \mathbb{Q}(\sqrt{l})$, and denote by ϵ its fundamental unit and by h_0 its class number. Then $K = k(\sqrt{-2p\epsilon\sqrt{l}})$ is a complex cyclic quartic field [9].

Let $K_2^{(1)}$ be the Hilbert 2-class field of K , $K_2^{(2)}$ the Hilbert 2-class field of $K_2^{(1)}$, and let $K^{(*)}$ be the absolute genus field of K . Since $(\frac{p}{l}) = 1$, so $K^{(*)} \subsetneq K_2^{(1)}$ (as mentioned in the introduction), and there exist two prime ideals $\mathcal{B}_1, \mathcal{B}_2$ of k such that $\mathcal{B}_1 \mathcal{B}_2 = (p)$. Thus $\mathcal{B}_1^{h_0}$ and $\mathcal{B}_2^{h_0}$ are principal in k , say $\mathcal{B}_1^{h_0} = (\pi_1)$ and $\mathcal{B}_2^{h_0} = (\pi_2)$, where

$$\pi_1 = a + b\sqrt{l}, \quad \pi_2 = a - b\sqrt{l} \quad \text{and} \quad p^{h_0} = a^2 - lb^2.$$

One can easily show that four prime ideals of k ramify in K , which are the prime ideal of k above 2, $(\sqrt{l})$,

$\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{P}_1, \mathcal{P}_2$ be the prime ideals in K above $\mathcal{B}_1, \mathcal{B}_2$, respectively, let $\mathcal{V}$ be the prime ideal in K above $\sqrt{l}$, and let $\mathcal{H}$ be the prime ideal in K above 2. Therefore, $\mathcal{P}_i^{2h_0} = \mathcal{B}_i^{h_0} \mathcal{O}_K = (\pi_i)$, $\mathcal{V}^2 = (\sqrt{l})$ and $\mathcal{H}^2 = (2)$. So, according to [9], the rank of the 2-class group of K is $4 - 1 = 3$, and the 2-class number h_2 of K is strictly greater than 4. Furthermore, by [9, Theorem 4, p. 68], $h_2 = 8$, i.e., $\mathbf{C}_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if $(\frac{2}{p}) = (\frac{p}{l})_4$.

The main results of this paper are Theorems 2.1, 2.2 and 2.3.

The unramified extensions of K

As $\mathbf{C}_{K,2} \simeq (2, 2, 2)$, by class field theory, K admits, within its first Hilbert 2-class field $K_2^{(1)}$, seven unramified quadratic extensions and seven unramified biquadratic extensions.

Theorem 2.1. *Let p and l be different primes as specified by equation (2.1). Recall that $\mathcal{P}_1, \mathcal{P}_2$ are the prime ideals in K above p , and $\mathcal{H}$ is the prime ideal in K above 2. Moreover, $\mathcal{P}_i^{2h_0} = (\pi_i)$ and $\mathcal{H}^2 = (2)$.*

(1) *The 2-class group of K is given by $\mathbf{C}_{K,2} = \langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}], [\mathcal{H}] \rangle \simeq (2, 2, 2)$.*

(2) *The seven unramified quadratic extensions of K are as follows:*

(a) *If $(\frac{2ab}{p}) = 1$, then*

$$\begin{aligned} F_1 &= K(\sqrt{-\epsilon\pi_1}), & F_2 &= K(\sqrt{-2\epsilon\pi_2}), & F_3 &= K(\sqrt{2p}), & F_4 &= K(\sqrt{-2}), \\ F_5 &= K(\sqrt{-p}), & F_6 &= K(\sqrt{2\epsilon\pi_1}) & \text{and} & F_7 &= K(\sqrt{\epsilon\pi_2}). \end{aligned}$$

(b) *If $(\frac{2ab}{p}) = -1$, then*

$$\begin{aligned} F_1 &= K(\sqrt{-2\epsilon\pi_1}), & F_2 &= K(\sqrt{-\epsilon\pi_2}), & F_3 &= K(\sqrt{2p}), & F_4 &= K(\sqrt{-2}), \\ F_5 &= K(\sqrt{-p}), & F_6 &= K(\sqrt{\epsilon\pi_1}) & \text{and} & F_7 &= K(\sqrt{2\epsilon\pi_2}). \end{aligned}$$

The fields F_3, F_4 and F_5 are intermediate between K and $K^{()}$. The fields $F_1 \simeq F_7$ and $F_2 \simeq F_6$ are pairwise conjugate and isomorphic. Consequently, F_3, F_4 and F_5 are absolutely abelian, whereas F_1, F_2, F_6 and F_7 are not.*

(3) *The seven unramified biquadratic extensions of K are*

$$\begin{aligned} L_2 &= F_3F_4F_5 = K^{(*)} = K(\sqrt{-p}, \sqrt{-2}), & \text{the absolute genus field of } K, \\ L_1 &= F_1F_2F_3, & L_3 &= F_3F_6F_7, & L_4 &= F_2F_5F_6, \\ L_5 &= F_1F_5F_7, & L_6 &= F_1F_4F_6, & L_7 &= F_2F_4F_7. \end{aligned}$$

Moreover, $L_1 \simeq L_3$ and $L_6 \simeq L_7$ are pairwise conjugate and thus isomorphic and absolutely non-normal, whereas L_4 and L_5 are absolutely Galois over $\mathbb{Q}$.

Structure of $G = \text{Gal}(K_2^{(2)}/K)$

Recall that $K_2^{(2)}$ is the second Hilbert 2-class field of K .

Theorem 2.2. *Keep the preceding assumptions and notations, and put $h_2(-pl) = 2^{n+1}$, where $n \geq 1$. Then:*

(1) *$\mathbf{C}_{F_5,2} \simeq (2, 2^{n+1})$, where $n \geq 2$ or $n = 1$, according as $p \equiv 3 \pmod{8}$ or $p \equiv 7 \pmod{8}$.*

(2) *The length of the 2-class field tower of K is 2.*

(3) *The second 2-class group $G = \text{Gal}(K_2^{(2)}/K)$ of K is given by*

$$G = \begin{cases} \langle \rho, \tau, \sigma : \rho^4 = \sigma^4 = \tau^{2^{n+1}} = 1, \sigma^2 = \rho^2 = \tau^{2^n}, n \geq 2, \\ \quad [\tau, \sigma] = 1, [\sigma, \rho] = \sigma^2, [\rho, \tau] = \tau^2 \rangle & \text{if } p \equiv 3 \pmod{8}, \\ \langle \rho, \tau, \sigma : \rho^4 = \sigma^4 = \tau^2 = 1, \sigma^2 = \rho^2, \\ \quad [\tau, \sigma] = [\tau, \rho] = 1, [\sigma, \rho] = \sigma^2 \rangle & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

(4) *The derived subgroup of G is $G' = \langle \tau^2 \rangle \simeq \mathbf{C}_{K_2^{(1)},2}$ or $G' = \langle \sigma^2 \rangle \simeq \mathbf{C}_{K_2^{(1)},2}$, according as $p \equiv 3 \pmod{8}$ or $p \equiv 7 \pmod{8}$. It is cyclic of order 2^n or 2, respectively.*

- (5) The coclass of G is equal to 2, and its nilpotency class is $n + 1$.
 (6) The center $Z(G)$ of G is of type $(2, 2)$.

Abelian-type invariants and capitulation kernels

Let $\kappa_{F/K}$ denote the capitulation kernel of the natural class extension homomorphism $J_{F/K} : \mathbf{C}_{K,2} \rightarrow \mathbf{C}_{F,2}$, where F is an unramified extension of K within $K_2^{(1)}$.

Theorem 2.3. *Keep the preceding assumptions and notations, then:*

- (1) The abelian-type invariants of the 2-class groups $\mathbf{C}_{F_i,2}$ are given by:
- (i) $\mathbf{C}_{F_3,2}$ and $\mathbf{C}_{F_4,2}$ are of type $(2, 2)$, and $\mathbf{C}_{F_5,2}$ is of type $(2, 2^{n+1})$ with $n \geq 1$.
 - (ii) If $\left(\frac{2ab}{p}\right) = 1$, then:
 - (a) $\mathbf{C}_{F_1,2} \simeq \mathbf{C}_{F_7,2} \simeq (2, 4)$ and $\mathbf{C}_{F_2,2} \simeq \mathbf{C}_{F_6,2} \simeq (2, 2)$ if $\left(\frac{2}{p}\right) = 1$.
 - (b) $\mathbf{C}_{F_1,2} \simeq \mathbf{C}_{F_7,2} \simeq (2, 2, 2)$ and $\mathbf{C}_{F_2,2} \simeq \mathbf{C}_{F_6,2} \simeq (2, 2)$ if $\left(\frac{2}{p}\right) = -1$.
 - (iii) If $\left(\frac{2ab}{p}\right) = -1$, then:
 - (a) $\mathbf{C}_{F_1,2} \simeq \mathbf{C}_{F_7,2} \simeq (2, 2)$ and $\mathbf{C}_{F_2,2} \simeq \mathbf{C}_{F_6,2} \simeq (2, 4)$ if $\left(\frac{2}{p}\right) = 1$.
 - (b) $\mathbf{C}_{F_1,2} \simeq \mathbf{C}_{F_7,2} \simeq (2, 2)$ and $\mathbf{C}_{F_2,2} \simeq \mathbf{C}_{F_6,2} \simeq (2, 2, 2)$ if $\left(\frac{2}{p}\right) = -1$.
- (2) The abelian-type invariants of the 2-class groups $\mathbf{C}_{L_i,2}$ are given by:
- (i) If $\left(\frac{2ab}{p}\right) = 1$, then:
 - (a) $\mathbf{C}_{L_1,2}, \mathbf{C}_{L_2,2}, \mathbf{C}_{L_3,2}, \mathbf{C}_{L_4,2}, \mathbf{C}_{L_6,2}$ and $\mathbf{C}_{L_7,2}$ are cyclic of order 4, and $\mathbf{C}_{L_5,2}$ is of type $(2, 2)$ if $\left(\frac{2}{p}\right) = 1$.
 - (b) $\mathbf{C}_{L_1,2} \simeq \mathbf{C}_{L_3,2} \simeq \mathbf{C}_{L_6,2} \simeq \mathbf{C}_{L_7,2} \simeq (2, 2)$, $\mathbf{C}_{L_5,2} \simeq (2, 2^n)$, and $\mathbf{C}_{L_2,2}$ and $\mathbf{C}_{L_4,2}$ are cyclic of order 2^{n+1} if $\left(\frac{2}{p}\right) = -1$.
 - (ii) If $\left(\frac{2ab}{p}\right) = -1$, then:
 - (a) $\mathbf{C}_{L_1,2}, \mathbf{C}_{L_2,2}, \mathbf{C}_{L_3,2}, \mathbf{C}_{L_5,2}, \mathbf{C}_{L_6,2}$ and $\mathbf{C}_{L_7,2}$ are cyclic of order 4, and $\mathbf{C}_{L_4,2}$ is of type $(2, 2)$ if $\left(\frac{2}{p}\right) = 1$.
 - (b) $\mathbf{C}_{L_1,2} \simeq \mathbf{C}_{L_3,2} \simeq \mathbf{C}_{L_6,2} \simeq \mathbf{C}_{L_7,2} \simeq (2, 2)$, $\mathbf{C}_{L_4,2} \simeq (2, 2^n)$, and $\mathbf{C}_{L_2,2}$ and $\mathbf{C}_{L_5,2}$ are cyclic of order 2^{n+1} if $\left(\frac{2}{p}\right) = -1$.
- (3) The capitulation is given by:
- (i) For all i , $\#\kappa_{F_i/K} = 4$ or 8 (for more details, see Tables 9 and 10), and all the extensions F_i satisfy Taussky's condition (A), i.e., $\#\kappa_{F_i/K} \cap N_i > 1$; see [16].
 - (ii) For all i , $\kappa_{L_i/K} = \mathbf{C}_{K,2}$ (total 2-capitulation), and each L_i is of type (A).

3 Preliminary results

Let us first collect some results that will be useful in the sequel.

Lemma 3.1. *Let p and l be different primes as specified by equation (2.1). Keep the previous notations, then:*

- (1) The equations $-\epsilon\pi_1 \equiv X^2 \pmod{4}$ and $\epsilon\pi_2 \equiv X^2 \pmod{4}$ have solutions in k if and only if $\left(\frac{2ab}{p}\right) = 1$.
 (2) The equations $\epsilon\pi_1 \equiv X^2 \pmod{4}$ and $-\epsilon\pi_2 \equiv X^2 \pmod{4}$ have solutions in k if and only if $\left(\frac{2ab}{p}\right) = -1$.

Proof. First, we put $\epsilon = x + y\sqrt{l}$, where x and y are two integers. Then $x \equiv 2 \pmod{4}$ and $y \equiv 1 \pmod{4}$ since $l \equiv 5 \pmod{8}$.

As $p^{h_0} = a^2 - lb^2$, then:

- If $p \equiv 3 \pmod{8}$, we have $a \equiv 0 \pmod{4}$ and $b \equiv \pm 1 \pmod{4}$.
- If $p \equiv 7 \pmod{8}$, we have $a \equiv 2 \pmod{4}$ and $b \equiv \pm 1 \pmod{4}$.

Thus we have four cases to discuss:

- (1) If $a \equiv 2 \pmod{4}$ and $b \equiv 1 \pmod{4}$, then we have $\pmod{4}$

$$\begin{aligned}\epsilon\pi_1 &= (x + y\sqrt{l})(a + b\sqrt{l}) \equiv (2 + \sqrt{l})(2 + \sqrt{l}) \equiv 4 + 4\sqrt{l} + l \equiv 1, \\ -\epsilon\pi_2 &= -(x + y\sqrt{l})(a - b\sqrt{l}) \equiv -(2 + \sqrt{l})(2 - \sqrt{l}) \equiv -4 + l \equiv 1.\end{aligned}$$

(2) If $a \equiv 2 \pmod{4}$ and $b \equiv -1 \pmod{4}$, then we have $\pmod{4}$

$$\begin{aligned} -\epsilon\pi_1 &= -(x+y\sqrt{l})(a+b\sqrt{l}) \equiv -(2+\sqrt{l})(2-\sqrt{l}) \equiv -4+l \equiv 1, \\ \epsilon\pi_2 &= (x+y\sqrt{l})(a-b\sqrt{l}) \equiv (2+\sqrt{l})(2+\sqrt{l}) \equiv 4+4\sqrt{l}+l \equiv 1. \end{aligned}$$

(3) If $a \equiv 0 \pmod{4}$ and $b \equiv 1 \pmod{4}$, then

$$\begin{aligned} -\epsilon\pi_1 &= -\epsilon(a+b\sqrt{l}) \equiv -\epsilon\sqrt{l} \pmod{4}, \\ \epsilon\pi_2 &= \epsilon(a-b\sqrt{l}) \equiv -\epsilon\sqrt{l} \pmod{4}, \end{aligned}$$

which is a square in k (see [9, Lemma A]).

(4) If $a \equiv 0 \pmod{4}$ and $b \equiv -1 \pmod{4}$, then

$$\begin{aligned} \epsilon\pi_1 &= \epsilon(a+b\sqrt{l}) \equiv -\epsilon\sqrt{l} \pmod{4}, \\ -\epsilon\pi_2 &= -\epsilon(a-b\sqrt{l}) \equiv -\epsilon\sqrt{l} \pmod{4}, \end{aligned}$$

which is a square in k (see [9, Lemma A]).

By the disjunction of cases, one can easily show the following:

- (1) If $-\epsilon\pi_1 \equiv X^2 \pmod{4}$ and $\epsilon\pi_2 \equiv X^2 \pmod{4}$ have solutions in k , then $(a \equiv 2 \text{ and } b \equiv -1 \pmod{4})$ or $(a \equiv 0 \text{ and } b \equiv 1 \pmod{4})$.
- (2) If $\epsilon\pi_1 \equiv X^2 \pmod{4}$ and $-\epsilon\pi_2 \equiv X^2 \pmod{4}$ have solutions in k , then $(a \equiv 2 \text{ and } b \equiv 1 \pmod{4})$ or $(a \equiv 0 \text{ and } b \equiv -1 \pmod{4})$.

On the other hand, we know from [2, Proof of Lemma 8.2] that

$$\left(\frac{ab}{p}\right) = (-1)^{\frac{b-1}{2}} (-1)^j \left(\frac{2}{p}\right)^j \left(\frac{p}{l}\right)_4,$$

where $j = v_2(a) \geq 1$ is the 2-adic valuation of a . So, under our conditions, we have

$$\left(\frac{ab}{p}\right) = (-1)^{\frac{b-1}{2}} (-1)^j \left(\frac{2}{p}\right)^{j+1},$$

and hence

$$\left(\frac{2ab}{p}\right) = (-1)^{\frac{b-1}{2}} (-1)^j \left(\frac{2}{p}\right)^j.$$

Finally, we have:

- (1) $-\epsilon\pi_1 \equiv X^2 \pmod{4}$ and $\epsilon\pi_2 \equiv X^2 \pmod{4}$ have solutions in k if and only if $(a \equiv 2 \text{ and } b \equiv -1 \pmod{4})$ or $(a \equiv 0 \text{ and } b \equiv 1 \pmod{4})$, i.e., $\left(\frac{2ab}{p}\right) = 1$.
- (2) $\epsilon\pi_1 \equiv X^2 \pmod{4}$ and $-\epsilon\pi_2 \equiv X^2 \pmod{4}$ have solutions in k if and only if $(a \equiv 2 \text{ and } b \equiv 1 \pmod{4})$ or $(a \equiv 0 \text{ and } b \equiv -1 \pmod{4})$, i.e., $\left(\frac{2ab}{p}\right) = -1$. $\square$

Lemma 3.2. Let $p \equiv 3 \pmod{4}$ and $l \equiv 5 \pmod{8}$ be different primes. Then the genus field of $K = k(\sqrt{-2p\epsilon\sqrt{l}})$ is $K^{(*)} = K(\sqrt{-p}, \sqrt{-2})$.

Proof. The proof is the same as the one for [2, Lemma 3.4]. $\square$

Let p and l be different primes as specified by equation (2.1). Then $h_2(-pl)$, the 2-class number of $k_1 = \mathbb{Q}(\sqrt{-pl})$, is 2^{n+1} , where $n \geq 1$ (see [13]). Put $k_2 = \mathbb{Q}(\sqrt{-p})$ and $R = kk_1k_2$. Then we get the following lemma:

Lemma 3.3. The 2-class group $\mathbf{C}_{R,2}$ of R is cyclic of order 2^n . It is generated by the class of an ideal $\mathfrak{J} \in R$ such that $\mathfrak{J}^{2^{n-1}} \sim \mathfrak{p}_i^{h_0} \sim V_i^v$, where $\mathfrak{p}_i$ (resp. V_i) is a prime ideal of R above p (resp. l) and v is the odd part of the class number of k_2 .

Proof. First, prove that P , the prime ideal of k_1 above p , is not principal. If not, there exists $\alpha \in k_1$ such that $P = (\alpha)$. This yields $P^2 = (p) = (\alpha^2)$. Hence, by putting $\alpha = x + y\sqrt{-pl}$ with $x, y \in \mathbb{Q}$, we get

$$x^2 - ply^2 + 2xy\sqrt{-pl} = \pm p.$$

This is equivalent to

$$\begin{cases} x = 0, \\ -ly^2 = \pm 1 \end{cases} \quad \text{or} \quad \begin{cases} y = 0, \\ x^2 = \pm p, \end{cases}$$

which is absurd. Similarly, one can prove that V , the prime ideal of k_1 above l , is not principal. Hence $P^2 \sim V^2 \sim 1$. Moreover, according to [13], $\mathbf{C}_{k_1,2}$ is cyclic of order 2^{n+1} , so generated by some class $[I]$ such that $P \sim V \sim I^{2^n}$.

Let $\mathfrak{p}_i$ and V_i be the prime ideals of R above p and l , respectively. Then $\mathfrak{p}_i^{h_0}$ and V_i^v are not principal in R since their norms are not principal in k_1 . Therefore, $(\mathfrak{p}_i^{h_0})^2 \sim (V_i^v)^2 \sim 1$.

Since the class number of k_2 is odd, then, from [11], the rank of the 2-class group $\mathbf{C}_{R,2}$ is $t - 1 - e$, where t is the number of primes (finite and infinite) of k_2 which ramify in R and $2^e = [E_{k_2} : E_{k_2} \cap N_{R/k_2}(R^\times)]$. Then we can easily prove that $\mathbf{C}_{R,2}$ is of rank 1, i.e., it is cyclic of order $\frac{h_2(-pl)}{2} = 2^n$.

Finally, the ideal I generates a subgroup of $\mathbf{C}_{R,2}$ of order 2^n since $I^{2^n} \sim P$ (in $\mathbf{C}_{R,2}$) and P capitulates in R . Then $\mathbf{C}_{R,2} = \langle [I] \rangle$ and $J^{2^{n-1}} \sim \mathfrak{p}_i^{h_0} \sim V_i^v$. $\square$

Remark 3.4. Note the following:

(1) If $p \equiv 3 \pmod{8}$, then, by genus theory, 2_{k_1} , a prime ideal of k_1 above 2, is not a square in k_1 since

$$\left(\frac{2, -pl}{p}\right) = \left(\frac{2}{p}\right) = -1 \quad \text{and} \quad \left(\frac{2, -pl}{l}\right) = \left(\frac{2}{l}\right) = -1.$$

Hence $I \sim 2_{k_1}^u$, where u is the odd part of the class number of k_1 . Thus $J \sim 2_R^u$, where 2_R is a prime ideal of R above 2. Moreover, $n \geq 2$ (see [13, Théorème 8]).

(2) If $p \equiv 7 \pmod{8}$, then $2_R^v \sim 1$ since $2_{k_2}^v \sim 1$ and 2_{k_2} remains inert in R . In this case, $n = 1$; see [13, Théorème 8].

Now, using the results of M. N. Gras [12], we will determine an FSU of some real cyclic quartic number field. Let L be a real cyclic quartic number field of Galois group $H = \langle \sigma_L \rangle$ and of quadratic subfield k , and let ϵ be the fundamental unit of k . Let f_L (resp. m_k) be the conductor of L (resp. k). Then $f_L = m_k \cdot g$, and the discriminant of L is equal to $m_k \cdot f_L^2$. It is known that $L \subset \mathbb{Q}(\xi_{f_L})$.

Denote by χ_L a rational character of $\mathbb{Q}(\xi_{f_L})$, and put $E_{\chi_L} = \{\omega \in E_L : \omega^{1+\sigma_L^2} = \pm 1\}$. Let $|E_L|$ (resp. $|E_k|$, $|E_{\chi_L}|$) be the group of the absolute values of E_L (resp. E_k , E_{χ_L}), $|E^L| = |E_k| \oplus |E_{\chi_L}|$, $Q_L = [|E_L| : |E^L|]$, and let ϵ_{χ_L} be a generator of E_{χ_L} . From [12], we have $Q_L = 1$ or 2, which gives two possible structures for the unit group of L :

- (i) If $Q_L = 1$, we have $E_L = \langle -1, \epsilon, \epsilon_{\chi_L}, \epsilon_{\chi_L}^{\sigma_L} \rangle$;
 - (ii) If $Q_L = 2$, we have $E_L = \langle -1, \xi_L, \xi_L^{\sigma_L}, \xi_L^{\sigma_L^2} \rangle$ with $\xi_L^2 = \pm \epsilon \epsilon_{\chi_L}^{1-\sigma_L}$, $\xi_L^{1+\sigma_L} = \pm \epsilon_{\chi_L}$ and $\xi_L^{1+\sigma_L^2} = \pm \epsilon$.
- Note that ξ_L is called a Minkowski unit.

Proposition 3.5. Let $l \equiv 5 \pmod{8}$ be a prime number. Put $L = k(\sqrt{2\epsilon\sqrt{l}})$; then $Q_L = 1$.

Proof. It is known that $Q_L = 2$ if and only if L admits a Minkowski unit, and, from [8, Théorème II.2], the field L admits a Minkowski unit if and only if the unit ϵ is norm in L .

Let $\mathfrak{p}$ be a prime ideal of k , so we have:

- If $\mathfrak{p}$ is not above l and 2, then $v_{\mathfrak{p}}(2\epsilon\sqrt{l}) = 0$, so

$$\left(\frac{\epsilon, 2\epsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{\epsilon}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(2\epsilon\sqrt{l})} = 1.$$

- If $\mathfrak{p}$ lies above l , then $v_{\mathfrak{p}}(2\epsilon\sqrt{l}) = 1$. So, by [2, Proof of Proposition 4.1], we get

$$\left(\frac{\epsilon, 2\epsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{\epsilon}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(2\epsilon\sqrt{l})} = \left(\frac{\epsilon}{\mathfrak{p}}\right) = \left(\frac{2}{l}\right) = -1.$$

This means that the unit ϵ is not norm in L . Thus the result follows. $\square$

Lemma 3.6. Let $p \equiv 3 \pmod{4}$ and $l \equiv 5 \pmod{8}$ be different primes. Then the Hasse unit index of

$$F_5 = k(\sqrt{2\epsilon\sqrt{l}}, \sqrt{-p})$$

is $Q_{F_5} = 1$.

Proof. Let λ be a unit of F_5^+ . Prove that the index Q_{F_5} equals 1, which is equivalent, by [1, Proposition 3], to proving that $p\lambda$ is not a square in F_5^+ . Suppose that $p\lambda$ is a square in F_5^+ . This is equivalent to $(p)_{F_5^+} = H^2$, where H is an ideal of F_5^+ , which contradicts the fact that p cannot ramify in F_5^+ . $\square$

Theorem 3.7. *Keep the assumptions and notations of Lemma 3.6. Then*

$$E_{F_5} = \begin{cases} \langle -1, \epsilon, \epsilon_{\chi_{F_5^+}}, \epsilon_{\chi_{F_5^+}}^{\sigma_{F_5^+}} \rangle & \text{if } p \neq 3, \\ \langle \xi_6, \epsilon, \epsilon_{\chi_{F_5^+}}, \epsilon_{\chi_{F_5^+}}^{\sigma_{F_5^+}} \rangle & \text{if } p = 3. \end{cases}$$

Proof. Since the Hasse unit index of F_5 is $Q_{F_5} = 1$, then $E_{F_5} = W_{F_5} E_{F_5^+}$. Thus the result follows. $\square$

4 The 2-class number of the genus field $K^{(*)}$

Let p and l be different primes as specified by equation (2.1). In this paragraph, we will compute the 2-class number of $K^{(*)}$, the genus field of K .

Proposition 4.1. *Put $F = k(\sqrt{2\epsilon\sqrt{l}}, \sqrt{-2})$. Then the class number $h(F^+)$ is odd. Moreover, $Q_F = 2$, and $h(F)$ is odd too.*

Proof. Since the class number of k is odd, then, by [11], the rank of 2-class group $\mathbf{C}_{F^+,2}$ of F^+ is given by $\text{rank}(\mathbf{C}_{F^+,2}) = t - 1 - e$, where t is the number of primes (finite and infinite) of k that ramify in F^+ and the integer e is defined by $2^e = [E_k : E_k \cap N_{F^+/k}(F^{+\times})]$. We easily check that the primes $(2)_k$ and $\sqrt{l}$ ramify in F^+ . Hence $t = 2$, and therefore, $\text{rank}(\mathbf{C}_{F^+,2}) = 1 - e$. We know, from the proof of Proposition 3.5, that ϵ is not a norm in F^+ , whereas -1 is. Then $E_k/E_k \cap N_{F^+/k}(F^{+\times}) = \{\bar{1}, \bar{\epsilon}\}$. Therefore, $e = 1$, so $\mathbf{C}_{F^+,2}$ is trivial.

Let us prove that $Q_F = 2$. For this, it suffices to prove that 2λ is a square in F^+ (see [1, Proposition 3]), where λ is a unit of F_5^+ . In other words, it suffices to prove that $(2)_{F^+} = H^2$, where H is a principal ideal in F^+ . As 2 ramifies in F^+ , there exists H in F^+ such that $(2)_{F^+} = H^2$. If H is not principal, then the class of H is of order 2, which is absurd since $\mathbf{C}_{F^+,2}$ is trivial.

Next, we will compute the 2-class number $h_2(F)$ of F . For this, notice that F/k is a V_4 -extension of CM-type fields (Figure 1), of quadratic subextensions F^+ , $K' = k(\sqrt{-2})$ and $L' = k(\sqrt{-\epsilon\sqrt{l}})$. Then, from [14], we have

$$h_2(F) = \frac{Q_F}{Q_{K'}Q_{L'}} \cdot \frac{w_F}{w_{L'}w_{K'}} \cdot \frac{h_2(F^+)h_2(K')h_2(L')}{h_2(l)^2}.$$

Note that $w_{K'} = w_F = w_{L'} = 2$. Furthermore, from [17, Theorem 10.4], we have $h_2(L') = 1$ since $f_{L'} = l$ and $Q_{K'} = Q_{L'} = 1$. From this, we deduce that $h_2(F) = h_2(K')$. As K' is an unramified extension of $\mathbb{Q}(\sqrt{-2}l)$ and the 2-class group of $\mathbb{Q}(\sqrt{-2}l)$ is cyclic of order 2, then $h_2(K') = \frac{h_2(-2l)}{2} = 1$. Thus $h_2(F) = 1$. $\square$

Figure 1: Subfields of F/k .

Proposition 4.2. *Put $F' = k(\sqrt{2\epsilon\sqrt{l}}, \sqrt{2p})$. Then the class number of F' is odd.*

Proof. By Proposition 4.1, the class number of $L = F^+ = k(\sqrt{2\epsilon\sqrt{l}})$ is odd, and, by Proposition 3.5,

$$E_L = \langle -1, \epsilon, \epsilon_{\chi_L}, \epsilon_{\chi_L}^{\sigma_L} \rangle.$$

Then, from [11] again, the rank of 2-class group $\mathbf{C}_{F',2}$ of F' is given by $\text{rank}(\mathbf{C}_{F',2}) = t - 1 - e$, where t is the number of primes (finite and infinite) of L that ramify in F' and $2^e = [E_L : E_L \cap N_{F'/L}(F'^{\times})]$. We easily check

that $t = 2$, and therefore, $\text{rank}(\mathbf{C}_{F',2}) = 1 - e$. In the following, we show that $e = 1$, which is equivalent to showing that $E_L \cap N_{F'/L}(F'^\times) = \langle -1, \epsilon, \epsilon_{\chi_L} \cdot \epsilon_{\chi_L}^{\sigma_L} \rangle$.

Let $\mathfrak{p}$ be a prime ideal of L . So we have:

- For all $\alpha \in \{-1, \epsilon\}$, α is a norm in F' . In fact, we have:

- If $\mathfrak{p}$ is not above p and 2 , then $v_{\mathfrak{p}}(2p) = 0$, so

$$\left(\frac{\alpha, 2p}{\mathfrak{p}}\right) = \left(\frac{\alpha}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(2p)} = 1.$$

- If $\mathfrak{p}$ lies above p , then $v_{\mathfrak{p}}(2p) = 1$, and thus

$$\left(\frac{\alpha, 2p}{\mathfrak{p}}\right) = \left(\frac{\alpha}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(2p)} = \left(\frac{\alpha}{\mathfrak{p}}\right) = \left(\frac{N_{L/k}(\alpha)}{\mathfrak{p}_k}\right) = \left(\frac{\alpha^2}{\mathfrak{p}_k}\right) = 1.$$

- If $\mathfrak{p}$ lies above 2 , then

$$\left(\frac{\alpha, 2p}{\mathfrak{p}}\right) = \left(\frac{N_{L/k}(\alpha), 2p}{\mathfrak{p}_k}\right) = \left(\frac{\alpha^2, 2p}{\mathfrak{p}_k}\right) = 1.$$

- ϵ_{χ_L} is not a norm in F' . In fact, we have:

- If $\mathfrak{p}$ is not above p and 2 , then $v_{\mathfrak{p}}(2p) = 0$, and thus

$$\left(\frac{\epsilon_{\chi_L}, 2p}{\mathfrak{p}}\right) = \left(\frac{\epsilon_{\chi_L}}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(2p)} = 1.$$

- If $\mathfrak{p}$ lies above p , then $v_{\mathfrak{p}}(2p) = 1$, and thus

$$\left(\frac{\epsilon_{\chi_L}, 2p}{\mathfrak{p}}\right) = \left(\frac{\epsilon_{\chi_L}}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(2p)} = \left(\frac{\epsilon_{\chi_L}}{\mathfrak{p}}\right) = \left(\frac{N_{L/k}(\epsilon_{\chi_L})}{\mathfrak{p}_k}\right) = \left(\frac{-1}{\mathfrak{p}_k}\right) = \left(\frac{-1}{p}\right) = -1.$$

- Similarly, we prove that $\epsilon_{\chi_L}^{\sigma_L}$ is not a norm in F' .

Consequently, $E_L/E_L \cap N_{F'/L}(F'^\times) = \{1, \epsilon_{\chi_L}\}$. Therefore, $e = 1$, and thus $\mathbf{C}_{F',2}$ is trivial. $\square$

We are now able to compute the 2-class number of $K^{(*)}$.

Theorem 4.3. Put $F_5 = K(\sqrt{-p})$. Then $h_2(K^{(*)}) = \frac{h_2(F_5)}{2} = 2^{n+1}$, where $n \geq 1$.

Proof. First, we compute the 2-class number $h_2(F_5)$ of F_5 . For this, consider Figure 2.

Figure 2: Subfields of F_5/k .

It is easy to see that F_5/k is a V_4 -extension of CM-type fields of quadratic subextensions L , K and $R = k(\sqrt{-p})$. Then, from [14], we have

$$h_2(F_5) = \frac{Q_{F_5}}{Q_K Q_R} \cdot \frac{w_{F_5}}{w_K w_R} \cdot \frac{h_2(R)h_2(K)h_2(L)}{h_2(l)^2}.$$

Note that $w_R = w_{F_5}$ and $w_K = 2$. Moreover, we have $Q_{F_5} = 1$ (see Lemma 3.6), and we easily see that $Q_K = Q_R = 1$. Then,

$$h_2(F_5) = \frac{1}{2} \cdot 8 \cdot h_2(R) = 4 \cdot h_2(R).$$

On the other hand, we easily check that $h_2(R) = \frac{h_2(-pl)}{2}$, and from [13], we have $h_2(-pl) = 2^{n+1}$, where $n \geq 1$ (under our conditions). Hence

$$h_2(F_5) = 2^{n+2}.$$

Figure 3: Subfields of $K^{(*)}/L$.

Now, we prove that $h_2(K^{(*)}) = \frac{h_2(F_5)}{2} = 2^{n+1}$ with $n \geq 1$. For this, note that $K^{(*)}/L$ is a V_4 -extension of CM-type fields (Figure 3), of quadratic subextensions $F_5, F' = L(\sqrt{2p})$ and $F = L(\sqrt{-2})$. Then, from [14] again, we have

$$h_2(K^{(*)}) = \frac{Q_{K^{(*)}}}{Q_{F_5} Q_F} \cdot \frac{w_{K^{(*)}}}{w_{F_5} w_F} \cdot \frac{h_2(F') h_2(F) h_2(F_5)}{h_2(L)^2}.$$

Note that

$$W_{F_5} = W_{K^{(*)}} = \begin{cases} \{+1, -1\} & \text{if } p \neq 3, \\ \langle \xi_6 \rangle & \text{if } p = 3, \end{cases} \quad \text{and} \quad W_F = \{+1, -1\}.$$

Then $w_{F_5} = w_{K^{(*)}}$ and $w_F = 2$. So Lemma 3.6 and Propositions 4.1 and 4.2 yield that $Q_{F_5} = 1$, $Q_F = 2$ and $h_2(F') = h_2(F) = h_2(L) = 1$, and therefore,

$$h_2(K^{(*)}) = \frac{Q_{K^{(*)}}}{2} \cdot \frac{1}{2} \cdot h_2(F_5).$$

Finally, from the proof of Proposition 4.1, we have $(2)_L = H_L^2$, where H_L is a principal ideal in L . On the other hand, H_L remains inert in $K^{(*)+}$, so $H_{K^{(*)+}}$, the prime ideal of $K^{(*)+}$ above 2, is principal. Therefore, $(2)_{K^{(*)+}} = H_{K^{(*)+}}^2$, and thus [1, Proposition 3] implies that $Q_{K^{(*)}} = 2$. From this, we deduce that

$$h_2(K^{(*)}) = \frac{h_2(F_5)}{2} = 2^{n+1}. \quad \square$$

5 Proof of Theorem 2.1

Let p and l be different primes as specified by equation (2.1). Denote by ϵ and by h_0 the fundamental unit and the class number of $k = \mathbb{Q}(\sqrt{l})$, respectively. Put $K = k(\sqrt{-2p\epsilon\sqrt{l}})$. Recall that $\mathcal{P}_1, \mathcal{P}_2$ are the prime ideals in K above p and $\mathcal{H}$ is the prime ideal in K above 2. Moreover, $\mathcal{P}_i^{2h_0} = (\pi_i)$ and $\mathcal{H}^2 = (2)$.

5.1 Generators of the 2-class group of K

Let us show that $\mathbf{C}_{K,2} = \langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}], [\mathcal{H}] \rangle$. For this, it suffices to show that $[\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}]$ and $[\mathcal{H}]$ are of order 2 and pairwise different.

- $[\mathcal{P}_1^{h_0} \mathcal{P}_2^{h_0}]$ is of order 2. In fact, if $\mathcal{P}_1^{h_0} \mathcal{P}_2^{h_0} \neq (\alpha)$ for some $\alpha \in K$, this yields that $(p^{h_0}) = (\alpha^2)$. Then there exists ϵ' , a unit of K , such that $p^{h_0} \epsilon' = \alpha^2$, and thus

$$p^{h_0} \epsilon' = (c + d\sqrt{-2p\epsilon\sqrt{l}})^2 = c^2 - 2p\epsilon\sqrt{l}d^2 + 2cd\sqrt{-2p\epsilon\sqrt{l}}$$

with c and d in k . As $\{\epsilon\}$ is an FSU of K and $\sqrt{-1} \notin K$, then $p^{h_0} \epsilon' \in k$, and therefore, $c = 0$ or $d = 0$. If $d = 0$, then $p^{h_0} \epsilon' = c^2$, and thus $\pm p^{h_0} = c^2$ or $p^{h_0} \epsilon = c^2$ in k , which gives that $\sqrt{\pm p} \in k$ in the first case and $\sqrt{-1} \in \mathbb{Q}$ in the second case, but this is impossible. Similarly, if $c = 0$, we find that $\pm l$ is a square in $\mathbb{Q}$, which is absurd.

- $[\mathcal{P}_i^{h_0}]$ is of order 2. In fact, assume that $\mathcal{P}_i^{h_0} = (\alpha)$ for $\alpha \in K$. Then $\mathcal{P}_i^{2h_0} = (\alpha)^2$, which is equivalent to $(\pi_i) = (\alpha^2)$. Then there exists ϵ' , a unit of K , such that

$$\pi_i \epsilon' = \alpha^2 = (c + d\sqrt{-2p\epsilon\sqrt{l}})^2 = c^2 - 2p\epsilon\sqrt{l}d^2 + 2cd\sqrt{-2p\epsilon\sqrt{l}}$$

with c and d in k , and then $c = 0$ or $d = 0$. If $d = 0$, multiplying by π_j for $j \neq i$, we get $p^{h_0}\epsilon' = \pi_j c^2$. So, by applying the norm in $k/\mathbb{Q}$, we find that $\pm p^{h_0} = N_{k/\mathbb{Q}}(c)^2$. This means that $\pm p$ is a square in $\mathbb{Q}$, which is impossible. In a similar way, if $c = 0$, we get $\pm lp$ is a square in $\mathbb{Q}$, which is impossible too.

Similarly, we can show that $[\mathcal{H}]$, $[\mathcal{H}\mathcal{P}_i^{h_0}]$ and $[\mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}]$ are of order 2. Thus $\langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}], [\mathcal{H}] \rangle$ is of type $(2, 2, 2)$, and then $\mathbf{C}_{K,2} = \langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}], [\mathcal{H}] \rangle$.

Remark 5.1. Let $\mathcal{V}$ be a prime ideal of K above l . Then $\mathcal{V} \sim \mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}$ since $\mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}\mathcal{V} = (\sqrt{-2p\epsilon\sqrt{l}})$, where $\sqrt{-2p\epsilon\sqrt{l}} \in K$.

5.2 The unramified extensions of K within $K_2^{(1)}$

We know from Lemma 3.2 that the genus field of K is $K^{(*)} = K(\sqrt{-p}, \sqrt{-2})$, and therefore, its subextensions $F_3 = K(\sqrt{2p})$, $F_4 = K(\sqrt{-2})$ and $F_5 = K(\sqrt{-p})$ are unramified over K and absolutely abelian.

From Lemma 3.1, if $(\frac{2ab}{p}) = 1$, then the equation $-\epsilon\pi_1 \equiv X^2 \pmod{4}$ admits a solution in k . Moreover, since $(-\epsilon\pi_1) = (\mathcal{P}_1^{h_0})^2$, where $\mathcal{P}_1$ is an ideal of K , so $K(\sqrt{-\epsilon\pi_1})$ is an unramified quadratic extension of K . This implies that $F_1 = K(\sqrt{-\epsilon\pi_1})$ is non-Galois over $\mathbb{Q}$.

Since the extensions F_i are pairwise different for $i \in \{1, 3, 4, 5\}$, then one can easily deduce that

$$F_2 = K(\sqrt{-2\epsilon\pi_2}), \quad F_6 = K(\sqrt{2\epsilon\pi_1}) \quad \text{and} \quad F_7 = K(\sqrt{\epsilon\pi_2})$$

are pairwise different and non-Galois over $\mathbb{Q}$.

Finally, as $-\epsilon = \epsilon'^{-1} = \epsilon'^{-2}\epsilon'$, where ϵ' is the conjugate of ϵ , hence

$$F_1 = K(\sqrt{-\epsilon\pi_1}) = K(\sqrt{\epsilon'\pi_1}) \quad \text{and} \quad F_2 = K(\sqrt{-2\epsilon\pi_2}) = K(\sqrt{2\epsilon'\pi_2}).$$

Thus $F_1 \simeq F_7$ and $F_2 \simeq F_6$.

Regarding the biquadratic extensions L_i , note that each extension is gotten by composing two different unramified quadratic extensions of K . Then L_i , $1 \leq i \leq 7$, are the unramified biquadratic extensions of K within $K_2^{(1)}$. By using the fact that $F_1 \simeq F_7$ and $F_2 \simeq F_6$, we deduce that $L_1 \simeq L_3$ and $L_6 \simeq L_7$.

If $(\frac{2ab}{p}) = -1$, the results are proved similarly.

6 Proof of Theorem 2.2

Let p and l be different primes as specified by equation (2.1). Recall that $R = k(\sqrt{-p})$ and $L = k(\sqrt{2\epsilon\sqrt{l}})$.

6.1 The 2-class group of F_5

From Figure 2, we see that F_5/R , F_5/L are ramified quadratic extensions, whereas F_5/K is not. Hence, according to class field theory,

$$[\mathbf{C}_{K,2} : N_{F_5/K}(\mathbf{C}_{F_5,2})] = 2, \quad \mathbf{C}_{R,2} = N_{F_5/R}(\mathbf{C}_{F_5,2}) \quad \text{and} \quad \mathbf{C}_{L,2} = N_{F_5/L}(\mathbf{C}_{F_5,2}).$$

Since $F_5 = K(\sqrt{-p}) = K(\sqrt{2\epsilon\sqrt{l}})$ and

$$N_{F_5/K}(\mathbf{C}_{F_5,2}) = \left\{ [Q] \in \mathbf{C}_{K,2} \mid \left(\frac{2\epsilon\sqrt{l}}{Q} \right) = 1 \right\} = \left\{ [Q] \in \mathbf{C}_{K,2} \mid \left(\frac{-p}{Q} \right) = 1 \right\},$$

then $N_{F_5/K}(\mathbf{C}_{F_5,2}) = \langle [\mathcal{H}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$.

In fact, from [9], we have

$$\left(\frac{2\epsilon\sqrt{l}}{\mathcal{P}_i^{h_0}}\right) = -\left(\frac{2}{p}\right)\left(\frac{p}{l}\right)_4 = -\left(\frac{2}{p}\right)^2 = -1,$$

and then

$$\left(\frac{2\epsilon\sqrt{l}}{\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}}\right) = \left(\frac{2\epsilon\sqrt{l}}{\mathcal{P}_1^{h_0}}\right)\left(\frac{2\epsilon\sqrt{l}}{\mathcal{P}_2^{h_0}}\right) = 1.$$

Moreover, let $\mathfrak{z}_k$ be a prime ideal of k above 2. So

$$\left(\frac{-p}{\mathcal{H}}\right) = \left(\frac{-p}{\mathfrak{z}_k}\right) = \left(\frac{-p, 2}{\mathfrak{z}_k}\right) = \left(\frac{N_{k/\mathbb{Q}}(-p), 2}{2}\right) = \left(\frac{p^2, 2}{2}\right) = 1.$$

Thus $[\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}]$ and $[\mathcal{H}]$ are norms in F_5/K .

Let us now compute $\kappa_{F_5/K}$. From Theorem 3.7, we get

$$E_{F_5} = \langle -1, \epsilon, \epsilon_{\chi_{F_5^+}}, \epsilon_{\chi_{F_5^+}}^{\sigma_{F_5^+}} \rangle \quad \text{and} \quad E_K = \langle -1, \epsilon \rangle, \quad \text{so} \quad N_{F_5/K}(E_{F_5}) = \langle -1, \epsilon^2 \rangle.$$

Therefore, $[E_K : N_{F_5/K}(E_{F_5})] = 2$, which implies that $\#\kappa_{F_5/K} = 4$. Moreover, we showed in the proof of Proposition 4.1 that $(2)_L = H^2$, H is principal in $L = F^+$ and $H\mathcal{O}_{F_5} = \mathcal{H}\mathcal{O}_{F_5}$. So, necessarily, $\mathcal{H}$ capitulates in F_5 . In addition, it is known that $\mathcal{P}_1^{2h_0}\mathcal{P}_2^{2h_0} = (p^{h_0})$, which means that $(\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}) = (\sqrt{p^{h_0}})$, and hence $\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}$ capitulates in F_5 . Thus $\kappa_{F_5/K} = \langle [\mathcal{H}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$.

On the other hand, we know from Proposition 4.1 and Lemma 3.3 that $\mathbf{C}_{L,2}$ is trivial and $\mathbf{C}_{R,2} = \langle [J] \rangle$. Let $\mathfrak{J}$ be a prime ideal of F_5 above l . Then we claim $(\mathfrak{J}^v)^4 \sim 1$.

To this end, let s and t be the elements of order 2 in $\text{Gal}(F_5/k)$ which fix R and K , respectively, so st fixes L . Using the relation $2 + (1 + s + t + st) = (1 + s) + (1 + t) + (1 + st)$ of the group ring $\mathbb{Z}[\text{Gal}(F_5/k)]$ and taking into account that the 2-class number of k is 1, we get

$$(\mathfrak{J}^v)^2 \sim (\mathfrak{J}^v)^{1+s}(\mathfrak{J}^v)^{1+t}(\mathfrak{J}^v)^{1+st} \sim V_i^v \mathcal{V}^v \sim V_i^v \mathcal{H} \mathcal{P}_1^{h_0} \mathcal{P}_2^{h_0} \sim V_i^v$$

since $\mathcal{V}^v \sim \mathcal{V} \sim \mathcal{H} \mathcal{P}_1^{h_0} \mathcal{P}_2^{h_0}$ (by Remark 5.1) and $\mathcal{H} \mathcal{P}_1^{h_0} \mathcal{P}_2^{h_0} \in \kappa_{F_5/K}$ (V_i is the prime ideal of $R = k_1 k_2$ above l , see Lemma 3.3). Therefore, $(\mathfrak{J}^v)^4 \sim (V_i^v)^2 \sim 1$.

Let $\mathcal{H}_0$ be a prime ideal of F_5 above 2. So, from Remark 3.4, we have:

- (1) If $p \equiv 3 \pmod{8}$, then $(\mathcal{H}_0^u)^2 \sim (\mathcal{H}_0^u)^{1+s}(\mathcal{H}_0^u)^{1+t}(\mathcal{H}_0^u)^{1+st} \sim 2_R^u \mathcal{H}^u \sim 2_R^u$, and therefore, $(\mathcal{H}_0^u)^{2^{n+1}} \sim (2_R^u)^{2^n} \sim 1$.
- (2) If $p \equiv 7 \pmod{8}$, then $(\mathcal{H}_0^v)^2 \sim (\mathcal{H}_0^v)^{1+s}(\mathcal{H}_0^v)^{1+t}(\mathcal{H}_0^v)^{1+st} \sim 2_R^v \mathcal{H}^v \sim 1$.

Moreover, we have:

- (1) If $p \equiv 3 \pmod{8}$, then $(\mathfrak{J}^v)^2 \sim V_i^v \sim \mathcal{J}^{2^{n-1}} \sim (\mathcal{H}_0^u)^{2^n}$. Furthermore, $\mathfrak{J}^v \nmid (\mathcal{H}_0^u)^{2^{n-1}}$. If not, by applying the norm of F_5/K , we get $\mathcal{H} \mathcal{P}_1^{h_0} \mathcal{P}_2^{h_0} \sim (\mathcal{H}^u)^{2^{n-1}} \sim 1$ (in K), which is absurd.
- (2) If $p \equiv 7 \pmod{8}$, then $(\mathfrak{J}^v)^2 \nmid \mathcal{H}_0^v$. If not, by applying the norm of F_5/K , we get $\mathcal{H}^2 \mathcal{P}_1^{2h_0} \mathcal{P}_2^{2h_0} \sim \mathcal{H} \sim 1$ (in K), which is absurd.

Consequently,

$$\mathbf{C}_{F_5,2} = \begin{cases} \langle [\mathcal{H}_0^u], [\mathfrak{J}^v] \rangle \simeq (2, 2^{n+1}) & \text{if } p \equiv 3 \pmod{8}, \\ \langle [\mathcal{H}_0^v], [\mathfrak{J}^v] \rangle \simeq (2, 4) & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Note that $\mathcal{P}_i$ remains inert and does not capitulate in F_5 . Then $\mathcal{P}_1^{2h_0} \sim 1$ (in F_5), and therefore,

$$\mathcal{P}_1^{h_0} \sim \begin{cases} (\mathfrak{J}^v)^2 \sim (\mathcal{H}_0^u)^{2^n} & \text{in the first case,} \\ \mathcal{H}_0^v \text{ or } (\mathfrak{J}^v)^2 & \text{in the second case.} \end{cases}$$

If $\mathcal{P}_1^{h_0} \sim \mathcal{H}_0^v$, by applying the norm of F_5/K , we get $\mathcal{P}_1^{2h_0} \sim \mathcal{H} \sim 1$, which is absurd. Thus $\mathcal{P}_1^{h_0} \sim (\mathfrak{J}^v)^2$.

6.2 The Hilbert 2-class field tower of K

Let us prove that the Hilbert 2-class field tower of K stops at $K_2^{(2)}$. From Theorem 4.3, we know that $h_2(K^{(*)}) = \frac{h_2(F_5)}{2}$ and the 2-rank of $\mathbf{C}_{F_5,2}$ equals 2. So, by [7, Proposition 7], the Hilbert 2-class field tower of F_5 stops at the first stage $F_{5,2}^{(1)}$. Therefore, the Hilbert 2-class field tower of K stops at $K_2^{(2)}$ since $K \subset F_5$.

6.3 The generators of $G = \text{Gal}(K_2^{(2)}/K)$

Let us compute the generators of $G = \text{Gal}(K_2^{(2)}/K)$. Let $\left(\frac{K_2^{(2)}/F_5}{\mathcal{H}_0^u}\right)$ denote the Artin symbol for $K_2^{(2)}/F_5$, and put

$$\sigma = \left(\frac{K_2^{(2)}/F_5}{\mathfrak{J}^v}\right) \quad \text{and} \quad \tau = \begin{cases} \left(\frac{K_2^{(2)}/F_5}{\mathcal{H}_0^u}\right) & \text{if } p \equiv 3 \pmod{8}, \\ \left(\frac{K_2^{(2)}/F_5}{\mathcal{H}_0^v}\right) & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Then σ and τ generate the abelian subgroup of G ,

$$\text{Gal}(K_2^{(2)}/F_5) \simeq \begin{cases} (2, 2^{n+1}) & \text{if } p \equiv 3 \pmod{8}, \\ (2, 4) & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

with

$$\begin{cases} \tau^{2^{n+1}} = \sigma^4 = 1 \text{ and } \tau^{2^n} = \sigma^2 & \text{if } p \equiv 3 \pmod{8}, \\ \tau^2 = \sigma^4 = 1 & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Also put

$$\rho = \left(\frac{K_2^{(2)}/K}{\mathcal{P}_1^{h_0}}\right),$$

so ρ restricts to the non-trivial automorphism of F_5/K since $\mathcal{P}_1^{h_0}$ is not a norm from F_5/K , and hence $G = \langle \sigma, \tau, \rho \rangle$.

Furthermore, the following relations hold:

- Since $\mathcal{P}_1^{h_0} \sim (\mathfrak{J}^v)^2$,

$$\rho^2 = \left(\frac{K_2^{(2)}/K}{\mathcal{P}_1^{2h_0}}\right) = \left(\frac{K_2^{(2)}/K}{N_{F_5/K}(\mathcal{P}_1^{h_0})}\right) = \left(\frac{K_2^{(2)}/F_5}{\mathcal{P}_1^{h_0}}\right),$$

and thus $\rho^2 = \sigma^2$.

- Since $(\mathfrak{J}^v)^{1+\rho} = N_{F_5/K}(\mathfrak{J}^v) \sim \mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0} \sim 1$ in F_5 ,

$$\sigma\rho^{-1}\sigma\rho = \left(\frac{K_2^{(2)}/F_5}{(\mathfrak{J}^v)^{1+\rho}}\right) = 1,$$

and thus $[\sigma, \rho] = [\rho, \sigma] = \sigma^2$.

- Since $(\mathcal{H}_0^u)^{1+\rho} = N_{F_5/K}(\mathcal{H}_0^u) \sim \mathcal{H} \sim 1$ and $(\mathcal{H}_0^v)^{1+\rho} = N_{F_5/K}(\mathcal{H}_0^v) \sim \mathcal{H} \sim 1$ in F_5 , according as $p \equiv 3 \pmod{8}$ and $p \equiv 7 \pmod{8}$,

$$\tau\rho^{-1}\tau\rho = \begin{cases} \left(\frac{K_2^{(2)}/F_5}{(\mathcal{H}_0^u)^{1+\rho}}\right) = 1 & \text{if } p \equiv 3 \pmod{8}, \\ \left(\frac{K_2^{(2)}/F_5}{(\mathcal{H}_0^v)^{1+\rho}}\right) = 1 & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

and thus $[\tau, \rho] = \tau^{-2}$.

Hence

$$G = \begin{cases} \langle \rho, \tau, \sigma : \rho^4 = \sigma^4 = \tau^{2^{n+1}} = 1, \sigma^2 = \rho^2 = \tau^{2^n}, n \geq 2, \\ \quad [\tau, \sigma] = 1, [\sigma, \rho] = \sigma^2, [\rho, \tau] = \tau^2 \rangle & \text{if } p \equiv 3 \pmod{8}, \\ \langle \rho, \tau, \sigma : \rho^4 = \sigma^4 = \tau^2 = 1, \sigma^2 = \rho^2, \\ \quad [\tau, \sigma] = [\tau, \rho] = 1, [\sigma, \rho] = \sigma^2 \rangle & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

- (1) If $p \equiv 3 \pmod{8}$, then $G' = \langle \tau^2 \rangle$ since $[\tau, \sigma] = 1$, $[\rho, \tau] = \tau^2$, $[\rho, \sigma] = \sigma^2$ and $\sigma^2 = (\tau^2)^{2^{n-1}}$, and therefore, $\mathbf{C}_{K_2^{(1)}, 2}$ is cyclic of order 2^n .

- (2) If $p \equiv 7 \pmod{8}$, we easily check that $G' = \langle \sigma^2 \rangle$, and thus $\mathbf{C}_{K_2^{(1)}, 2}$ is cyclic of order 2.

Now, compute the coclass of G . Recall that the lower central series of G is inductively defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$, which is the subgroup of G generated by the set $\{[a, b] = a^{-1}b^{-1}ab \mid a \in \gamma_i(G), b \in G\}$.

So the coclass of G is defined to be $cc(G) = h - c$, where $\#G = 2^h$ and $c = c(G)$ is the nilpotency class of G , which is the smallest positive integer satisfying $\gamma_c(G) \neq \{1\}$ and $\gamma_{c+1}(G) = \{1\}$.

(1) If $p \equiv 3 \pmod{8}$, then $\gamma_1(G) = G$, $\gamma_2(G) = G' = \langle \tau^2 \rangle$ and

$$\gamma_{n+1}(G) = [\gamma_n(G), G] = \langle \tau^{2^n} \rangle, \quad \gamma_{n+2}(G) = [\gamma_{n+1}(G), G] = \langle \tau^{2^{n+1}} \rangle = \langle 1 \rangle.$$

So $c(G) = n + 1$, and $cc(G) = 2$.

(2) If $p \equiv 7 \pmod{8}$, then $\gamma_1(G) = G$, $\gamma_2(G) = G' = \langle \sigma^2 \rangle$ and $\gamma_3(G) = [G', G] = \langle 1 \rangle$. So $cc(G) = c(G) = 2$.

Finally, prove assertion (6) of Theorem 2.2.

(1) If $p \equiv 3 \pmod{8}$, then $[\rho, \sigma^2] = \sigma^4 = 1$, $[\rho, \sigma\tau^{2^{n-1}}] = 1$ and $\sigma^2 \neq \sigma\tau^{2^{n-1}}$. So $\langle \sigma^2, \sigma\tau^{2^{n-1}} \rangle = Z(G) \simeq (2, 2)$.

(2) If $p \equiv 7 \pmod{8}$, then $[\rho, \sigma^2] = \sigma^4 = 1$, $[\rho, \tau] = 1$ and $\sigma^2 \neq \tau$. So $\langle \sigma^2, \tau \rangle = Z(G) \simeq (2, 2)$.

7 Proof of Theorem 2.3

To prove this theorem, we need the following norm class groups $N_{F_i/K}(\mathbf{C}_{F_i,2})$. The results are summarized in Tables 1 and 2.

F_i	$N_{F_i/K}(\mathbf{C}_{F_i,2})$	
	$\left(\frac{2}{p}\right) = -1$	$\left(\frac{2}{p}\right) = 1$
$F_1 = K(\sqrt{-\epsilon\pi_1})$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_1^{h_0}] \rangle$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$
$F_2 = K(\sqrt{-2\epsilon\pi_2})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_2^{h_0}] \rangle$
$F_6 = K(\sqrt{2\epsilon\pi_1})$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_1^{h_0}] \rangle$
$F_7 = K(\sqrt{\epsilon\pi_2})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$
$F_3 = K(\sqrt{2p})$	$\langle [\mathcal{H}\mathcal{C}\mathcal{P}_i^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$
$F_4 = K(\sqrt{-2})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}\mathcal{C}\mathcal{P}_i^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$
$F_5 = K(\sqrt{-p})$	$\langle [\mathcal{H}\mathcal{C}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$	

Table 1: Norm class groups for the case $\left(\frac{2ab}{p}\right) = 1$.

F_i	$N_{F_i/K}(\mathbf{C}_{F_i,2})$	
	$\left(\frac{2}{p}\right) = -1$	$\left(\frac{2}{p}\right) = 1$
$F_1 = K(\sqrt{-2\epsilon\pi_1})$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_1^{h_0}] \rangle$
$F_2 = K(\sqrt{-\epsilon\pi_2})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$
$F_6 = K(\sqrt{\epsilon\pi_1})$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_1^{h_0}] \rangle$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$
$F_7 = K(\sqrt{2\epsilon\pi_2})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}] \rangle$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{C}\mathcal{P}_2^{h_0}] \rangle$
$F_3 = K(\sqrt{2p})$	$\langle [\mathcal{H}\mathcal{C}\mathcal{P}_i^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$
$F_4 = K(\sqrt{-2})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}\mathcal{C}\mathcal{P}_i^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$
$F_5 = K(\sqrt{-p})$	$\langle [\mathcal{H}\mathcal{C}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$	

Table 2: Norm class groups for the case $\left(\frac{2ab}{p}\right) = -1$.

Let us check some examples. Recall that, for $1 \leq i \leq 7$, $[\mathbf{C}_{K,2} : N_{F_i/K}(\mathbf{C}_{F_i,2})] = 2$.

Example. For F_5 , we already proved that $N_{F_5/K}(\mathbf{C}_{F_5,2}) = \langle [\mathcal{H}\mathcal{C}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$.

Example. For $F_3 = K(\sqrt{2p}) = K(\sqrt{-\epsilon\sqrt{l}})$, we have

$$\begin{aligned} \left(\frac{-\epsilon\sqrt{l}}{\mathcal{P}_i^{h_0}}\right) &= \left(\frac{-1}{\mathcal{P}_i^{h_0}}\right)\left(\frac{\epsilon\sqrt{l}}{\mathcal{P}_i^{h_0}}\right) = -\left(\frac{-1}{p^{h_0}}\right)\left(\frac{p}{l}\right)_4 = \left(\frac{2}{p}\right), \\ \left(\frac{-\epsilon\sqrt{l}}{\mathcal{H}}\right) &= \left(\frac{-\epsilon\sqrt{l}}{2k}\right) = \left(\frac{-\epsilon\sqrt{l}, 2}{2k}\right) = \left(\frac{N_{K/\mathbb{Q}}(-\epsilon\sqrt{l}), 2}{2}\right) = \left(\frac{l, 2}{2}\right) = \left(\frac{2}{l}\right) = -1. \end{aligned}$$

Then $N_{F_3/K}(\mathbf{C}_{F_3,2}) = \langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$ or $\langle [\mathcal{H}\mathcal{C}\mathcal{P}_i^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$, according as $\left(\frac{2}{p}\right) = 1$ or -1 .

Example. For $F_4 = K(\sqrt{-2}) = K(\sqrt{p\epsilon\sqrt{l}})$, we have

$$\begin{aligned} \left(\frac{-2}{\mathcal{P}_i^{h_0}}\right) &= -\left(\frac{2}{p}\right), \\ \left(\frac{p\epsilon\sqrt{l}}{\mathcal{H}}\right) &= \left(\frac{p\epsilon\sqrt{l}}{2k}\right) = \left(\frac{p\epsilon\sqrt{l}, 2}{2k}\right) = \left(\frac{N_{K/\mathbb{Q}}(p\epsilon\sqrt{l}), 2}{2}\right) = \left(\frac{p^2l, 2}{2}\right) = \left(\frac{l, 2}{2}\right) = \left(\frac{2}{l}\right) = -1. \end{aligned}$$

Then $N_{F_4/K}(\mathbf{C}_{F_4,2}) = \langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$ or $\langle [\mathcal{H}\mathcal{C}\mathcal{P}_i^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$, according as $\left(\frac{2}{p}\right) = -1$ or 1 .

Example. Consider as a last example $F_6 = K(\sqrt{2\epsilon\pi_1}) = K(\sqrt{-\pi_2\sqrt{l}})$, i.e., the case

$$\left(\frac{2ab}{p}\right) = 1 \iff \left(\frac{ab}{p}\right) = \left(\frac{2}{p}\right).$$

- The quadratic residue symbol is $(\frac{-\pi_2 \sqrt{l}}{\mathcal{P}_1^{h_0}}) = -1$. In fact,

$$\left(\frac{-\pi_2 \sqrt{l}}{\mathcal{P}_1^{h_0}}\right) = -\left(\frac{-1}{p^{h_0}}\right)\left(\frac{p}{l}\right)_4 \left(\frac{\sqrt{l}}{\mathcal{P}_1^{h_0}}\right) = \left(\frac{2}{p}\right)\left(\frac{\sqrt{l}}{\mathcal{P}_1^{h_0}}\right)$$

since $\pi_1 = a + b\sqrt{l}$. Then

$$\left(\frac{\pi_1 - a}{\mathcal{P}_1^{h_0}}\right) = \left(\frac{-a}{\mathcal{P}_1^{h_0}}\right) = \left(\frac{b\sqrt{l}}{\mathcal{P}_1^{h_0}}\right),$$

which implies

$$\left(\frac{\sqrt{l}}{\mathcal{P}_1^{h_0}}\right) = \left(\frac{-ab}{\mathcal{P}_1^{h_0}}\right) = -\left(\frac{2}{p}\right),$$

and thus

$$\left(\frac{-\pi_2 \sqrt{l}}{\mathcal{P}_1^{h_0}}\right) = -\left(\frac{2}{p}\right)^2 = -1.$$

- The quadratic residue symbol is $(\frac{2\epsilon\pi_1}{\mathcal{P}_2^{h_0}}) = 1$. In fact,

$$\left(\frac{2\epsilon\pi_1}{\mathcal{P}_2^{h_0}}\right) = \left(\frac{2\epsilon}{\mathcal{P}_2^{h_0}}\right)\left(\frac{\pi_1}{\mathcal{P}_2^{h_0}}\right) = -\left(\frac{p}{l}\right)_4 \left(\frac{2}{p}\right)\left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right) = -\left(\frac{2}{p}\right)^2 \left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right) = -\left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right).$$

Similarly, as we just did for $(\frac{\sqrt{l}}{\mathcal{P}_1^{h_0}})$, we get

$$\left(\frac{\sqrt{l}}{\mathcal{P}_2^{h_0}}\right) = \left(\frac{ab}{p}\right) = \left(\frac{2}{p}\right),$$

so

$$\left(\frac{\epsilon\sqrt{l}}{\mathcal{P}_2^{h_0}}\right) = \left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right)\left(\frac{\sqrt{l}}{\mathcal{P}_2^{h_0}}\right) = \left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right)\left(\frac{2}{p}\right),$$

Thus

$$\left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right) = \left(\frac{-p}{l}\right)_4 \left(\frac{2}{p}\right) = -\left(\frac{2}{p}\right)^2 = -1,$$

and finally,

$$\left(\frac{2\epsilon\pi_1}{\mathcal{P}_2^{h_0}}\right) = -\left(\frac{\epsilon}{\mathcal{P}_2^{h_0}}\right) = 1.$$

- The quadratic residue symbol is $(\frac{-\pi_2 \sqrt{l}}{\mathcal{H}}) = -(\frac{2}{p})$. In fact,

$$\left(\frac{-\pi_2 \sqrt{l}}{\mathcal{H}}\right) = \left(\frac{-\pi_2 \sqrt{l}}{2k}\right) = \left(\frac{-\pi_2 \sqrt{l}, 2}{2k}\right) = \left(\frac{N_{k/\mathbb{Q}}(-\pi_2 \sqrt{l}), 2}{2}\right) = \left(\frac{-pl, 2}{2}\right) = \left(\frac{2}{p}\right)\left(\frac{2}{l}\right) = -\left(\frac{2}{p}\right).$$

Thus

- if $(\frac{2}{p}) = 1$, then $N_{F_6/K}(\mathbf{C}_{F_6,2}) = \langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{P}_1^{h_0}] \rangle$;
- if $(\frac{2}{p}) = -1$, then $N_{F_6/K}(\mathbf{C}_{F_6,2}) = \langle [\mathcal{P}_2^{h_0}], [\mathcal{H}] \rangle$.

Proceeding similarly, we check the other tables entries.

7.1 The abelian-type invariants of $\mathbf{C}_{F_i,2}$ and capitulation kernels $\kappa_{F_i/K}$

We now compute the Galois groups $G_i = \text{Gal}(K_2^{(2)}/F_i)$, their derived groups, the abelian-type invariants of $\mathbf{C}_{F_i,2}$ and the transfer kernels $\ker V_{G \rightarrow G_i}$, and we deduce $\kappa_{F_i/K}$ the capitulation kernels of F_i/K . The results are summarized in Tables 3–10.

Tables 3 and 4 are filled using Tables 1 and 2, the relations in Remark 7.1 and Proposition 7.2.

F_i	G_i		G'_i	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$F_1 = K(\sqrt{-\epsilon\pi_1})$	$\langle \sigma, \tau\rho, \tau^2 \rangle$	$\langle \rho, \tau \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$F_2 = K(\sqrt{-2\epsilon\pi_2})$	$\langle \rho, \tau \rangle$	$\langle \rho, \sigma \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_6 = K(\sqrt{2\epsilon\pi_1})$	$\langle \sigma\rho, \tau \rangle$	$\langle \sigma, \tau\rho \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_7 = K(\sqrt{\epsilon\pi_2})$	$\langle \rho, \sigma, \tau^2 \rangle$	$\langle \sigma\rho, \tau \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$F_3 = K(\sqrt{2p})$	$\langle \tau\rho, \sigma\rho, \tau^2 \rangle$	$\langle \rho, \tau\sigma \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_4 = K(\sqrt{-2})$	$\langle \rho, \tau\sigma, \tau^2 \rangle$	$\langle \tau\rho, \sigma\rho \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_5 = K(\sqrt{-p})$	$\langle \tau, \sigma \rangle$		$\langle 1 \rangle$	

Table 3: $G_i = \text{Gal}(K_2^{(2)}/F_i)$ for the case $(\frac{2ab}{p}) = 1$.

F_i	G_i		G'_i	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$F_1 = K(\sqrt{-2\epsilon\pi_1})$	$\langle \sigma\rho, \tau \rangle$	$\langle \sigma, \tau\rho \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_2 = K(\sqrt{-\epsilon\pi_2})$	$\langle \rho, \sigma, \tau^2 \rangle$	$\langle \sigma\rho, \tau \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$F_6 = K(\sqrt{\epsilon\pi_1})$	$\langle \sigma, \tau\rho, \tau^2 \rangle$	$\langle \rho, \tau \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$F_7 = K(\sqrt{2\epsilon\pi_2})$	$\langle \rho, \tau \rangle$	$\langle \rho, \sigma \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_3 = K(\sqrt{2p})$	$\langle \tau\rho, \sigma\rho, \tau^2 \rangle$	$\langle \rho, \tau\sigma \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_4 = K(\sqrt{-2})$	$\langle \rho, \tau\sigma, \tau^2 \rangle$	$\langle \tau\rho, \sigma\rho \rangle$	$\langle \tau^2 \rangle$	$\langle \sigma^2 \rangle$
$F_5 = K(\sqrt{-p})$	$\langle \tau, \sigma \rangle$		$\langle 1 \rangle$	

Table 4: $G_i = \text{Gal}(K_2^{(2)}/F_i)$ for the case $(\frac{2ab}{p}) = -1$.

Remark 7.1. Note the following relations:

$$\sigma = \left(\frac{K_2^{(2)}/F_5}{\mathfrak{J}^v} \right) = \left(\frac{K_2^{(2)}/K}{N_{F_5/K}(\mathfrak{J}^v)} \right) = \left(\frac{K_2^{(2)}/K}{\mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}} \right),$$

$$\tau = \left(\frac{K_2^{(2)}/F_5}{\mathcal{H}_0} \right) = \left(\frac{K_2^{(2)}/K}{N_{F_5/K}(\mathcal{H}_0)} \right) = \left(\frac{K_2^{(2)}/K}{\mathcal{H}} \right)$$

and

$$\rho^{-1}\tau^{-1}\sigma = \left(\frac{K_2^{(2)}/K}{\mathcal{P}_2^{h_0}} \right)$$

since

$$\sigma = \left(\frac{K_2^{(2)}/K}{\mathcal{H}\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}} \right), \quad \tau = \left(\frac{K_2^{(2)}/K}{\mathcal{H}} \right) \quad \text{and} \quad \rho = \left(\frac{K_2^{(2)}/K}{\mathcal{P}_1^{h_0}} \right).$$

Proposition 7.2. Let $G = \langle \rho, \tau, \sigma \rangle$ denote the group defined in Section 6.3. Then the following relations hold:

$$\begin{aligned} \rho^{-1}\sigma\rho &= \sigma^{-1}, \quad \rho^{-1}\tau\rho = \tau^{-1}, \\ (\tau\sigma)^2 &= \begin{cases} \tau^2\sigma^2 = \tau^{2n+2} & \text{if } p \equiv 3 \pmod{8}, \\ \sigma^2 & \text{if } p \equiv 7 \pmod{8}, \end{cases} \\ (\rho\tau\sigma)^2 &= (\tau\rho)^2 = (\rho\sigma)^2 = \rho^2, \\ \rho\sigma\rho &= \sigma \end{aligned}$$

and

$$(\tau\sigma)^{2n+2} = \tau^{2n}\tau^2\sigma^{2n}\sigma^2 = \tau^{2n+1}\tau^2 = \tau^2 \quad \text{if } p \equiv 3 \pmod{8}$$

since $\sigma^2 = \tau^{2n}$ and $\sigma^{2n} = 1$.

Tables 5 and 6 are easily deduced by calculating $\mathbf{C}_{F_i,2} \simeq G_i/G'_i$. Take, for example, $G_7 = \langle \rho, \sigma, \tau^2 \rangle$ for the case

$$\left(\frac{2ab}{p} \right) = 1 \quad \text{and} \quad \left(\frac{2}{p} \right) = -1.$$

Then $G'_7 = \langle \tau^4 \rangle$ since $[\rho, \sigma] = \sigma^2$, $[\sigma, \tau^2] = 1$, $[\rho, \tau^2] = \tau^4$ and $\sigma^2 = (\tau^4)^{2^{n-2}}$. Thus

$$\mathbf{C}_{F_7,2} \simeq G_7/G'_7 = \langle \overline{\rho}, \overline{\sigma}, \overline{\tau^2} \rangle \simeq (2, 2, 2).$$

Compute now the transfer kernels $\ker V_{G \rightarrow G_i}$ (Tables 7 and 8). First, recall that if $G/G_i = \{\overline{1}, \overline{z}\}$, then

$$V_{G \rightarrow G_i}(gG'_i) = \begin{cases} gz^{-1}gzG'_i = g^2[g, z]G'_i & \text{if } g \in G_i, \\ g^2G'_i & \text{if } g \notin G_i. \end{cases}$$

Tables 9 and 10 are deduced from Tables 7 and 8 and Remark 7.1.

F_i	$\mathbf{C}_{F_i,2}$		F_i	$\mathbf{C}_{F_i,2}$	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$		$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$F_1 = K(\sqrt{-\epsilon\pi_1})$	(2, 2, 2)	(2, 4)	$F_1 = K(\sqrt{-2\epsilon\pi_1})$	(2, 2)	(2, 2)
$F_2 = K(\sqrt{-2\epsilon\pi_2})$	(2, 2)	(2, 2)	$F_2 = K(\sqrt{-\epsilon\pi_2})$	(2, 2, 2)	(2, 4)
$F_6 = K(\sqrt{2\epsilon\pi_1})$	(2, 2)	(2, 2)	$F_6 = K(\sqrt{\epsilon\pi_1})$	(2, 2, 2)	(2, 4)
$F_7 = K(\sqrt{\epsilon\pi_2})$	(2, 2, 2)	(2, 4)	$F_7 = K(\sqrt{2\epsilon\pi_2})$	(2, 2)	(2, 2)
$F_3 = K(\sqrt{2p})$	(2, 2)	(2, 2)	$F_3 = K(\sqrt{2p})$	(2, 2)	(2, 2)
$F_4 = K(\sqrt{-2})$	(2, 2)	(2, 2)	$F_4 = K(\sqrt{-2})$	(2, 2)	(2, 2)
$F_5 = K(\sqrt{-p})$	(2, 2^{n+1})	(2, 4)	$F_5 = K(\sqrt{-p})$	(2, 2^{n+1})	(2, 4)

Table 5: $\mathbf{C}_{F_i,2}$ for the case $(\frac{2ab}{p}) = 1$.Table 6: $\mathbf{C}_{F_i,2}$ for the case $(\frac{2ab}{p}) = -1$.

F_i	$\ker V_{G \rightarrow G_i}$		F_i	$\ker V_{G \rightarrow G_i}$	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$		$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$F_1 = K(\sqrt{-\epsilon\pi_1})$	$\langle \sigma G', \rho G' \rangle$	$\langle \rho G', \tau G' \rangle$	$F_1 = K(\sqrt{-2\epsilon\pi_1})$	$\langle \rho G', \sigma G', \tau G' \rangle$	$\langle \rho G', \sigma G', \tau G' \rangle$
$F_2 = K(\sqrt{-2\epsilon\pi_2})$	$\langle \rho G', \sigma G', \tau G' \rangle$	$\langle \rho G', \sigma G', \tau G' \rangle$	$F_2 = K(\sqrt{-\epsilon\pi_2})$	$\langle \sigma G', \rho \tau G' \rangle$	$\langle \sigma \rho G', \tau G' \rangle$
$F_6 = K(\sqrt{2\epsilon\pi_1})$	$\langle \rho G', \sigma G', \tau G' \rangle$	$\langle \rho G', \sigma G', \tau G' \rangle$	$F_6 = K(\sqrt{\epsilon\pi_1})$	$\langle \sigma G', \rho G' \rangle$	$\langle \rho G', \tau G' \rangle$
$F_7 = K(\sqrt{\epsilon\pi_2})$	$\langle \sigma G', \rho \tau G' \rangle$	$\langle \sigma \rho G', \tau G' \rangle$	$F_7 = K(\sqrt{2\epsilon\pi_2})$	$\langle \rho G', \sigma G', \tau G' \rangle$	$\langle \rho G', \sigma G', \tau G' \rangle$
$F_3 = K(\sqrt{2p})$		$\langle \rho G', \sigma G', \tau G' \rangle$	$F_3 = K(\sqrt{2p})$		$\langle \rho G', \sigma G', \tau G' \rangle$
$F_4 = K(\sqrt{-2})$		$\langle \rho G', \sigma G', \tau G' \rangle$	$F_4 = K(\sqrt{-2})$		$\langle \rho G', \sigma G', \tau G' \rangle$
$F_5 = K(\sqrt{-p})$		$\langle \tau G', \sigma G' \rangle$	$F_5 = K(\sqrt{-p})$		$\langle \tau G', \sigma G' \rangle$

Table 7: $\ker V_{G \rightarrow G_i}$ for the case $(\frac{2ab}{p}) = 1$.Table 8: $\ker V_{G \rightarrow G_i}$ for the case $(\frac{2ab}{p}) = -1$.

F_i	$\kappa_{F_i/K}$		F_i	$\kappa_{F_i/K}$	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$		$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$F_1 = K(\sqrt{-\epsilon\pi_1})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}] \rangle$	$F_1 = K(\sqrt{-2\epsilon\pi_1})$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$
$F_2 = K(\sqrt{-2\epsilon\pi_2})$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$F_2 = K(\sqrt{-\epsilon\pi_2})$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{P}_1^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_2^{h_0}] \rangle$
$F_6 = K(\sqrt{2\epsilon\pi_1})$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$F_6 = K(\sqrt{\epsilon\pi_1})$	$\langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}] \rangle$
$F_7 = K(\sqrt{\epsilon\pi_2})$	$\langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{P}_1^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_2^{h_0}] \rangle$	$F_7 = K(\sqrt{2\epsilon\pi_2})$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$
$F_3 = K(\sqrt{2p})$		$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$F_3 = K(\sqrt{2p})$		$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$
$F_4 = K(\sqrt{-2})$		$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$	$F_4 = K(\sqrt{-2})$		$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle$
$F_5 = K(\sqrt{-p})$		$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$	$F_5 = K(\sqrt{-p})$		$\langle [\mathcal{H}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle$

Table 9: $\kappa_{F_i/K}$ for the case $(\frac{2ab}{p}) = 1$.Table 10: $\kappa_{F_i/K}$ for the case $(\frac{2ab}{p}) = -1$.

Take some examples:

Example. Consider F_4 . Then we know from Table 1 or 2 that

$$N_{F_4/K}(\mathbf{C}_{F_4,2}) = \begin{cases} \langle [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle & \text{if } (\frac{2}{p}) = -1, \\ \langle [\mathcal{H}\mathcal{P}_1^{h_0}], [\mathcal{P}_1^{h_0}\mathcal{P}_2^{h_0}] \rangle & \text{if } (\frac{2}{p}) = 1. \end{cases}$$

Then $G_4 = \langle \rho, \tau^{-1}\sigma, \tau^2 \rangle = \langle \rho, \tau\sigma, \tau^2 \rangle$ or $G_4 = \langle \tau\rho, \tau^{-1}\sigma \rangle = \langle \tau\rho, \sigma\rho \rangle$ and $G'_4 = \langle \tau^2 \rangle$ or $G'_4 = \langle \sigma^2 \rangle$, respectively, and hence $G/G_4 = \{\bar{1}, \bar{\tau} = \bar{\sigma}\}$ or $G/G_4 = \{\bar{1}, \bar{\tau} = \bar{\sigma} = \bar{\rho}\}$. Thus

- $V_{G \rightarrow G_4}(\rho G') = \rho^2[\rho, \sigma]G'_4 = \rho^2\sigma^2G'_4 = G'_4$ since $\rho^2 = \sigma^2$ and $\sigma^4 = 1$,
- $V_{G \rightarrow G_4}(\sigma G') = \sigma^2G'_4 = G'_4$ since $\sigma^2 = (\tau^2)^{2^{n-1}} \in G'_4$,
- $V_{G \rightarrow G_4}(\tau G') = \tau^2G'_4 = G'_4$;

or

- $V_{G \rightarrow G_4}(\rho G') = \rho^2G'_4 = \sigma^2G'_4 = G'_4$ since $\rho^2 = \sigma^2$,
- $V_{G \rightarrow G_4}(\sigma G') = \sigma^2G'_4 = G'_4$,
- $V_{G \rightarrow G_4}(\tau G') = \tau^2G'_4 = G'_4$ since $\tau^2 = 1$.

So $\ker V_{G \rightarrow G_4} = \langle \tau G', \rho G', \sigma G' \rangle$, which gives

$$\kappa_{F_4/K} = \langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle = \mathbf{C}_{K,2}.$$

Example. Now consider F_1 .

- If $(\frac{2ab}{p}) = 1$, from Table 1, we have

$$N_{F_1/K}(\mathbf{C}_{F_1,2}) = \begin{cases} \langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{P}_1^{h_0}] \rangle & \text{if } (\frac{2}{p}) = -1, \\ \langle [\mathcal{P}_1^{h_0}], [\mathcal{H}] \rangle & \text{if } (\frac{2}{p}) = 1. \end{cases}$$

Then $G_1 = \langle \rho^{-1}\tau^{-1}\sigma, \tau\rho, \tau^2 \rangle = \langle \sigma, \tau\rho, \tau^2 \rangle$ or $G_1 = \langle \rho, \tau \rangle$, and $G'_1 = \langle \tau^4 \rangle$ or $G'_1 = \langle 1 \rangle$, respectively, and hence $G/G_1 = \{\bar{1}, \bar{\tau} = \bar{\rho}\}$ or $G/G_1 = \{\bar{1}, \bar{\sigma}\}$. Thus

- $V_{G \rightarrow G_1}(\sigma G') = \sigma^2[\sigma, \tau]G'_1 = \sigma^2 G'_1 = \tau^{2^n} G'_1 = G'_1$ since $n \geq 2$,
- $V_{G \rightarrow G_1}(\tau G') = \tau^2 G'_1 \neq G'_1$,
- $V_{G \rightarrow G_1}(\rho G') = \rho^2 G'_1 = \tau^{2^n} G'_1 = G'_1$,

so $\ker V_{G \rightarrow G_1} = \langle \sigma G', \rho G' \rangle$, which implies $\kappa_{F_1/K} = \langle [\mathcal{P}_1^{h_0}], [\mathcal{H}\mathcal{P}_2^{h_0}] \rangle$;

or

- $V_{G \rightarrow G_1}(\rho G') = \rho^2[\rho, \sigma] = \rho^2 \sigma^2 = 1$,
- $V_{G \rightarrow G_1}(\tau G') = \tau^2[\tau, \sigma] = \tau^2 = 1$,
- $V_{G \rightarrow G_1}(\sigma G') = \sigma^2 \neq 1$,

so $\ker V_{G \rightarrow G_1} = \langle \tau G', \rho G' \rangle$, which implies $\kappa_{F_1/K} = \langle [\mathcal{H}], [\mathcal{P}_1^{h_0}] \rangle$.

- If $(\frac{2ab}{p}) = -1$, from Table 2, we have

$$N_{F_1/K}(\mathbf{C}_{F_1,2}) = \begin{cases} \langle [\mathcal{P}_2^{h_0}], [\mathcal{H}] \rangle & \text{if } (\frac{2}{p}) = -1, \\ \langle [\mathcal{P}_2^{h_0}], [\mathcal{H}\mathcal{P}_1^{h_0}] \rangle & \text{if } (\frac{2}{p}) = 1. \end{cases}$$

Then $G_1 = \langle \sigma\rho, \tau \rangle$ or $G_1 = \langle \sigma, \rho\tau \rangle$, and $G'_1 = \langle \tau^2 \rangle$ or $G'_1 = \langle \sigma^2 \rangle$, respectively, and hence $G/G_1 = \{\bar{1}, \bar{\sigma} = \bar{\rho}\}$ or $G/G_1 = \{\bar{1}, \bar{\tau} = \bar{\rho}\}$. Thus

- $V_{G \rightarrow G_1}(\tau G') = \tau^2[\tau, \sigma]G'_1 = \tau^2 G'_1 = G'_1$,
- $V_{G \rightarrow G_1}(\sigma G') = \sigma^2 G'_1 = \tau^{2^n} G'_1 = G'_1$,
- $V_{G \rightarrow G_1}(\rho G') = \rho^2 G'_1 = \tau^{2^n} G'_1 = G'_1$;

or

- $V_{G \rightarrow G_1}(\sigma G') = \sigma^2[\sigma, \tau]G'_1 = \sigma^2 G'_1 = G'_1$,
- $V_{G \rightarrow G_1}(\tau G') = \tau^2 G'_1 = G'_1$ since $\tau^2 = 1$,
- $V_{G \rightarrow G_1}(\rho G') = \rho^2 G'_1 = G'_1$ since $\rho^2 = \sigma^2$,

so $\ker V_{G \rightarrow G_1} = \langle \tau G', \rho G', \sigma G' \rangle$, which implies $\kappa_{F_1/K} = \langle [\mathcal{H}], [\mathcal{P}_1^{h_0}], [\mathcal{P}_2^{h_0}] \rangle = \mathbf{C}_{K,2}$.

We similarly check the remaining tables entries.

7.2 The abelian-type invariants of $\mathbf{C}_{L_i,2}$ and capitulation kernels $\kappa_{L_i/K}$

Finally, we easily compute the Galois groups $\mathcal{G}_i = \text{Gal}(K_2^{(2)}/L_i)$, their derived groups, the abelian-type invariants of $\mathbf{C}_{L_i,2}$, the transfer kernels $\ker V_{G \rightarrow \mathcal{G}_i}$ and $\kappa_{L_i/K}$, the capitulation kernels of L_i/K . The results are summarized in Tables 11, 12, 13 and 14.

Let us take just one example:

Example. Consider $L_1 = F_1 F_2 F_3$ in the case where $(\frac{2ab}{p}) = 1$. Then $\text{Gal}(K_2^{(2)}/L_1) = \mathcal{G}_1 = G_1 \cap G_2 \cap G_3$, and hence

$$\mathcal{G}_1 = \begin{cases} \langle \rho, \tau \rangle \cap \langle \rho, \sigma \rangle \cap \langle \rho, \tau\sigma \rangle = \langle \rho \rangle & \text{if } (\frac{2}{p}) = 1. \\ \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \rho, \tau \rangle \cap \langle \tau\rho, \sigma\rho, \tau^2 \rangle = \langle \tau\rho, \tau^2 \rangle & \text{if } (\frac{2}{p}) = -1. \end{cases}$$

Thus $\mathcal{G}'_1 = \langle 1 \rangle$ or $\mathcal{G}'_1 = \langle \tau^4 \rangle$ according as $(\frac{2}{p}) = 1$ or -1 , which implies that $\mathbf{C}_{L_1,2} \simeq \mathbb{Z}/4\mathbb{Z}$ or $\mathbf{C}_{L_1,2} \simeq (2, 2)$, respectively.

On the other hand, $\kappa_{L_1/K} = \mathbf{C}_{K,2}$ since $\kappa_{F_3/K} = \mathbf{C}_{K,2}$ and $\kappa_{F_3/K} \subset \kappa_{L_1/K}$.

L_i	$\mathcal{G}_i$		$\mathcal{G}'_i$	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$L_1 = K(\sqrt{-\epsilon\pi_1}, \sqrt{-2\epsilon\pi_2})$	$\langle \tau\rho, \tau^2 \rangle$	$\langle \rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$L_2 = K(\sqrt{-p}, \sqrt{-2})$	$\langle \tau\sigma \rangle$	$\langle \tau\sigma \rangle$	$\langle 1 \rangle$	$\langle 1 \rangle$
$L_3 = K(\sqrt{2\epsilon\pi_1}, \sqrt{\epsilon\pi_2})$	$\langle \sigma\rho, \tau^2 \rangle$	$\langle \tau\rho\sigma \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$L_4 = K(\sqrt{-2\epsilon\pi_2}, \sqrt{2\epsilon\pi_1})$	$\langle \tau \rangle$	$\langle \sigma \rangle$	$\langle 1 \rangle$	$\langle 1 \rangle$
$L_5 = K(\sqrt{-\epsilon\pi_1}, \sqrt{\epsilon\pi_2})$	$\langle \sigma, \tau^2 \rangle$	$\langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$	$\langle 1 \rangle$
$L_6 = K(\sqrt{-\epsilon\pi_1}, \sqrt{2\epsilon\pi_1})$	$\langle \tau\sigma\rho, \tau^2 \rangle$	$\langle \tau\rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$L_7 = K(\sqrt{-2\epsilon\pi_2}, \sqrt{\epsilon\pi_2})$	$\langle \rho, \tau^2 \rangle$	$\langle \sigma\rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$

Table 11: $\mathcal{G}_i = \text{Gal}(K_2^{(2)}/L_i)$ and $\mathcal{G}'_i$ for the case $(\frac{2ab}{p}) = 1$.

L_i	$\mathbf{C}_{L_i,2}$		$\ker V_{G \rightarrow \mathcal{G}_i}$	$\kappa_{L_i/K}$
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$		
$L_1 = K(\sqrt{-\epsilon\pi_1}, \sqrt{-2\epsilon\pi_2})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_2 = K(\sqrt{-p}, \sqrt{-2})$	$\mathbb{Z}/2^{n+1}\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_3 = K(\sqrt{2\epsilon\pi_1}, \sqrt{\epsilon\pi_2})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_4 = K(\sqrt{-2\epsilon\pi_2}, \sqrt{2\epsilon\pi_1})$	$\mathbb{Z}/2^{n+1}\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_5 = K(\sqrt{-\epsilon\pi_1}, \sqrt{\epsilon\pi_2})$	$(2, 2^n)$	$(2, 2)$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_6 = K(\sqrt{-\epsilon\pi_1}, \sqrt{2\epsilon\pi_1})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_7 = K(\sqrt{-2\epsilon\pi_2}, \sqrt{\epsilon\pi_2})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$

Table 12: $\mathbf{C}_{L_i,2}$, $\ker V_{G \rightarrow \mathcal{G}_i}$ and $\kappa_{L_i/K}$ for the case $(\frac{2ab}{p}) = 1$.

L_i	$\mathcal{G}_i$		$\mathcal{G}'_i$	
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$
$L_1 = K(\sqrt{-2\epsilon\pi_1}, \sqrt{-\epsilon\pi_2})$	$\langle \sigma\rho, \tau^2 \rangle$	$\langle \tau\sigma\rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$L_2 = K(\sqrt{-p}, \sqrt{-2})$	$\langle \tau\sigma \rangle$	$\langle \tau\sigma \rangle$	$\langle 1 \rangle$	$\langle 1 \rangle$
$L_3 = K(\sqrt{\epsilon\pi_1}, \sqrt{2\epsilon\pi_2})$	$\langle \tau\rho, \tau^2 \rangle$	$\langle \rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$L_4 = K(\sqrt{-\epsilon\pi_2}, \sqrt{\epsilon\pi_1})$	$\langle \sigma, \tau^2 \rangle$	$\langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$	$\langle 1 \rangle$
$L_5 = K(\sqrt{-2\epsilon\pi_1}, \sqrt{2\epsilon\pi_2})$	$\langle \tau \rangle$	$\langle \sigma \rangle$	$\langle 1 \rangle$	$\langle 1 \rangle$
$L_6 = K(\sqrt{-2\epsilon\pi_1}, \sqrt{\epsilon\pi_1})$	$\langle \tau\sigma\rho, \tau^2 \rangle$	$\langle \tau\rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$
$L_7 = K(\sqrt{-\epsilon\pi_2}, \sqrt{2\epsilon\pi_2})$	$\langle \rho, \tau^2 \rangle$	$\langle \sigma\rho \rangle$	$\langle \tau^4 \rangle$	$\langle 1 \rangle$

Table 13: $\mathcal{G}_i = \text{Gal}(K_2^{(2)}/L_i)$ and $\mathcal{G}'_i$ for the case $(\frac{2ab}{p}) = -1$.

L_i	$\mathbf{C}_{L_i,2}$		$\ker V_{G \rightarrow \mathcal{G}_i}$	$\kappa_{L_i/K}$
	$(\frac{2}{p}) = -1$	$(\frac{2}{p}) = 1$		
$L_1 = K(\sqrt{-2\epsilon\pi_1}, \sqrt{-\epsilon\pi_2})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_2 = K(\sqrt{-p}, \sqrt{-2})$	$\mathbb{Z}/2^{n+1}\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_3 = K(\sqrt{\epsilon\pi_1}, \sqrt{2\epsilon\pi_2})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_4 = K(\sqrt{-\epsilon\pi_2}, \sqrt{\epsilon\pi_1})$	$(2, 2^n)$	$(2, 2)$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_5 = K(\sqrt{-2\epsilon\pi_1}, \sqrt{2\epsilon\pi_2})$	$\mathbb{Z}/2^{n+1}\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_6 = K(\sqrt{-2\epsilon\pi_1}, \sqrt{\epsilon\pi_1})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$
$L_7 = K(\sqrt{-\epsilon\pi_2}, \sqrt{2\epsilon\pi_2})$	$(2, 2)$	$\mathbb{Z}/4\mathbb{Z}$	$\langle \sigma G', \tau G', \rho G' \rangle$	$\mathbf{C}_{K,2}$

Table 14: $\mathbf{C}_{L_i,2}$, $\ker V_{G \rightarrow \mathcal{G}_i}$ and $\kappa_{L_i/K}$ for the case $(\frac{2ab}{p}) = -1$.

Numerical examples

The examples in Tables 15 and 16 are computed by using PARI/GP.

$l.p$	$(\frac{2ab}{p})$	$(\frac{2}{p})$	$C_{F_{1,2}}$	$C_{F_{2,2}}$	$C_{F_{3,2}}$	$C_{F_{4,2}}$	$C_{F_{5,2}}$	$C_{F_{6,2}}$	$C_{F_{7,2}}$
5.19	-1	-1	(2, 2)	(2, 2, 2)	(2, 2)	(2, 2)	(2, 8)	(2, 2, 2)	(2, 2)
5.19	1	-1	(2, 2, 2)	(2, 2)	(2, 2)	(2, 2)	(2, 8)	(2, 2)	(2, 2, 2)
5.31	-1	1	(2, 2)	(2, 4)	(2, 2)	(2, 2)	(2, 4)	(2, 4)	(2, 2)
5.31	1	1	(2, 4)	(2, 2)	(2, 2)	(2, 2)	(2, 4)	(2, 2)	(2, 4)
101.43	1	-1	(2, 2, 2)	(2, 2)	(2, 2)	(2, 2)	(2, 64)	(2, 2)	(2, 2, 2)
5.59	1	-1	(2, 2, 2)	(2, 2)	(2, 2)	(2, 2)	(2, 8)	(2, 2)	(2, 2, 2)
5.59	-1	-1	(2, 2)	(2, 2, 2)	(2, 2)	(2, 2)	(2, 8)	(2, 2, 2)	(2, 2)
37.67	-1	-1	(2, 2)	(2, 2, 2)	(2, 2)	(2, 2)	(2, 8)	(2, 2, 2)	(2, 2)

Table 15: Invariants of F_i .

$l.p$	$(\frac{2ab}{p})$	$(\frac{2}{p})$	$C_{L_{1,2}}$	$C_{L_{2,2}}$	$C_{L_{3,2}}$	$C_{L_{4,2}}$	$C_{L_{5,2}}$	$C_{L_{6,2}}$	$C_{L_{7,2}}$
5.19	-1	-1	(2, 2)	(8)	(2, 2)	(2, 4)	(8)	(2, 2)	(2, 2)
5.19	1	-1	(2, 2)	(8)	(2, 2)	(8)	(2, 4)	(2, 2)	(2, 2)
5.31	-1	1	(4)	(4)	(4)	(2, 2)	(4)	(4)	(4)
5.31	1	1	(4)	(4)	(4)	(4)	(2, 2)	(4)	(4)
101.43	1	-1	(2, 2)	(64)	(2, 2)	(64)	(2, 32)	(2, 2)	(2, 2)
5.59	1	-1	(2, 2)	(8)	(2, 2)	(8)	(2, 4)	(2, 2)	(2, 2)
5.59	-1	-1	(2, 2)	(8)	(2, 2)	(2, 4)	(8)	(2, 2)	(2, 2)
37.67	-1	-1	(2, 2)	(8)	(2, 2)	(2, 4)	(8)	(2, 2)	(2, 2)

Table 16: Invariants of L_i .

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