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Dynamic D-Vine Copula Model with Applications to Value-at-Risk (VaR)

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Abstract:

Vine copulas are multivariate dependence models constructed from pair-copulas (bivariate copulas). In this paper, we allow the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, following a restricted ARMA(1, m) process, in order to obtain a very flexible dependence model for applications to multivariate financial return data. We investigate the dependence among the broad stock market indexes from Germany (DAX), France (CAC 40), Britain (FTSE 100), the United States (S&P 500) and Brazil (IBOVESPA) both in a crisis and in a non-crisis period. We find evidence of stronger dependence among the indexes in bear markets. Surprisingly, though, the dynamic D-vine copula indicates the occurrence of a sharp decrease in dependence between the indexes FTSE and CAC in the beginning of 2011, and also between CAC and DAX during mid-2011 and in the beginning of 2008, suggesting the absence of contagion in these cases. We evaluate the dynamic D-vine copula with respect to Value-at-Risk (VaR) forecasting accuracy in crisis periods. The dynamic D-vine outperforms the static D-vine in terms of predictive accuracy for our real data sets. We also investigate the dynamic D-vine copula in a simulation study and the overall results of the Monte Carlo experiments are quite favorable to the dynamic D-vine copula in comparison with a static D-vine copula.

Keywords: regular vine, pair-copula constructions, time-varying copulas, value-at-risk

JEL classification: C13, C51, C52, C58, G15

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1 Introduction

A realistic modeling of the dependence structure of multivariate financial return data is fundamental in finance for the accurate computation of Value-at-Risk, the construction of optimal portfolios, the pricing of financial products, amongst other applications. In particular, it is essential for the correct assessment of financial risk, and the subprime crisis has made it clear.

The linear correlation coefficient has long time been adopted as a measure of dependence in finance. However, there are stylized facts observed in the dependence structure of multivariate financial returns which are not captured by elliptical distributions. One such feature is asymmetric dependence, meaning that negative returns tend to be more dependent between themselves than positive returns. Another one is tail dependence, which refers to the dependence in extreme values. Also tail dependencies exhibit asymmetries, i.e. lower tail dependence can be larger than upper tail dependence and vice versa.

The famous theorem by Sklar (1959) introduced the copulas as a tool to model more intricate patterns of dependence. It states that any n -dimensional joint distribution function can be decomposed into its n marginal distributions and a copula, where the latter completely characterizes the dependence among the variables.

Early applications of copula models are based on bivariate copulas. Larger dimensional copulas other than Gaussian or Student t have become popular only recently and have a tendency towards hierarchical structures. Joe (1996) proposed a probabilistic construction of multivariate distributions based on pair-copulas (bivariate copulas), later extended and systematically organized by Bedford and Cooke (2001, 2002) through the specification of a graphical model called regular vine. The regular vine copula model (R-vine copula), also called pair-copula constructions (PCC), is hierarchical in nature and consists in decomposing a multivariate density into a cascade of pair-copulas and the marginal densities. It is a more flexible method to model multivariate distributions, since pair-copulas belonging to different families may be mixed in a vine copula, matching any

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possible dependence structure. Inference for two special cases of regular vine copulas, the canonical vine (C-vine) and the drawable vine (D-vine) copulas, was introduced by Aas et al. (2009).

Research in multivariate dependence modeling using copulas is focused mostly on the case of time-homogeneous dependence structures, however promising approaches for allowing time variation in dependence have been put forth (see Manner and Reznikova (2012) for a recent survey of time-varying copula models with focus on the bivariate case). The dependence among variables can be rendered time-varying by allowing either the dependence parameter or the copula function to vary over time. The first line of research includes fully parametric models, as the one proposed by Patton (2006), who allows the dependence parameters of bivariate copulas to follow a kind of restricted ARMA(1, 10) process. There are also semi-parametric models (Hafner and Reznikova 2010) and adaptive approaches (Giacomini, Härdle, and Spokoiny 2009), both applied to the bivariate case. The copulas parameters can also be influenced by a Markov chain, as in Jondeau and Rockinger (2006) and Silva Filho, Ziegelmann, and Dueker (2012). To the best of our knowledge, Heinen and Valdesogo (2009, 2011) were the first ones to introduce time variation in the vine copula context by specifying a law of motion for the pair-copulas parameters, based on the DCC equations. For each pair-copula, the dynamics is driven by a variation of the DCC equation, which captures the correlation coefficient between the variables at each period t , and subsequently it is converted to the Kendall's tau and transformed into the parameter of the pair-copula. More recently, So and Yeung (2014) proposed that the dependence measures given by the Kendall's tau, the rank correlation and the linear correlation, associated with the pair-copulas of C and D-vines, followed a DCC like equation. The other direction of research combines copula models with regime switching to allow for changes in the whole dependence structure, represented by the copula function, according to the regimes characterizing the international financial markets. Chollete, Heinen, and Valdesogo (2009) and Garcia and Tsafack (2011) and Stöber and Czado (2014) are examples of publications combining the regular vine copula model with the Markov switching model.

In this paper, we introduce dynamics into the vine copula model according to the first approach above-mentioned, specifying an evolution equation directly for the pair-copulas parameters, in order to obtain a very flexible dependence model for applications to multivariate financial return data. We allow the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, following a restricted ARMA(1, m) process as in Patton (2006). Hereafter we will call this time-varying or dynamic D-vine copula model.

The dynamics specified here circumvents certain limitations inherent both in the dynamic vine copula model of Heinen and Valdesogo (2009, 2011) and in the vine-copula GARCH model of So and Yeung (2014). Because their specifications involve converting the Kendall's tau, and also the rank correlation in the case of So and Yeung (2014), to the parameter of the pair-copula at each period t , at least two difficulties arise. First, since the non-linear transformation from the Kendall's tau and the rank correlation to the copula parameter cannot be done in closed form for all copulas, estimation becomes a difficult task when the parameters have to be obtained by solving numerically for the solution. Second, it is only possible to adopt one-parameter copula families as building blocks in the vine construction, and, according to Joe (2011), it is important to have copulas with flexible lower and upper tail dependencies, such as the two-parameter copulas BB1 and BB7 (see Joe 1997), for making inferences on joint tail probabilities, which are related to joint risks.

We first evaluate the performance of the dynamic D-vine copula model in a simulation study. The overall findings of the Monte Carlo experiments are quite favorable to the dynamic D-vine copula in comparison with a static D-vine copula. When the data generating process is the dynamic D-vine copula, the dependence parameters estimates from this same model are superior to those from the static model in terms of both the mean errors and the root mean squared errors. When the samples are drawn from the static D-vine copula, both models have similar performance in terms of the mean errors, nonetheless the dynamic D-vine copula performs worse in terms of the root mean squared errors.

We also investigate both the static and the dynamic D-vine copula models in an empirical study, using two data sets of daily log-returns of the broad stock market indexes from Germany (DAX), France (CAC 40), Britain (FTSE 100), the United States (S&P 500) and Brazil (IBOVESPA), one comprising the period from January 03, 2003 to December 28, 2007 and another one from January 02, 2008 to May 04, 2012, which we denominate "non-crisis period" and "crisis period", respectively. Besides of analyzing the different patterns of dependence characterizing these periods, the intention is to further evaluate the dynamic D-vine copula model with respect to Value-at-Risk (VaR) forecasting accuracy in crisis periods. We find evidence of greater dependence among the indexes in bear markets. Surprisingly, though, the dynamic model indicates the occurrence of a strong decrease in dependence between the indexes FTSE and CAC in the beginning of 2011, and also between CAC and DAX during mid-2011 and in the beginning of 2008, suggesting the absence of contagion in these cases. In an out-of-sample exercise, the estimated models are used to forecast one-day VaR for an equally weighted portfolio of the aforementioned indexes in the period from January 02, 2008 to August 19, 2008 (150 days) and also from May 08, 2012 to December 28, 2012 (149 days). Based on the results of the superior predictive ability (SPA) test of

Hansen (2005), the dynamic D-vine copula model outperforms the static D-vine copula in terms of predictive accuracy.

The remainder of this paper is organized as follows. In the next section, we present the dynamic D-vine copula model by first providing necessary background on regular vine copulas in Section 2.1 and then specifying the dynamic structure of dependence in Section 2.2. In Section 3, we focus on inference of the dynamic D-vine copula, describing the sequential estimation procedure. The dynamic copula model is first evaluated in a simulation study in Section 4 before we turn to the empirical application in Section 5, where the dependence structure of the above-mentioned indexes is investigated both in the non-crisis and in the crisis period and the dynamic D-vine copula is assessed in terms of the accuracy of VaR forecasts. Section 6 provides some concluding remarks and an outlook to future research.

2 The Dynamic D-Vine Copula Model

In this section, we present the dynamic D-vine copula model. We first provide a brief account of the regular vine copula theory and, then, we give details on the specification of the dynamic structure of dependence.

2.1 Regular Vine Copulas

Sklar's Theorem (Sklar 1959) states that every multivariate cumulative probability distribution function F with marginals $F_1, \dots, F_n$ may be written as

$$F(x_1, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)), \quad (1)$$

for some appropriate n -dimensional copula C . In terms of the joint probability density function f , for an absolutely continuous F with strictly increasing continuous marginals $F_1, \dots, F_n$, we have

$$f(x_1, \dots, x_n) = c_{12\dots n}(F_1(x_1), \dots, F_n(x_n)) \cdot f_1(x_1) \dots f_n(x_n). \quad (2)$$

Consider now, for example, a trivariate random vector $X = (X_1, X_2, X_3)$. Its density can be factorized as

$$f(x_1, x_2, x_3) = f(x_3) \cdot f(x_2|x_3) \cdot f(x_1|x_2, x_3). \quad (3)$$

According to eq. (2), we can write

$$f(x_2|x_3) = c_{23}(F_2(x_2), F_3(x_3)) \cdot f_2(x_2). \quad (4)$$

Similarly, it is possible to decompose the conditional density of X_1 given X_2 and X_3 as

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot f(x_1|x_2). \quad (5)$$

Now, decomposing $f(x_1|x_2)$ in eq. (5) further, we have

$$f(x_1|x_2, x_3) = c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1). \quad (6)$$

Finally, from eqs. (4) and (6), the joint density function for the trivariate case can be written as

$$f(x_1, x_2, x_3) = f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{23}(F_2(x_2), F_3(x_3)) \cdot c_{13|2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)). \quad (7)$$

That is, the trivariate density can be factorized as a product of the marginals, two bivariate copulas, c_{12} and c_{23} , and a third copula $c_{13|2}$ named conditional because its arguments are conditional distributions.

The previous results for the trivariate case can be generalized for an n -dimensional vector, using the following formula:

$$f(x|\mathbf{v}) = c_{xv_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})) \cdot f(x|\mathbf{v}_{-j}), \quad (8)$$

for a vector $\mathbf{v}$ with dimension d . Here v_j is an arbitrarily chosen component of $\mathbf{v}$ and $\mathbf{v}_{-j}$ corresponds to the vector $\mathbf{v}$ excluding this component. It follows that the multivariate density function with dimension n can be decomposed into its marginal densities and a set of iteratively conditioned bivariate copulas.

The pair-copula decomposition of a multivariate density involves marginal conditional distributions of the form $F(x|\mathbf{v})$, computed using a formula of Joe (1996):

$$F(x|\mathbf{v}) = \frac{\partial C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial F(v_j|\mathbf{v}_{-j})}. \quad (9)$$

As the number of variables grows, the different possibilities of decomposition in pair-copulas also increase. To organize these possibilities, Bedford and Cooke (2001, 2002) introduced a graphical model called regular vine (R-vine). The R-vines are a sequence of nested trees that facilitate the identification of the needed pairs of variables and their corresponding set of conditioning variables (we refer the reader to Bedford and Cooke 2001, 2002, for more details on general R-vines, and to Dißmann et al. 2013, for inference of R-vines). Two boundary cases, popularized by Aas et al. (2009), are the canonical vine (C-vine) and the drawable vine (D-vine). Canonical vines resemble factor models, with a particular variable playing the role of pivot (factor) in every tree. Because there is no economic reason to think that a factor structure should be relevant in our data, we will focus our attention on the D-vine.

An n -dimensional D-vine consists of $n - 1$ hierarchical trees (or levels), with path structures in their sequences and increasing conditional sets, and $n(n - 1)/2$ edges corresponding to a pair-copula (for a more detailed description, see Aas et al. 2009). Define the index sets $v_{ij} = \{i + 1, \dots, i + j - 1\}$, with $v_{i1} = \emptyset$, and $w_{ij} = \{i, v_{ij}, i + j\}$, for $1 \leq i \leq n - j$, $1 \leq j \leq n - 1$. Let α and θ denote the parameters of the marginals and the n -dimensional copula, respectively, and $\theta_{i,i+j|v_{ij}}$ be the parameters of the copula density $c_{i,i+j|v_{ij}}$. Finally, define $\theta_{i \rightarrow i+j} = \{\theta_{s,s+t|v_{st}} : (s, s+t) \in w_{ij}\}$, with $\theta_{i \rightarrow i} = \emptyset$, and $\theta_j = \{\theta_{s,s+t|v_{st}} : |v_{st}| = j - 1\}$, where $|\cdot|$ denotes the cardinality, i.e. θ_j gathers all parameters at level j of the structure. The density $f(x_1, \dots, x_n; \alpha, \theta)$ associated with a D-vine may be written as¹

$$\begin{aligned} f(x_1, \dots, x_n; \alpha, \theta) = & \prod_{k=1}^n f(x_k; \alpha_k) \\ & \cdot \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,i+j|v_{ij}}(F_{i|v_{ij}}(x_i|\mathbf{x}_{v_{ij}}; \alpha_{w_{i,j-1}}), \theta_{i \rightarrow i+j-1}), \\ & F_{i+j|v_{ij}}(x_{i+j}|\mathbf{x}_{v_{ij}}; \alpha_{w_{i+1,j-1}}, \theta_{i+1 \rightarrow i+j}); \theta_{i,i+j|v_{ij}}), \end{aligned} \quad (10)$$

where index j identifies the trees, whereas i runs over the edges in each tree. The whole decomposition is given by the $n(n - 1)/2$ pair-copulas and the marginal densities of each variable.

Figure D.1 in the online Appendix depicts a five-dimensional D-vine. A simple manner of decomposing the density $f(x_1, x_2, x_3, x_4, x_5; \alpha, \theta)$ is by multiplying the edges of the nested set of trees and the marginal densities $f(\cdot)$, as indicated below

$$\begin{aligned} & f_1(x_1; \alpha_1) \cdot f_2(x_2; \alpha_2) \cdot f_3(x_3; \alpha_3) \cdot f_4(x_4; \alpha_4) \cdot f_5(x_5; \alpha_5) \\ & \cdot c_{12}(F_1(x_1; \alpha_1), F_2(x_2; \alpha_2); \theta_{12}) \cdot c_{23}(F_2(x_2; \alpha_2), F_3(x_3; \alpha_3); \theta_{23}) \\ & \cdot c_{34}(F_3(x_3; \alpha_3), F_4(x_4; \alpha_4); \theta_{34}) \cdot c_{45}(F_4(x_4; \alpha_4), F_5(x_5; \alpha_5); \theta_{45}) \\ & \cdot c_{13|2}(F_{1|2}(x_1|x_2; \alpha_1, \alpha_2, \theta_{12}), F_{3|2}(x_3|x_2; \alpha_2, \alpha_3, \theta_{23}); \theta_{13|2}) \\ & \cdot c_{24|3}(F_{2|3}(x_2|x_3; \alpha_2, \alpha_3, \theta_{23}), F_{4|3}(x_4|x_3; \alpha_3, \alpha_4, \theta_{34}); \theta_{24|3}) \\ & \cdot c_{35|4}(F_{3|4}(x_3|x_4; \alpha_3, \alpha_4, \theta_{34}), F_{5|4}(x_5|x_4; \alpha_4, \alpha_5, \theta_{45}); \theta_{35|4}) \\ & \cdot c_{14|23}(F_{1|23}(x_1|x_2, x_3; \alpha_1, \alpha_2, \alpha_3, \theta_{12}, \theta_{23}, \theta_{13|2}), \\ & F_{4|23}(x_4|x_2, x_3; \alpha_2, \alpha_3, \alpha_4, \theta_{23}, \theta_{34}, \theta_{24|3}); \theta_{14|23}) \\ & \cdot c_{25|34}(F_{2|34}(x_2|x_3, x_4; \alpha_2, \alpha_3, \alpha_4, \theta_{23}, \theta_{34}, \theta_{24|3}), \\ & F_{5|34}(x_5|x_3, x_4; \alpha_3, \alpha_4, \alpha_5, \theta_{34}, \theta_{45}, \theta_{35|4}); \theta_{25|34}) \\ & \cdot c_{15|234}(F_{1|234}(x_1|x_2, x_3, x_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \theta_{12}, \theta_{23}, \theta_{34}, \theta_{13|2}, \theta_{24|3}, \theta_{14|23}), \\ & F_{5|234}(x_5|x_2, x_3, x_4; \alpha_2, \alpha_3, \alpha_4, \alpha_5, \theta_{23}, \theta_{34}, \theta_{45}, \theta_{24|3}, \theta_{35|4}, \theta_{25|34}); \theta_{15|234}). \end{aligned} \quad (11)$$

2.1.1 Copula-based Dependence Measures and Tail Dependence in Regular Vine Copulas

Because copulas describe the dependence structure among random variables, it is natural to think of dependence measures expressible in terms of the copula function. The Kendall's tau and the tail dependence² are useful copula-based dependence measures.

The Kendall's tau is defined as the difference between the probability of concordance and the probability of discordance. Let (X, Y) be a vector of continuous random variables, then the population version of Kendall's tau for X and Y is given by

$$\begin{aligned}\tau = \tau_{X,Y} &= P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0] \\ &= 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1,\end{aligned}$$

where C is the copula of X and Y .

Tail dependence measures the dependence in extreme values, for this reason it is an important measure for risk management. If the limit

$$\lim_{\varepsilon \rightarrow 0} \Pr[U_1 \leq \varepsilon | U_2 \leq \varepsilon] = \lim_{\varepsilon \rightarrow 0} \Pr[U_2 \leq \varepsilon | U_1 \leq \varepsilon] = \lim_{\varepsilon \rightarrow 0} C(\varepsilon, \varepsilon) / \varepsilon = \lambda_L$$

exists, then the copula C has lower tail dependence if $\lambda_L \in (0, 1]$ and no lower tail dependence if $\lambda_L = 0$. Similarly, if the limit

$$\lim_{\delta \rightarrow 1} \Pr[U_1 > \delta | U_2 > \delta] = \lim_{\delta \rightarrow 1} \Pr[U_2 > \delta | U_1 > \delta] = \lim_{\delta \rightarrow 1} (1 - 2\delta + C(\delta, \delta)) / (1 - \delta) = \lambda_U$$

exists, then the copula C has upper tail dependence if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. In other words, the lower (upper) tail dependence is the probability that one variable takes an extremely large negative (positive) value, given that the other variable took an extremely large negative (positive) value.

Recently, Joe, Li, and Nikoloulopoulos (2010) have found interesting results concerning tail dependence in vine copulas. They have a main theorem which states that if the supports of the pair-copulas in a vine are the entire $(0, 1)^2$ and all the pair-copulas in level 1 have lower (upper) tail dependence, then the vine copula C has lower (upper) tail dependence. If a copula C has multivariate lower (upper) tail dependence, then all bivariate and lower-dimensional margins have lower (upper) tail dependence. Another important finding is concerned with tail asymmetry of the vine copulas. They show that vine copulas can have different upper and lower tail dependence for each bivariate margin when asymmetric bivariate copulas with upper/lower tail dependence are used in level 1 of the vine.

2.2 Introducing Dynamics into the Vine Copula

Most of the works on vine copulas applied to financial data are focused on time-homogeneous models, however evidence found in the literature suggests that dependence among returns is not constant over time (see e.g., Longin and Solnik 2001; Ang and Bekaert 2002; Ang and Chen 2002). Therefore we introduce dynamics into the D-vine copula model, by allowing the dependence parameters of the pair-copulas to be potentially time-varying, evolving through time according to an equation that follows a restricted ARMA(1, m) process as in Patton (2006). The evolution equation of the dependence parameter $\theta_{i,i+j|v_{ij},t}$ of the pair-copula $c_{i,i+j|v_{ij}}$, with $v_{ij} = \{i+1, \dots, i+j-1\}$ and $v_{i1} = \emptyset$, for $1 \leq i \leq n-j, 1 \leq j \leq n-1$, may be written as

$$\theta_{i,i+j|v_{ij},t} = \Lambda(\omega + \beta \theta_{i,i+j|v_{ij},t-1} + \alpha \psi_t), \quad (12)$$

where Λ is a logistic transformation used to keep the parameter in its interval at all times and ψ_t is a forcing variable. The latter is defined as the mean absolute difference between the transformed data $u_{i|v_{ij},t} = F_{i|v_{ij}}(x_{i,t} | \mathbf{x}_{v_{ij},t})$ and $u_{i+j|v_{ij},t} = F_{i+j|v_{ij}}(x_{i+j,t} | \mathbf{x}_{v_{ij},t})$ over the past m observations, $\frac{1}{m} \sum_{k=1}^m |u_{i|v_{ij},t-k} - u_{i+j|v_{ij},t-k}|$. The idea is to use this measure as an indication of how far the data was from comonotonicity: if $X_i | \mathbf{X}_{v_{ij}}$ and $X_{i+j} | \mathbf{X}_{v_{ij}}$ are comonotonic, $|u_{i|v_{ij},t} - u_{i+j|v_{ij},t}|$ is close to zero. For the Gaussian copula, the cross-product $\frac{1}{m} \sum_{k=1}^m \Phi^{-1}(u_{i|v_{ij},t-k}) \cdot \Phi^{-1}(u_{i+j|v_{ij},t-k})$, where Φ^{-1} is the inverse of the standard Normal *c.d.f.*, is used as the forcing variable. For the Student- t , the forcing variable is $\frac{1}{m} \sum_{k=1}^m T_\nu^{-1}(u_{i|v_{ij},t-k}) \cdot T_\nu^{-1}(u_{i+j|v_{ij},t-k})$, where T_ν^{-1} is the inverse of the Student- t .*c.d.f.* with ν degrees of freedom. If data is positively dependent, the inverse of the transforms of both variables will have the same sign, thus, α is expected to be positive. Patton restricted m to be equal to 10, but here we do allow for different window lengths for the forcing variable. We assume $m = 5, 10$ or 15 in order to investigate the comovements over three different periods, which correspond to the last 1, 2 and 3 weeks, respectively, for daily returns.

3 Inference Procedure: Sequential Estimation

According to eq. (10), the log-likelihood function corresponding to a D-vine is given by

$$\begin{aligned}
 \ell(\boldsymbol{\alpha}, \boldsymbol{\theta}; \mathbf{x}) &= \\
 &= \sum_{t=1}^T \log(f(x_{1,t}, \dots, x_{n,t}; \boldsymbol{\alpha}, \boldsymbol{\theta})) \\
 &= \sum_{t=1}^T \sum_{k=1}^n \log(f_k(x_{k,t}; \boldsymbol{\alpha}_k)) \\
 &+ \sum_{t=1}^T \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \log(c_{i,i+j|v_{ij}}(F_{i|v_{ij}}(x_{i,t} | \mathbf{x}_{v_{ij},t}; \boldsymbol{\alpha}_{w_{i,j-1}}, \boldsymbol{\theta}_{i \rightarrow i+j-1}), \\
 &F_{i+j|v_{ij}}(x_{i+j,t} | \mathbf{x}_{v_{ij},t}; \boldsymbol{\alpha}_{w_{i+1,j-1}}, \boldsymbol{\theta}_{i+1 \rightarrow i+j}); \boldsymbol{\theta}_{i,i+j|v_{ij}})) \\
 &= \ell_M(\boldsymbol{\alpha}; \mathbf{x}) + \ell_C(\boldsymbol{\alpha}, \boldsymbol{\theta}; \mathbf{x})
 \end{aligned} \tag{13}$$

In particular, for the five-dimensional case, based on the density eq. (11), we have

$$\begin{aligned}
 &\sum_{t=1}^T \left(\log f_1(x_{1,t}; \boldsymbol{\alpha}_1) + \log f_2(x_{2,t}; \boldsymbol{\alpha}_2) + \log f_3(x_{3,t}; \boldsymbol{\alpha}_3) + \log f_4(x_{4,t}; \boldsymbol{\alpha}_4) + \log f_5(x_{5,t}; \boldsymbol{\alpha}_5) \right) \\
 &+ \sum_{t=1}^T \left(\log(c_{12}(F_1(x_{1,t}; \boldsymbol{\alpha}_1), F_2(x_{2,t}; \boldsymbol{\alpha}_2); \boldsymbol{\theta}_{12})) + \log(c_{23}(F_2(x_{2,t}; \boldsymbol{\alpha}_2), F_3(x_{3,t}; \boldsymbol{\alpha}_3); \boldsymbol{\theta}_{23})) \right. \\
 &\quad \left. + \log(c_{34}(F_3(x_{3,t}; \boldsymbol{\alpha}_3), F_4(x_{4,t}; \boldsymbol{\alpha}_4); \boldsymbol{\theta}_{34})) + \log(c_{45}(F_4(x_{4,t}; \boldsymbol{\alpha}_4), F_5(x_{5,t}; \boldsymbol{\alpha}_5); \boldsymbol{\theta}_{45})) \right) \\
 &+ \sum_{t=1}^T \left(\log(c_{13|2}(F_{1|2}(x_{1,t} | x_{2,t}; \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\theta}_{12}), F_{3|2}(x_{3,t} | x_{2,t}; \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\theta}_{23}); \boldsymbol{\theta}_{13|2})) \right. \\
 &\quad \left. + \log(c_{24|3}(F_{2|3}(x_{2,t} | x_{3,t}; \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\theta}_{23}), F_{4|3}(x_{4,t} | x_{3,t}; \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\theta}_{34}); \boldsymbol{\theta}_{24|3})) \right. \\
 &\quad \left. + \log(c_{35|4}(F_{3|4}(x_{3,t} | x_{4,t}; \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\theta}_{34}), F_{5|4}(x_{5,t} | x_{4,t}; \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5, \boldsymbol{\theta}_{45}); \boldsymbol{\theta}_{35|4})) \right) \\
 &+ \sum_{t=1}^T \left(\log(c_{14|23}(F_{1|23}(x_{1,t} | x_{2,t}, x_{3,t}; \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\theta}_{12}, \boldsymbol{\theta}_{23}, \boldsymbol{\theta}_{13|2}), \right. \\
 &\quad \left. F_{4|23}(x_{4,t} | x_{2,t}, x_{3,t}; \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\theta}_{23}, \boldsymbol{\theta}_{34}, \boldsymbol{\theta}_{24|3}); \boldsymbol{\theta}_{14|23})) \right. \\
 &\quad \left. + \log(c_{25|34}(F_{2|34}(x_{2,t} | x_{3,t}, x_{4,t}; \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\theta}_{23}, \boldsymbol{\theta}_{34}, \boldsymbol{\theta}_{24|3}), \right. \\
 &\quad \left. F_{5|34}(x_{5,t} | x_{3,t}, x_{4,t}; \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5, \boldsymbol{\theta}_{34}, \boldsymbol{\theta}_{45}, \boldsymbol{\theta}_{35|4}); \boldsymbol{\theta}_{25|34})) \right) \\
 &+ \sum_{t=1}^T \left(\log(c_{15|234}(F_{1|234}(x_{1,t} | x_{2,t}, x_{3,t}, x_{4,t}; \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\theta}_{12}, \boldsymbol{\theta}_{23}, \boldsymbol{\theta}_{34}, \boldsymbol{\theta}_{13|2}, \boldsymbol{\theta}_{24|3}, \boldsymbol{\theta}_{14|23}), \right. \\
 &\quad \left. F_{5|234}(x_{5,t} | x_{2,t}, x_{3,t}, x_{4,t}; \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_5, \boldsymbol{\theta}_{23}, \boldsymbol{\theta}_{34}, \boldsymbol{\theta}_{45}, \boldsymbol{\theta}_{24|3}, \boldsymbol{\theta}_{35|4}, \boldsymbol{\theta}_{25|34}); \boldsymbol{\theta}_{15|234})) \right).
 \end{aligned} \tag{14}$$

The hierarchical structure of the vine copula allows us to adopt a very fast still asymptotically efficient sequential estimation procedure (Haff 2013). It is clear from the decomposition eq. (14) above that the joint log-likelihood function of a five-dimensional D-vine can be separated in, at least, five parts: the one corresponding to the log-likelihoods of the marginal distributions, and those corresponding to the log-likelihoods associated with the different levels of the vine, i.e. with the pair-copulas with increasing conditioning sets. In this case, estimation can proceed in five steps, and, at each step, it is carried out conditionally on the parameters estimated in earlier steps. In the first step, the parameters of the marginal distributions are estimated via maximum likelihood and the log-returns are transformed into uniforms, which become inputs for the pair-copulas in the first level of the D-vine. In the second step, the parameters of the pair-copulas in the first tree are estimated via maximum likelihood, taking the parameters of the marginals as fixed at the estimated values from the first step. Additionally, the transformed data (i.e. conditional distributions) necessary for tree 2 are computed using eq. (9). In the sequel, the parameters of the pair-copulas in level 2 are estimated, given the estimates of the parameters in the previous levels. We proceed like this till we have the parameters estimates for all trees. It means that the whole procedure consists of a series of optimizations for the marginals and for iteratively conditioned bivariate copulas. The levelwise estimation improves significantly the computational efficiency, which is very important because the number of parameters to be estimated increases rapidly with the number of variables. Following, we give details on the model specification.

3.1 GARCH Models for the Univariate Distributions

A univariate ARMA(p, q)-GARCH(m, n)³ specification is usually chosen to model the marginal distributions of return data. It can be described by the following equations:

$$x_t = \phi_0 + \sum_{i=1}^p \phi_i x_{t-i} + a_t - \sum_{j=1}^q \theta_j a_{t-j} = \mu_t + a_t \quad (15)$$

$$a_t = h_t^{1/2} \varepsilon_t, \quad (16)$$

$$h_t = \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2 + \sum_{j=1}^n \beta_j h_{t-j}, \quad (17)$$

where μ_t and h_t are the conditional mean and variance given past information, respectively. ε_t is the innovation process and, in this paper, we assume that it may have a standard Normal distribution, $\varepsilon_t \sim \text{Normal}(0, 1)$, a Student- t distribution, $\varepsilon_t \sim \text{Student-}t(\nu)$, or a Skewed- t distribution proposed by Hansen (1994), $\varepsilon_t \sim \text{Skewed-}t(\nu, \lambda)$.

If, for example, $\varepsilon_t \sim \text{Skewed-}t(\nu, \lambda)$, then the conditional distribution function of X_t is given by $F(x_t | \mu_t, h_t) = \text{Skewed-}t_{\nu, \lambda}((x_t - \mu_t)h_t^{-1/2})$. $u_t = F(x_t | \mu_t, h_t) \sim U[0, 1]$ is referred to as a probability integral transform (PIT) variable. Thus, if the marginal distribution is well specified, the PIT variable has a uniform distribution in $[0, 1]$. This is a necessary result to identify the copulas in the second step of the estimation procedure, since they are joint distribution functions defined over $u_{i,t} = F_i(x_{i,t} | \mu_{i,t}, h_{i,t})$ and $u_{i+1,t} = F_{i+1}(x_{i+1,t} | \mu_{i+1,t}, h_{i+1,t})$, $i = 1, \dots, n-1$, with $u_{i,t}$ and $u_{i+1,t}$ uniforms in $[0, 1]$. To test whether the PIT variable has distribution $U[0, 1]$, we use the Kolmogorov-Smirnov test of goodness-of-fit.

In summary, in the first step of the sequential estimation procedure, the ARMA-GARCH filter is applied to the return data to obtain the univariate parameters estimates and the transformed data (uniforms) necessary for the first level of the D-vine.

3.2 Building Blocks: Bivariate Copulas

For fitting multivariate financial data with flexible lower/upper tail dependence, possibly with stronger tail dependence in the joint lower tail than upper tail, we consider copula families with different strengths of tail behavior in the estimation of the D-vine copula:

1. BB1, BB7 and Symmetrized Joe-Clayton (SJC) copulas, which have different upper and lower tail dependence.
2. Gumbel copula, with only upper tail dependence.
3. Rotated-Gumbel and Clayton copulas, with only lower tail dependence.
4. Student- t copula, with reflection symmetric⁴ upper and lower tail dependence.
5. Normal copula, also reflection symmetric, but with no tail dependence at all.

For each pair of transformed data, we estimate both a static and a dynamic version of such copulas⁵ and we use the AIC to choose the best of all of them. In addition, to verify whether the dependence structure between the data was appropriately modeled, we apply the Kolmogorov-Smirnov and the Anderson-Darling goodness-of-fit tests.

4 Simulation Study

In this section, we carry out a simulation study in order to evaluate the performance of the dynamic D-vine copula model. We consider two data generating processes (DGPs) in the Monte Carlo study: (i) the dynamic D-vine and (ii) a static D-vine copula. In both cases, we choose a decomposition with all Rotated-Gumbel pair-copulas and also one with all BB1 pair-copulas to account for the evidence of asymmetric tail dependence in financial data. In addition, we consider a decomposition with all Normal pair-copulas to allow for reflection symmetry

too. Using the simulation algorithm for a D-vine in Aas et al. (2009)⁶, we replicate 1,000 four-dimensional time series, with $T = 1,000$, $T = 2,000$ and $T = 5,000$ observations.

The dependence dynamics of the time-varying D-vine copula model is as defined in Section 2.2, with $\theta_{i,i+j|v_{ij},t}$, the dependence parameter of the pair-copula density $c_{i,i+j|v_{ij}}$ in t , with $v_{ij} = \{i+1, \dots, i+j-1\}$ and $v_{i1} = \emptyset$, for $1 \leq i \leq 4-j$, $1 \leq j \leq 3$, following a restricted ARMA(1, 10) process. The coefficients of the evolution equation of the time-varying parameters are as set in Table B.1 of the online Appendix. This dynamic structure comprises better defined time paths for the dependence parameters of the pair-copulas in the first level of the vine and noisier time paths in higher levels. Furthermore, the degree of dependence is higher in the first tree than in the second and third ones. These characteristics are intended to reproduce evidences found for real data. Table B.1 also presents the values of the dependence parameters of the static D-vine copula. Again, in this case, we have greater dependence in the first level of the vine.

Having simulated the data sets, for each time series replicated from both DGPs, we estimate models (i) and (ii)⁷ above-mentioned. To simplify the exercise, in the identification we model the correct pair-copula function. We then compute the mean errors (ME) and root mean squared errors (RMSE) based on the difference between the estimates and the true values of the pair-copulas dependence parameters over time.

Table 1 and Table 2 report the mean, median and standard deviation of the mean errors of the Monte Carlo study considering a D-vine decomposition with all Rotated-Gumbel pair-copulas. For the decompositions with all BB1 and all Normal pair-copulas, descriptive statistics of the mean errors are presented in Tables B.2 to B.7 of the online Appendix to save space. One can easily notice that, when the DGP is the dynamic D-vine copula with all Rotated-Gumbel pair-copulas (Table 1), the static D-vine copula tends to underestimate the dependence parameters. It can be inferred from the negative medians of the mean errors associated with the estimates from the static model. Though the negative bias is greater in the estimates of the pair-copulas parameters in the first tree, which is expected, given that the dynamics is better defined at this level in the source model, it also occurs in the estimates of the pair-copulas parameters in higher levels. The same is true when the samples are drawn from the dynamic D-vine with BB1 pair-copulas: the static model tends to underestimate parameter γ , with the exception of γ_{34} , as can be observed in Table B.4. The static D-vine copula also provides biased estimates of parameter κ (Table B.2). Moreover, the median of the mean errors associated with the estimates from the static model barely changes given increases in the sample size, although the variability of the mean errors decreases, which means that the bias does not diminish as the number of observations in the sample increases. Taking the dynamic D-vine with Normal pair-copulas as the DGP (Table B.6), the static model provides unbiased estimates of the pair-copulas parameters in level one, but tends to underestimate the dependence parameters of levels two and three, and the bias is persistent, i.e. it does not diminish as the sample size increases. On the other hand, when the samples drawn from the dynamic D-vine copula are estimated with the same model, for the parameters of all pair-copulas, the mean errors have median closer to zero and their variability decreases as the sample size increases. Considering, now, that the static D-vine copula is the DGP (Table 2 and Tables B.3, B.5 and B.7 in the online Appendix), we can observe that, for either the dynamic or the static model estimates, the mean errors have medians close to zero, which means that the estimates from both models are unbiased, and their variabilities tend to decrease as the sample size increases. So, the previous results suggest that the dynamic D-vine copula model outperforms the static D-vine copula in terms of the mean errors, or bias.

Table 1: Statistics of the mean errors of the Monte Carlo study – DGP dynamic D-vine with Rotated-Gumbel pair-copulas.

	$T = 1,000$			$T = 2,000$			$T = 5,000$		
	M	Md	SD	M	Md	SD	M	Md	SD
<i>Tree 1</i>									
ME12,s	−0.1321	−0.1338	0.0905	−0.1372	−0.1392	0.0633	−0.1408	−0.1423	0.0400
ME12,d	0.0025	−0.0013	0.0761	0.0031	0.0048	0.0542	0.0016	−0.0004	0.0346
ME23,s	−0.1043	−0.0942	0.1281	−0.1173	−0.1161	0.0877	−0.1165	−0.1153	0.0557
ME23,d	0.0106	0.0126	0.1002	−0.0006	−0.0016	0.0656	0.0016	0.0013	0.0430
ME34,s	−0.1029	−0.1047	0.0872	−0.1050	−0.1030	0.0656	−0.1096	−0.1103	0.0408
ME34,d	0.0043	−0.0012	0.0771	0.0037	0.0028	0.0562	−0.0004	−0.0018	0.0358
<i>Tree 2</i>									
ME13 2,s	−0.0073	−0.0079	0.0225	−0.0075	−0.0078	0.0152	−0.0086	−0.0089	0.0098
ME13 2,d	0.0026	0.0021	0.0221	0.0013	0.0011	0.0153	5.86E−05	−4.0E−06	0.0096
ME24 3,s	−0.0055	−0.0057	0.0201	−0.0066	−0.0069	0.0144	−0.0065	−0.0069	0.0090
ME24 3,d	0.0042	0.0029	0.0193	0.0015	0.0009	0.0140	0.0007	0.0004	0.0091

Tree 3

ME14 23,s	−0.0092	−0.0100	0.0237	−0.0103	−0.0104	0.0164	−0.0111	−0.0111	0.0103
ME14 23,d	0.0022	0.0014	0.0235	0.0010	0.0007	0.0167	−3.3E−05	−0.0001	0.0104

Note: $ME_{i,i+j|v_{ij},d}$ and $ME_{i,i+j|v_{ij},s}$ refer to the mean errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i,i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the mean errors, respectively.

Table 2: Statistics of the mean errors of the Monte Carlo study – DGP static D-vine with Rotated-Gumbel pair-copulas.

	T = 1,000			T = 2,000			T = 5,000		
	M	Md	SD	M	Md	SD	M	Md	SD
<i>Tree 1</i>									
ME12,s	-0.0009	-0.0007	0.0809	0.0016	0.0045	0.0571	0.0006	-0.0011	0.0365
ME12,d	0.0054	0.0043	0.0821	0.0046	0.0070	0.0573	0.0018	-6.4E-05	0.0368
ME23,s	0.0079	0.0107	0.1041	-0.0007	0.0008	0.0673	0.0016	0.0014	0.0442
ME23,d	0.0153	0.0191	0.1047	0.0031	0.0041	0.0686	0.0030	0.0027	0.0444
ME34,s	0.0014	-0.0025	0.0799	0.0027	0.0017	0.0588	-0.0014	-0.0025	0.0360
ME34,d	0.0080	0.0031	0.0822	0.0052	0.0036	0.0590	-1.4E-05	-0.0013	0.0366
<i>Tree 2</i>									
ME13 2,s	0.0013	0.0002	0.0288	0.0004	0.0001	0.0203	-0.0003	-0.0001	0.0127
ME13 2,d	0.0030	0.0015	0.0297	0.0012	0.0007	0.0206	0.0001	0.0002	0.0128
ME24 3,s	0.0003	-0.0001	0.0262	-0.0001	-0.0007	0.0192	0.0004	0.0003	0.0124
ME24 3,d	0.0020	0.0013	0.0268	0.0007	0.0001	0.0193	0.0009	0.0007	0.0124
<i>Tree 3</i>									
ME14 23,s	0.0011	0.0012	0.0233	8.34E-05	-0.0004	0.0162	-0.0002	-0.0002	0.0103
ME14 23,d	0.0021	0.0013	0.0238	0.0007	-2.2E-05	0.0167	-2.6E-05	-0.0002	0.0104

Note: $MEi, i + j|v_{ij}, d$ and $MEi, i + j|v_{ij}, s$ refer to the mean errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i,i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the mean errors, respectively.

Summary statistics of the root mean squared errors are presented in Table 3 and Table 4 for the decomposition with all Rotated-Gumbel pair-copulas, whereas for the decompositions with BB1 and Normal pair-copulas, these statistics are reported in Tables B.8–B.13 in the online Appendix to save space. When the underlying data generating model is the dynamic D-vine copula with Rotated-Gumbel or BB1 pair-copulas, regarding the parameters of the first level of the vine, the medians of the errors associated with the static D-vine copula estimates are higher than those related to the errors of the dynamic D-vine copula, as can be noted in Table 3 and Tables B.8 and B.10 of the online Appendix, except for parameter γ_{34} . Furthermore, they do not seem to decrease as the number of observations in the samples increases, although the errors variabilities diminish. It suggests that, even though the estimates from the static model show less variability as the sample size increases, the bias, previously analyzed, does not allow for precise estimates. Contrary to what happens to the static model, both the median and dispersion of the errors from the dynamic model decrease as the number of observations in the samples increases, which indicates that the estimates become more precise. Concerning the parameters of the pair-copulas in the second and third levels of the vine, both static and dynamic models display smaller values for the errors medians and these are closer to each other, if compared with the first level. Usually the dynamic model displays smaller values for the errors medians in comparison with the static model but, in some cases, the opposite happens. However, while the estimates from the dynamic model become more precise, with both median and variability of the errors decreasing as the number of observations in the sample increases, the errors medians from the static model barely change, which may be explained by the persistent bias. When the DGP is the dynamic D-vine with Normal pair-copulas (Table B.12), the medians of the errors from the static model are also higher than those of the errors from the dynamic model, for all pair-copulas parameters, even the ones of the first level, whose estimates are unbiased. When the samples are drawn from the static D-vine copula (Table 4 and Tables B.9, B.11 and B.13 of the online Appendix), both the median and the variability of the errors from the dynamic D-vine are higher than those of the static model, for all pair-copulas parameters. Consequently, although the dynamic D-vine copula provides unbiased estimates when the DGP is static, the variability of its estimates is higher than that of the static D-vine estimates. One should take note, though, that both the median and the variability of the errors from the dynamic model decrease as the sample size increases. In summary, in terms of the root mean squared errors, the dynamic D-vine copula is superior to the static D-vine when the data comes from the dynamic model. Nevertheless, the dynamic D-vine is not able to outperform the static D-vine when the latter is the DGP, given the higher variability of the former's estimates in this case.

Table 3: Statistics of the root mean squared errors of the Monte Carlo study – DGP dynamic D-vine with Rotated-Gumbel pair-copulas.

	T = 1,000				T = 2,000				T = 5,000			
	M	Md	SD		M	Md	SD		M	Md	SD	
<i>Tree 1</i>												
RMSE12,s	0.6008	0.5977	0.0645		0.6070	0.6055	0.0449		0.6079	0.6071	0.0277	
RMSE12,d	0.1281	0.1237	0.0579		0.0900	0.0835	0.0448		0.0570	0.0528	0.0276	
RMSE23,s	0.5779	0.5595	0.0865		0.5783	0.5700	0.0626		0.5757	0.5731	0.0388	
RMSE23,d	0.1399	0.1327	0.0716		0.0953	0.0906	0.0478		0.0587	0.0547	0.0288	
RMSE34,s	0.5299	0.5218	0.0540		0.5312	0.5272	0.0408		0.5311	0.5301	0.0255	
RMSE34,d	0.1252	0.1193	0.0554		0.0900	0.0861	0.0398		0.0552	0.0512	0.0254	
<i>Tree 2</i>												
RMSE13 2,s	0.0380	0.0352	0.0084		0.0349	0.0334	0.0050		0.0333	0.0323	0.0030	
RMSE13 2,d	0.0356	0.0322	0.0193		0.0247	0.0223	0.0130		0.0156	0.0147	0.0069	
RMSE24 3,s	0.0400	0.0388	0.0073		0.0375	0.0364	0.0050		0.0357	0.0353	0.0027	
RMSE24 3,d	0.0358	0.0318	0.0187		0.0255	0.0227	0.0136		0.0159	0.0148	0.0074	
<i>Tree 3</i>												
RMSE14 23,s	0.0379	0.0344	0.0098		0.0350	0.0330	0.0061		0.0331	0.0321	0.0037	
RMSE14 23,d	0.0373	0.0330	0.0205		0.0253	0.0230	0.0129		0.0162	0.0153	0.0073	

Note: RMSE $_{i, i+j|v_{ij}, d}$ and RMSE $_{i, i+j|v_{ij}, s}$ refer to the root mean squared errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i, i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the root mean squared errors, respectively.

Table 4: Statistics of the root mean squared errors of the Monte Carlo study – DGP static D-vine with Rotated-Gumbel pair-copulas.

	T = 1,000				T = 2,000				T = 5,000			
	M	Md	SD		M	Md	SD		M	Md	SD	
<i>Tree 1</i>												
RMSE12,s	0.0646	0.0540	0.0488		0.0460	0.0398	0.0338		0.0288	0.0244	0.0224	
RMSE12,d	0.1297	0.1252	0.0597		0.0889	0.0859	0.0414		0.0559	0.0533	0.0254	
RMSE23,s	0.0831	0.0699	0.0631		0.0535	0.0443	0.0407		0.0352	0.0303	0.0268	
RMSE23,d	0.1600	0.1489	0.0800		0.1107	0.1049	0.0513		0.0693	0.0661	0.0314	
RMSE34,s	0.0631	0.0524	0.0490		0.0475	0.0403	0.0347		0.0284	0.0238	0.0222	
RMSE34,d	0.1298	0.1251	0.0631		0.0911	0.0879	0.0430		0.0561	0.0526	0.0270	
<i>Tree 2</i>												
RMSE13 2,s	0.0229	0.0192	0.0176		0.0162	0.0134	0.0122		0.0102	0.0086	0.0076	
RMSE13 2,d	0.0446	0.0422	0.0225		0.0309	0.0289	0.0148		0.0194	0.0183	0.0091	
RMSE24 3,s	0.0209	0.0174	0.0159		0.0150	0.0123	0.0119		0.0100	0.0088	0.0074	
RMSE24 3,d	0.0432	0.0404	0.0225		0.0305	0.0286	0.0178		0.0185	0.0178	0.0082	
<i>Tree 3</i>												
RMSE14 23,s	0.0181	0.0149	0.0147		0.0130	0.0116	0.0095		0.0082	0.0069	0.0063	
RMSE14 23,d	0.0386	0.0350	0.0207		0.0260	0.0237	0.0135		0.0158	0.0149	0.0076	

Note: RMSE $_{i, i+j|v_{ij}, d}$ and RMSE $_{i, i+j|v_{ij}, s}$ refer to the root mean squared errors corresponding to the estimates from the (i) dynamic D-vine copula and (ii) static D-vine copula, respectively, for the parameter of the pair-copula $c_{i, i+j|v_{ij}}$. M, Md and SD denote the mean, median and standard deviation of the root mean squared errors, respectively.

Overall, the findings of the Monte Carlo experiments are quite favorable to the dynamic D-vine copula. Notedly, when the DGP is the time-varying model, the static model tends to provide biased estimates of the pair-copulas dependence parameters. Furthermore, the bias does not seem to diminish as the number of observations in the samples increases. The estimates from the dynamic D-vine copula, in this case, are superior to the estimates from the static D-vine, both in terms of the mean errors and the root mean squared errors. When the data comes from the static D-vine copula, both models have similar performance in terms of the mean errors, with unbiased estimates. However, the dynamic D-vine copula performs worse in terms of the root mean squared errors, what suggests that its estimates display higher variability, though it tends to diminish as the sample size increases.

5 Empirical Application: Dependence Modeling and VaR Backtesting

In this section, we model the dependence among the returns of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA indexes, using both the dynamic D-vine copula model and a static D-vine copula. We consider two distinct periods, one from January 03, 2003 to December 28, 2007 and another one from January 02, 2008 to May 04, 2012, which we denominate “non-crisis period” and “crisis period”, respectively. Besides of investigating the different patterns of dependence characterizing these periods, the intention here is to evaluate the dynamic D-vine copula model concerning the accuracy of the VaR forecasts in crisis periods.

5.1 Return Data

In our empirical study, we use two data sets of daily log-returns of the indexes DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA: one comprising the period from January 03, 2003 to December 28, 2007, which we call “non-crisis period” because the financial markets were in an upturn trend till September, 2007, with a total of 1,178 observations; and another one spanning the period from January 02, 2008 to May 04, 2012, considered a “crisis period” because it coincides with the subprime crisis till June, 2009, and with the European sovereign debt crisis from early 2010 and thereafter, with 1,029 observations. We use close-to-close returns, meaning that the daily returns are those observed for trading days occurring simultaneously in all five stock markets considered.

Table 5 provides a few descriptive statistics of our data sets. We can see from the table that the average returns of all indexes become negative in the crisis period and the standard deviations increase. It is also possible to notice that both data sets present signs of non-normality. All returns series have kurtosis above 3, and the excess kurtosis is higher in the crisis period. FTSE 100 and S&P 500 returns display negative skewness in both periods, whereas DAX, CAC 40 and IBOVESPA returns change from negative to positive skewness. Also, according to the Jarque-Bera test statistics, it is possible to reject the null hypothesis of normality for all indexes returns in both periods.

Table 5: Summary statistics of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA log-returns.

	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
Non-crisis period					
Mean	8.1810E-04	4.8050E-04	4.0711E-04	4.1290E-04	0.0014
Median	0.0013	9.0553E-04	7.8933E-04	8.4515E-04	0.0020
Maximum	0.0661	0.0700	0.0590	0.0348	0.0516
Minimum	-0.0579	-0.0583	-0.0492	-0.0359	-0.0686
Std. Deviation	0.0123	0.0112	0.0093	0.0085	0.0168
Asymmetry	-0.2246	-0.1134	-0.1191	-0.2174	-0.3382
Kurtosis	5.8964	6.4352	6.7222	4.6685	3.6245
Jarque-Bera	418.1 (0.0000)	577.1 (0.0000)	677.5 (0.0000)	144.3 (0.0000)	41.1 (0.0000)
Crisis period					
Mean	-2.0078E-04	-5.6018E-04	-1.3186E-04	-7.4701E-05	-4.7780E-05
Median	4.1623E-04	-2.6411E-04	-4.8268E-05	9.4609E-04	7.1406E-04
Maximum	0.1080	0.1059	0.0938	0.1096	0.1368
Minimum	-0.0774	-0.0947	-0.0926	-0.0947	-0.1210
Std. Deviation	0.0187	0.0195	0.0163	0.0177	0.0215
Asymmetry	0.1161	0.1458	-0.0381	-0.2247	0.0723
Kurtosis	7.3303	7.0833	8.2463	8.9963	9.1444
Jarque-Bera	799.4 (0.0000)	712.3 (0.0000)	1,170.8 (0.0000)	1,538.3 (0.0000)	1,607.1 (0.0000)

Note: Jarque-Bera corresponds to the Jarque-Bera test statistics, with *p*-values in parentheses.

5.2 Marginal Models

We first proceed to the modeling of the marginal distributions using the ARMA-GARCH specification. To account for the leverage effect, present in financial time series, we also consider asymmetric GARCH specifications, such as the EGARCH and GJR models⁸.

We choose the best specifications for the marginals based on the information criteria AIC and BIC. In the non-crisis period, we choose an AR(1)-EGARCH(1,1) for S&P 500, an AR(1)-GARCH(1,1) for FTSE 100, a GARCH(1,1) for both CAC 40 and IBOVESPA, and, finally, an EGARCH(1,1) for DAX. In the crisis period, we choose an AR(1)-EGARCH(2,1) for S&P 500, an AR(3)-EGARCH(1,1) for IBOVESPA, and a GARCH(1,1) for FTSE 100, CAC 40 and DAX, with conditional means modeled by an AR(2) in the first two cases. Because our data sets display clear signs of asymmetry and excess kurtosis, we use *Skewed* – $t(\nu, \lambda)$ distributed iid innovations.⁹ The estimates from the ARMA-GARCH fits¹⁰ are presented in Table 6 and Table 7. We can observe that the estimated asymmetry coefficient, $\hat{\lambda}$, is negative and statistically significant for all series in both periods, suggesting a heavy tail to the left for the marginal distributions, i.e., large negative returns are more likely than large positive returns. The estimated degrees of freedom, $\hat{\nu}$, range from 11.5421 for S&P 500 to 18.8979 for FTSE 100 in the non-crisis period, and from 8.7683 for DAX to 14.3925 for FTSE 100 in the crisis period, suggesting heavier tails in the second period, which is in accordance with the descriptive statistics of the unconditional distributions of the returns series. Regarding the conditional variances, in some cases, we choose the EGARCH model as the best specification, suggesting the presence of some sort of leverage effect. This is the case, for example, of the S&P 500 returns, which display an asymmetric effect over the volatility, with greater impact induced by big negative returns, captured by the estimates of γ_1 , -0.0820 in the non-crisis period and -0.1642 in the crisis period. Note that the leverage effect is intensified in the crisis period. We also find evidence of leverage effect for IBOVESPA returns in the crisis period, with $\hat{\gamma}_1 = -0.0963$, and for DAX returns in the non-crisis period, with $\hat{\gamma}_1 = -0.1127$. Table 6 and Table 7 also provide the p -values of the Ljung-Box test of autocorrelation in the standardized and squared standardized residuals with 15 lags, $Q(15)$ and $Q^2(15)$, respectively. For all series, the null hypothesis of no autocorrelation left cannot be rejected at the 5% level, indicating that the ARMA-GARCH specifications are successful at modeling the serial correlation in the conditional mean and variance. Additionally, these tables report the p -values of the Kolmogorov–Smirnov test of uniformity of the PIT of the standardized residuals. For all series, there is no evidence against uniformity, so all marginal distributions seem to be well specified, which is very important, since, otherwise, the copula estimation would be affected.

Table 6: Estimates from the univariate ARMA-GARCH models for the non-crisis period.

Parameter	Conditional mean equation				
	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
ϕ_0	0.0008 (0.0003)	0.0005 (0.0002)	...	0.0005 (0.0002)	0.0014 (0.0005)
ϕ_1	...	...	-0.1162 (0.0230)	-0.1039 (0.0005)	...
Parameter	Conditional variance equation				
	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
α_0	-0.1586 (0.0520)	$2.3943\text{E}-06$ (0.0000)	$1.9933\text{E}-06$ (0.0000)	-0.1036 (0.0668)	$8.5316\text{E}-06$ (0.0000)
α_1	0.1182 (0.0201)	0.0797 (0.0165)	0.1052 (0.0225)	0.0702 (0.0207)	0.0437 (0.0106)
γ_1	-0.1127 (0.0243)	...	...	-0.0820 (0.0175)	...
β_1	0.9825 (0.0057)	0.8978 (0.0206)	0.8705 (0.0265)	0.9893 (0.0069)	0.9258 (0.0173)
ν	11.8568 (4.3112)	14.8531 (5.4413)	18.8979 (8.5510)	11.5421 (4.4127)	16.3456 (6.8297)
λ	-0.1736 (0.0352)	-0.1421 (0.0403)	-0.1907 (0.0384)	-0.1485 (0.0301)	-0.1670 (0.0439)
$Q(15)$	0.5760	0.5948	0.5433	0.1916	0.9503

$Q^2(15)$	0.8239	0.9945	0.6798	0.1666	0.8767
K-S test	0.4648	0.9996	0.2570	0.3906	0.8364

Note: Standard errors in parentheses. $Q(15)$, $Q^2(15)$ and K-S test are p -values.

Table 7: Estimates from the univariate ARMA-GARCH models for the crisis period.

Parameter	Conditional mean equation				
	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
ϕ_1	...	...	...	−0.1336 (0.0249)	...
ϕ_2	...	−0.0763 (0.0038)	−0.0933 (0.0020)	...	...
ϕ_3	...	...	...	...	−0.0667 (0.0025)
	...	...	...	...	...
Parameter	Conditional variance equation				
	DAX	CAC 40	FTSE 100	S&P 500	IBOVESPA
α_0	3.8372E−06 (0.0000)	5.5846E−06 (0.0000)	3.1461E−06 (0.0000)	−0.2446 (0.0650)	−0.1092 (0.0742)
α_1	0.0872 (0.0192)	0.0945 (0.0237)	0.0971 (0.0222)	−0.1913 (0.0680)	0.1565 (0.0471)
α_2	...	...	...	0.3375 (0.0730)	...
γ_1	...	...	...	−0.1642 (0.0235)	−0.0963 (0.0285)
β_1	0.9020 (0.0193)	0.8900 (0.0240)	0.8901 (0.0227)	0.9709 (0.0076)	0.9861 (0.0093)
ν	8.7683 (2.4751)	11.1004 (3.6827)	14.3925 (6.5502)	8.9613 (2.4611)	10.4290 (3.0912)
λ	−0.0730 (0.0329)	−0.0914 (0.0394)	−0.1111 (0.0361)	−0.2012 (0.0337)	−0.0975 (0.0377)
$Q(15)$	0.9581	0.9914	0.7119	0.6389	0.9749
$Q^2(15)$	0.4818	0.3397	0.4298	0.1873	0.8453
K-S test	0.3417	0.2214	0.7452	0.4970	0.7881

Note: Standard errors in parentheses. $Q(15)$, $Q^2(15)$ and K-S test are p -values.

5.3 Copula Structure

Having chosen the D-vine decomposition of the multivariate copula, we still have to match the indexes returns to the labels 1, ..., 5, since there are $5!/2$ possible distinct permutations. A rule to select the best permutation for D-vines, according to Nikoloulopoulos, Joe, and Li (2012), consists of choosing and connecting the most dependent pairs in the first tree. Using the sample Kendall's taus computed based on the PIT of the ARMA-GARCH residuals, reported in Tables C.2 and C.3 of the online Appendix, we choose as the best permutation for the first level of the D-vines of both periods under analysis $(1, 2, 3, 4, 5) = (\text{FTSE 100}, \text{CAC 40}, \text{DAX}, \text{S\&P 500}, \text{IBOVESPA})$, since it comprises the largest possible dependencies.

Table 8 and Table 9 report the estimates of the pair-copulas chosen to compose the dynamic D-vine copula in the non-crisis period and in the crisis period, respectively.¹¹ We can observe that time-varying pair-copulas are selected only in the first tree.¹² The dependence between FTSE and CAC during the non-crisis period is characterized by the BB1 copula, with the estimated parameter $\hat{\gamma}_t$ following an ARMA(1, 10), whereas $\hat{\kappa}$ remains constant, equal to 0.7813. The same copula characterizes their dependence during the crisis period too, however, in this case, both estimated parameters $\hat{\gamma}_t$ and $\hat{\kappa}_t$ evolve through time according to an ARMA(1, 5). We also choose a BB1 copula for the pair CAC-DAX: in the non-crisis period, $\hat{\gamma}_t$ follows a MA(10) and $\hat{\kappa}$ is constant,

equal to 0.7252, whereas in the crisis period, $\hat{\gamma}_t$ evolves according to an ARMA(1, 10) and $\hat{\kappa}$ remains constant, equal to 0.9915. The reflection symmetric Student- t copula is selected for both pairs DAX-S&P500 and S&P500-IBOVESPA in both periods. With regard to the dynamics, for the pair DAX-S&P500, during the non-crisis period, there is evidence of time variation for the correlation coefficient, with $\hat{\rho}_t$ following an ARMA(1, 10), whereas the estimated degrees of freedom remain constant, equal to 8.7969; in the crisis period, we find no dynamics at all for both estimated parameters, $\hat{\rho} = 0.7088$ and $\hat{\nu} = 18.8746$. For S&P500-IBOVESPA, $\hat{\rho}_t$ follows an ARMA(1, 10) and $\hat{\nu} = 6.9966$, in the non-crisis period, whereas $\hat{\rho}_t$ follows an ARMA(1, 15) and $\hat{\nu} = 9.5523$, in the crisis period.

Table 8: Estimation results of the dynamic D-vine copula for the non-crisis period.

Bivariate Copula		θ_1	θ_2	ω_1	α_1	β_1	ω_2	α_2	β_2
Tree 1									
FTSE, CAC	BB1 tvp	0.7813 (0.0863)					0.3303 (0.0635)	-0.7987 (0.2189)	0.3795 (0.0195)
CAC, DAX	BB1 tvp	0.7252 (0.0905)					1.6422 (0.0790)	-4.3897 (0.6589)	-
DAX, S&P500	Student- <i>t</i> tvp		8.7969 (2.2623)	1.7175 (0.3564)	0.2465* (0.1343)	-1.2680* (0.7013)			
S&P500, IBOVESPA	Student- <i>t</i> tvp		6.9966 (1.4740)	-0.3757 (0.0016)	0.0584 (0.0041)	2.8970 (0.0005)			
Tree 2									
FTSE, DAX CAC	Student- <i>t</i>	0.0723 (0.0314)	10.6398 (3.3715)						
CAC, S&P500 DAX	Normal	0.0847 (0.0297)							
DAX, IBOVESPA S&P500	Rotated-Gumbel	1.0568 (0.0187)							
Tree 3									
FTSE, S&P500 CAC, DAX	Normal	0.0978 (0.0293)							
CAC, IBOVESPA DAX, S&P500	Normal	0.0660 (0.0290)							
Tree 4									
FTSE, IBOVESPA CAC, DAX, S&P500	Normal	0.0754 (0.0300)							

Note: Estimates obtained using the sequential estimation procedure. Standard errors in parentheses. (*) stands for significant only at the 10% level. In the second column, we have the pair-copula selected based on the AIC criterion. The copula name followed by “tv” means that at least one of the copula’s parameters is time-varying. The columns labeled θ_i , $i = 1, 2$ (considering two-parameter copulas), present the estimates of the constant parameters. ω_i , α_i and β_i , refer to the coefficients of the evolution equation of the time-varying parameter, $\theta_{i,t} = \Lambda(\omega_i + \beta_i \theta_{i,t-1} + \alpha_i \psi_t)$, $i = 1, 2$, as defined in Section 2.2 and presented in Appendix A for the time-varying parameters of the different copula functions adopted in this paper. With regard to the BB1 tvp copula selected for both FTSE-CAC and CAC-DAX, only the estimated parameter γ_t displays time variation, following an ARMA(1, 10) for FTSE-CAC and MA(10) for CAC-DAX, whereas $\bar{\kappa}$ remains constant. Regarding the Student- t tvp chosen for both DAX-S&P500 and S&P500-IBOVESPA, only the estimated correlation coefficient is time-varying, following an ARMA(1, 10), whereas the estimated degrees of freedom remain constant.

Table 9: Estimation results of the dynamic D-vine copula for the crisis period.

Bivariate Copula		θ_1	θ_2	ω_1	α_1	β_1	ω_2	α_2	β_2
<i>Tree 1</i>									
FTSE, CAC	BB1 tvp			0.7046 (0.1020)	-1.3185* (0.7325)	0.3499 (0.0602)	0.6366 (0.0980)	-0.9050 (0.4269)	0.2801 (0.0220)
CAC, DAX	BB1 tvp	0.9915 (0.1265)					2.4714 (0.3390)	-8.0118 (2.4776)	-0.1519* (0.0784)
DAX, S&P500	Student- <i>t</i>	0.7088 (0.0137)	18.8746* (10.5248)						
S&P500, IBOVESPA	Student- <i>t</i> tvp		9.5523 (3.1905)	-0.4945 (0.1019)	0.2145 (0.0583)	3.0149 (0.1991)			
<i>Tree 2</i>									
FTSE, DAX CAC	Rotated-Gumbel	1.1117 (0.0253)							
CAC, S&P500 DAX	BB1	0.1706 (0.0494)	1.0443 (0.0221)						
DAX, IBOVESPA S&P500	Gumbel	1.0776 (0.0216)							
<i>Tree 3</i>									
FTSE, S&P500 CAC, DAX	Gumbel	1.0607 (0.0198)							
CAC, IBOVESPA DAX, S&P500	Normal	0.1142 (0.0296)							
<i>Tree 4</i>									
FTSE, IBOVESPA CAC, DAX, S&P500	Normal	0.1118 (0.0315)							

Note: Estimates obtained using the sequential estimation procedure. Standard errors in parentheses. (*) stands for significant only at the 10% level. In the second column, we have the pair-copula selected based on the AIC criterion. The copula name followed by "tvp" means that at least one of the copula's parameters is time-varying. The columns labeled θ_i , $i = 1, 2$ (considering two-parameter copulas), present the estimates of the constant parameters. ω_i , α_i and β_i , refer to the coefficients of the evolution equation of the time-varying parameter, $\theta_{i,t} = \Lambda(\omega_i + \beta_i \theta_{i,t-1} + \alpha_i \psi_t)$, $i = 1, 2$, as defined in Section 2.2 and presented in Appendix A for the time-varying parameters of the different copula functions adopted in this paper. Here, both estimated parameters of the BB1 tvp copula chosen for the pair FTSE-CAC follow an ARMA(1, 5). As far as the BB1 tvp copula of CAC-DAX is concerned, only the estimated parameter γ_t displays time variation, following an ARMA(1, 10), whereas $\hat{\kappa}$ remains constant. With regard to the Student- t tvp selected for the pair S&P500-IBOVESPA, only the estimated correlation coefficient is time-varying, following an ARMA(1, 15), whereas the degrees of freedom remain constant.

The dynamics of the dependencies in the first level of the estimated D-vines can be observed in Figure 1–Figure 4, which display the evolutions of the Kendall's tau and the tail dependence parameters computed based on the pair-copulas of the first tree. In Figure 1, panel (a), the dependence between FTSE and CAC measured by the Kendall's tau oscillates around 0.6372 from January 03, 2003 to December 28, 2007, when it increases and begins to oscillate around 0.7359 from January 02, 2008 to May 04, 2012. Note that the Kendall's tau path is a bit noisier during the crisis period. Also the tail dependence parameters, in panel (b), increase from the non-crisis period to the crisis period. Interestingly, the lower tail dependence steadily fluctuates above the path of the upper tail dependence all over the non-crisis period, becoming quite volatile during the crisis period. Curiously we can observe a strong decrease in dependence, measured both by the Kendall's tau and the lower tail dependence parameter, in the beginning of 2011, during the European crisis, which is an evidence of no contagion at this moment. For the pair CAC-DAX, in Figure 2, panel (a), the Kendall's tau follows a path that fluctuates around 0.7031 till the end of 2007, when it reaches a higher level and begins oscillating closely to 0.7743 during the crisis period. In panel (b), the lower and upper tail dependence parameters evolve near each other during the non-crisis period, fluctuating around 0.6785, and move apart from 2008 on, with the upper tail dependence oscillating around 0.7382, whereas the lower tail dependence oscillates around a higher level, 0.7873. We also find evidence of no contagion for the pair CAC-DAX during mid-2011 and in the beginning of 2008, given the strong decrease in dependence during these moments. Figure 3 presents the evolution of the dependence between DAX and S&P500. In panel (a), the Kendall's tau varies over time along the first period, moving around 0.3609, but changes to a constant path in the crisis period, assuming the value 0.5015, estimated from the static Student-*t* copula. In panel (b), the tail dependence oscillates around 0.1180 till the end of 2007, when it, surprisingly, experiences a decrease, assuming a constant value of 0.0807 during the crisis period. Finally, the dependence between S&P500 and IBOVESPA, when measured by the Kendall's tau, in panel (a) of Figure 4, experiences an increase from the non-crisis to the crisis period: it oscillates around 0.4328 over the first period, whereas, in the second one, it fluctuates around 0.5207. On the other hand, although the tail dependence parameters, in panel (b), experience a meaningful increase during the end of 2008 and 2009, on average, they do not reach much higher a baseline during the crisis period in comparison with the previous period. From 2003 to 2007, the tail dependence oscillates close to 0.2134, and, from 2008 on, it oscillates around 0.2387, under the estimated value from the static Student-*t*, 0.3038.

Figure 1: The dynamics of the Kendall's tau and the tail dependence parameters for the pair FTSE-CAC.

Figure 2: The dynamics of the Kendall's tau and the tail dependence parameters for the pair CAC-DAX.

Figure 3: The dynamics of the Kendall's tau and the tail dependence parameters for the pair DAX-S&P500.

Figure 4: The dynamics of the Kendall's tau and the tail dependence parameters for the pair S&P500-IBOVESPA.

Looking at Table 8 and Table 9 once more, for higher levels of the D-vine, we choose mainly symmetric copulas in the non-crisis period, and asymmetric copulas in the crisis period. For example, the conditional copula of FTSE, DAX|CAC changes from the Student- t in the non-crisis period to the Rotated-Gumbel in the crisis period, with an implied increase in the lower tail dependence from 0.0083 to 0.1345. Also the type of dependence

between the French and the North American markets, given information on the German market, captured by the conditional copula of CAC, S&P500|DAX, changes from symmetrical and with no tail dependence at all, in the non-crisis period, to asymmetrical featuring upper tail dependence of 0.0579 and lower tail dependence equal to 0.0204, in the crisis period.

Important features of the joint dependence among the indexes can be inferred from the preceding estimation results, based on the findings of Joe, Li, and Nikoloulopoulos (2010). Because the pair-copulas in the first level of the estimated D-vines have upper and lower tail dependence, both multivariate copulas also have upper and lower tail dependence. Moreover, since two of these pair-copulas are tail asymmetric, the range of upper/lower tail dependence for the bivariate (and lower-dimensional) margins is quite flexible. These characteristics are in accordance with empirical evidence found in the literature that financial data tends to exhibit tail dependence and asymmetries. The previous findings suggest that the overall dependence structure of the indexes does not change dramatically from the non-crisis to the crisis period, although the predominance of asymmetric pair-copulas in higher levels of the estimated D-vine for the crisis period may create a little more asymmetric dependence structure.

Notably, the estimated dynamic D-vines differ in terms of the dependence strength that they describe, with stronger overall as well as tail dependencies captured by the estimated D-vine for the crisis period.

For the purpose of comparison, we also estimate a static D-vine copula for the two investigated data sets. To obtain the estimates of the pair-copulas parameters in this case, as it is usual in the literature of static vine copulas, we first estimate the parameters using the sequential estimation procedure¹³ and, then, we maximize the D-vine copula log-likelihood over all dependence parameters, using as starting values the parameters obtained from the stepwise procedure. It corresponds to applying the two-step estimation procedure of Joe and Xu (1996), the Inference Function for Margins (IFM) method. The estimates of the pair-copulas composing the static D-vine copula in the non-crisis period and in the crisis period are presented in Table 10 and Table 11, respectively. These tables also report the estimated Kendall's tau and tail dependence parameters, computed based on the estimated pair-copulas. The estimation results suggest that both D-vines display lower and upper tail dependence, since the pair-copulas in the first tree of both constructions are all Student-*t*. Furthermore, the dependence structure characterizing the crisis period is more asymmetric than the one of the non-crisis period, given the prevalence of asymmetric pair-copulas in the second level of the estimated D-vine. Concerning the degree of dependence described by the estimated D-vines, it is clear from the dependence measures reported in the tables that the estimated D-vine for the crisis period captures a stronger dependence among the indexes. In comparison with the estimated dynamic D-vines, the range of upper/lower tail dependencies of the margins, in this case, is less flexible, since only symmetric copulas are selected in the first level of the D-vines.

Table 10: Estimation results of the static D-vine copula for the non-crisis period.

Bivariate Copula		θ_1	θ_2	τ	λ_L	λ_U
<i>Tree 1</i>						
FTSE, CAC	Student- <i>t</i>	0.8562 (0.0066)	14.8356 (6.3513)	0.6544	0.2846	0.2846
CAC, DAX	Student- <i>t</i>	0.9035 (0.0049)	8.9643 (2.1594)	0.7180	0.4935	0.4935
DAX, S&P500	Student- <i>t</i>	0.5370 (0.0204)	8.7557 (2.2666)	0.3609	0.1180	0.1180
S&P500, IBOVESPA	Student- <i>t</i>	0.6287 (0.0177)	7.0128 (1.6477)	0.4328	0.2134	0.2134
<i>Tree 2</i>						
FTSE, DAX CAC	Student- <i>t</i>	0.0669 (0.0317)	9.5239 (2.4388)	0.0426	0.0119	0.0119
CAC, S&P500 DAX	Normal	0.0819 (0.0294)		0.0522		
DAX, IBOVESPA S&P500	Rotated-Gumbel	1.0580 (0.0187)		0.0548	0.0746	
<i>Tree 3</i>						
FTSE, S&P500 CAC, DAX	Normal	0.1053 (0.0288)		0.0672		

CAC, IBOVESPA DAX, S&P500	Normal	0.0630 (0.0292)	0.0402
<i>Tree 4</i>			
FTSE, IBOVESPA CAC, DAX, S&P500	Normal	0.0854 (0.0305)	0.0544

Note: Estimates obtained using the IFM method. Standard errors in parentheses. In the second column, we have the pair-copula selected based on the AIC criterion. The columns labeled θ_i , $i = 1, 2$ (considering two-parameter copulas), present the estimates of the copula parameters. τ corresponds to the estimate of the Kendall's tau, whereas λ_L and λ_U refer to the estimates of the lower and upper tail dependence parameters, respectively.

Table 11: Estimation results of the static D-vine copula for the crisis period.

Bivariate Copula		θ_1	θ_2	τ	λ_L	λ_U
<i>Tree 1</i>						
FTSE, CAC	Student- t	0.9231 (0.0038)	13.9509 (5.5628)	0.7487	0.4515	0.4515
CAC, DAX	Student- t	0.9434 (0.0035)	5.4932 (1.4046)	0.7848	0.6777	0.6777
DAX, S&P500	Student- t	0.7088 (0.0137)	18.8746* (10.5248)	0.5015	0.0807	0.0807
S&P500, IBOVESPA	Student- t	0.7297 (0.0143)	6.7582 (1.6322)	0.5207	0.3038	0.3038
<i>Tree 2</i>						
FTSE, DAX CAC	BB1	0.1156 (0.0440)	1.0494 (0.0223)	0.0990	0.0033	0.0642
CAC, S&P500 DAX	BB1	0.1983 (0.0481)	1.0354 (0.0211)	0.1213	0.0342	0.0468
DAX, IBOVESPA S&P500	Gumbel	1.0763 (0.0215)		0.0709		0.0959
<i>Tree 3</i>						
FTSE, S&P500 CAC, DAX	Normal	0.1239 (0.0306)		0.0791		
CAC, IBOVESPA DAX, S&P500	Normal	0.1146 (0.0302)		0.0731		
<i>Tree 4</i>						
FTSE, IBOVESPA CAC, DAX, S&P500	Student- t	0.1140 (0.0323)	17.8143* (10.0107)	0.0728	0.0011	0.0011

Note: Estimates obtained using the IFM method. Standard errors in parentheses. (*) stands for significant only at the 10% level. In the second column, we have the pair-copula selected based on the AIC criterion. The columns labeled θ_i , $i = 1, 2$ (considering two-parameter copulas), present the estimates of the copula parameters. τ corresponds to the estimate of the Kendall's tau, whereas λ_L and λ_U refer to the estimates of the lower and upper tail dependence parameters, respectively.

5.4 VaR Backtesting

We are interested in comparing the static and the dynamic D-vine copula models' abilities to forecast capital losses in the occurrence of extreme events, more specifically, crisis. For this purpose, we compare their performance in an out-of-sample exercise. The estimated models for the period from January 03, 2003 to December

28, 2007 are used to forecast one-day VaR at the 1%, 5% and 10% significance levels for an equally weighted portfolio of the indexes DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA in the period from January 02, 2008 to August 19, 2008 (150 days). Additionally, the estimated models for the period from January 02, 2008 to May 04, 2012 are used for VaR forecasting from May 08, 2012 to December 28, 2012 (149 days). Notice that both testing periods belong to the crisis period. In the former case, there is an additional motivation regarding the models' abilities to forecast extremal losses in bear markets, given that the copulas parameters were estimated in a different context, of bull markets. Given the estimation set of $\{1, \dots, T\}$ daily observations for the copula model and the testing set $\{T + 1, \dots, T + h\}$, the exercise is done as follows:

1. For $k = 1, \dots, 1,000$:

- From the fitted copula model, we simulate a sample $u_{1,t}^{(k)}, \dots, u_{5,t}^{(k)}, t = 1, \dots, h$.
- For $j = 1, \dots, 5$, we convert $u_{j,t}^{(k)}$ to $\hat{\varepsilon}_{j,t}^{(k)}, t = 1, \dots, h$, using the inverse Skewed- t cdf's, i.e., $\hat{\varepsilon}_{j,t}^{(k)} = F_j^{-1}(u_{j,t}^{(k)})$.
- For $j = 1, \dots, 5$, we convert $\hat{\varepsilon}_{j,t}^{(k)}$ to the return forecasts as

$$\hat{x}_{j,T+t}^{(k)} = \hat{\mu}_{j,T+t} + \sqrt{\hat{h}_{j,T+t}} \cdot \hat{\varepsilon}_{j,t}^{(k)}, t = 1, \dots, h,$$

where $\hat{\mu}_{j,T+t}$ and $\hat{h}_{j,T+t}$ correspond to the one-step ahead forecasts of the conditional mean and variance, respectively.¹⁴

- Then we compute the portfolio return forecasts as $\hat{x}_{P,T+t}^{(k)} = \sum_{j=1}^5 \hat{x}_{j,T+t}^{(k)} / 5, t = 1, \dots, h$.

2. For significance levels $\alpha \in \{0.01, 0.05, 0.1\}$, we compute the one-day $\text{VaR}_{\alpha,1}$ forecast for the day $T + t$ as the 100α th-percentile of $\hat{x}_{P,T+t}^{(k)}, k = 1, \dots, 1,000$. If the observed value of the portfolio return for the day $T + t$, $x_{P,T+t}$, is less than $\widehat{\text{VaR}}_{\alpha,1}$, then a violation (or exceedance) is said to occur.

To evaluate the VaR forecasts, we initially use the likelihood ratio tests proposed by Kupiec (1995) and Christoffersen (1998). Based on the previous procedure, it is possible to construct an indicator sequence of violations $I_t, t = 1, \dots, h$, called hits. If the forecasts are accurate, the hit sequence should exhibit two properties. First, the proportion of violations should approximately equal the VaR significance level α . The unconditional coverage test of Kupiec is a test of the null hypothesis that the expected violation rate is equal to the theoretical rate α of the VaR and the test statistic is defined as

$$LR_{uc} = -2 \ln[\alpha^n (1 - \alpha)^{h-n}] + 2 \ln[(n/h)^n (1 - n/h)^{h-n}] \sim_{H_0} \chi_1^2,$$

where n is the number of VaR violations, h is the size of the testing sample and n/h is the observed proportion of violations. Second, the exceedances should occur independently, i.e., not in clusters. Christoffersen (1998) proposed a combined test for both unconditional coverage and serial independence. He considers a binary first-order Markov chain for the hits, with transition probability matrix $\pi_{ij} = \Pr(I_t = j | I_{t-1} = i)$, with $i, j = 0, 1$ ("0" means no VaR violation and "1" means VaR violation), $\hat{\pi}_{ij} = (n_{ij} / \sum_j n_{ij})$, where n_{ij} is the number of hits with indicator i followed by hits with indicator j . Under $H_0 : \pi_{01} = \pi_{11} = \alpha$, the test statistic is the conditional coverage statistic given by

$$LR_{cc} = -2 \ln[\alpha^n (1 - \alpha)^{h-n}] + 2 \ln[\hat{\pi}_{01}^{n_{01}} (1 - \hat{\pi}_{01})^{n_{00}} \hat{\pi}_{11}^{n_{11}} (1 - \hat{\pi}_{11})^{n_{10}}] \sim_{H_0} \chi_2^2.$$

According to Lopez (1999), the statistical tests proposed by Kupiec and Christoffersen to evaluate the accuracy of VaR models can have relatively low power against inaccurate VaR models. For this reason, he proposed an alternative methodology based not on a statistical testing framework, but instead on standard forecast evaluation techniques: the accuracy of the VaR forecasts is determined by how well they minimize a certain regulatory loss function. We implement this additional procedure proposed by Lopez, adopting the capital requirement loss function (CR) defined at the Basel II Accord¹⁵:

$$CR_t = \max \left[\frac{(3 + \delta)}{60} \sum_{i=0}^{59} \text{VaR}_{\alpha, t-i}, \text{VaR}_{\alpha, t} \right], \delta = \begin{cases} 0, & \text{if } \zeta \leq 4; \\ 0.3 + 0.1(\zeta - 4), & \text{if } 5 \leq \zeta \leq 6; \\ 0.65, & \text{if } \zeta = 7; \\ 0.65 + 0.1(\zeta - 7), & \text{if } 8 \leq \zeta \leq 9; \\ 1, & \text{if } \zeta \geq 10. \end{cases}$$

where δ is a multiplicative factor that depends on the number of violations of the VaR in the previous 250 trading days (ζ).

To compare the VaR models performance by using the CR loss function, we apply the superior predictive ability (SPA) test statistic proposed by Hansen (2005). Testing for SPA is to test whether a particular forecasting procedure is outperformed by alternative forecasts. The relevant question is whether an observed excess performance by an alternative model is significant or not. In Hansen's framework, the interest is to know whether any of the alternative models, $k = 1, \dots, m$, are better than the benchmark, bch , in terms of expected loss L . So he tests the null hypothesis that the best alternative model is not better than the benchmark. The performance of model k relative to the benchmark at time t may be defined as $d_{k,t} = L_{bch,t} - L_{k,t}$. Provided that $E(d_{k,t}) = \mu_{k,t}$ is well defined, the null hypothesis of interest can be formulated as

$$H_0 : \max_{k=1,\dots,m} \mu_k \leq 0,$$

whereas the alternative hypothesis is that the best alternative model is superior to the benchmark. A k model is better than the benchmark if and only if $E(d_{k,t}) > 0$. The test statistic is given by

$$T^{SPA} \equiv \max \left[\max_{k=1,\dots,m} \frac{T^{1/2} \bar{d}_k}{\hat{\omega}_k}, 0 \right],$$

where $\bar{d}_k \equiv T^{-1} \sum_{t=1}^T d_{k,t}$ and $\hat{\omega}_k^2$ is some consistent estimator of $\omega_k^2 \equiv \text{var}(T^{1/2} \bar{d}_k)$. The test is implemented via stationary bootstrap of Politis and Romano (1994).

For the testing period from January 02, 2008 to August 19, 2008 (150 days), the results of the Kupiec and Christoffersen tests are reported in Table 12. For a testing period of 150 days and significance levels of 10%, 5% and 1%, we expect 15, 7.5 and 1.5 exceedances, respectively. Both estimated copula models produced the same hit sequences. For the 1% and 5% significance levels, there is a (non-significant) lack of coverage, since the numbers of exceedances are slightly increased in comparison with the expected ones. For the 10% significance level, the VaR forecasts are too conservative, and the null hypotheses of the Kupiec and Christoffersen tests are rejected using a 5% level for the LR_{uc} and LR_{cc} statistics. To some extent, the increased number of violations was expected, since we used the estimated models for the non-crisis period, with lower degree of dependence and less asymmetric dependence structure, to forecast VaR into the crisis period, with higher degree of dependence and more asymmetric dependence structure. Table 13 reports the average (capital requirement) losses computed based on the VaR forecasts and the results of the SPA test.¹⁶ The latter is implemented considering each copula model at a time as the benchmark. According to the test results, based on the 1%-VaR forecasts, both models display similar performance in terms of predictive accuracy, however, with regard to the 5%-VaR, the static D-vine copula model performs worse than the dynamic D-vine copula.

Table 12: Results of the VaR backtests for the testing period from January 02, 2008 to August 19, 2008 (150 days).

Dynamic D-Vine Copula			
α	1%	5%	10%
n	2	9	23
n/h	0.0133	0.0600	0.1533
Kupiec	0.6962	0.5854	0.0417
Christoffersen	0.8897	0.4539	0.0144
Static D-Vine Copula			
α	1%	5%	10%
n	2	9	23
n/h	0.0133	0.0600	0.1533
Kupiec	0.6962	0.5854	0.0417
Christoffersen	0.8897	0.4539	0.0144

Note: Kupiec and Christoffersen correspond to the p -values of the respective tests.

Table 13: Average losses computed based on the VaR forecasts for the testing period from January 02, 2008 to August 19, 2008 (150 days) and the results of the SPA test.

1%-VaR		5%-VaR	
Benchmark	Average loss (%)	Benchmark	Average loss (%)

Dynamic D-Vine	10.1303 (0.4583)	Dynamic D-Vine	7.2424 (1.0000)
Static D-Vine	10.1266 (1.0000)	Static D-Vine	7.3551 (0.0000)

Note: In parentheses, we have the p -value of the SPA test.

The results of the VaR backtests for the period from May 08, 2012 to December 28, 2012 (149 days) are provided in Table 14. For a testing period of 149 days and significance levels of 10%, 5% and 1%, we expect 14.9, 7.45 and 1.49 exceedances, respectively. The tests results suggest that the forecasts of all three quantiles from both models are accurate, since the null hypotheses of unconditional and conditional coverage cannot be rejected. Nevertheless, it is worthy noticing that the observed numbers of exceedances of the 5%-VaR and 10%-VaR forecasts from the dynamic D-vine copula are closer to the expected numbers. Further, the results of the SPA test in Table 15 indicate that the forecasting performance of the static D-vine copula is inferior to the dynamic D-vine copula performance for the first and tenth percentiles.

Table 14: Results of the VaR backtests for the testing period from May 08, 2012 to December 28, 2012 (149 days).

Dynamic D-Vine Copula			
α	1%	5%	10%
n	0	7	13
n/h	0.0000	0.0470	0.0872
Kupiec	0.0835	0.8644	0.5966
Christoffersen	0.2237	0.6633	0.6699

Static D-Vine Copula			
α	1%	5%	10%
n	0	5	11
n/h	0.0000	0.0336	0.0738
Kupiec	0.0835	0.3286	0.2662
Christoffersen	0.2237	0.5034	0.5345

Note: Kupiec and Christoffersen correspond to the p -values of the respective tests.

Table 15: Average losses computed based on the VaR forecasts for the testing period from May 08, 2012 to December 28, 2012 (149 days) and the results of the SPA test.

1%-VaR		5%-VaR		10%-VaR	
Benchmark	Average loss (%)	Benchmark	Average loss (%)	Benchmark	Average loss (%)
Dynamic D-Vine	9.3932 (1.0000)	Dynamic D-Vine	6.2782 (1.0000)	Dynamic D-Vine	5.2018 (1.0000)
Static D-Vine	9.6120 (0.0000)	Static D-Vine	6.2908 (0.4648)	Static D-Vine	5.3651 (0.0000)

Note: In parentheses, we have the p -value of the SPA test.

Overall, the dynamic D-vine copula seems to work very well out-of-sample, in crisis periods, usually outperforming the static D-vine copula, with more accurate VaR forecasts. It is true even in the adverse situation when we use the estimated copula corresponding to the non-crisis period to forecast VaR in a crisis context.

6 Concluding Remarks and Outlook

In this paper, we introduce dynamics into the state-of-the-art model for multivariate dependencies, the vine copula model. We allow the dependence parameters of the pair-copulas in a D-vine decomposition to be potentially time-varying, evolving through time according to an equation that follows a restricted ARMA(1, m)

process as in Patton (2006). Our contribution is towards assessing the performance of the dynamic D-vine copula model both in a simulation and in an empirical study.

The overall findings of the Monte Carlo study are quite favorable to the dynamic D-vine copula. When the data generating process is the time-varying model, the static model tends to provide biased estimates of the pair-copulas dependence parameters. Furthermore, the bias does not seem to diminish as the number of observations in the samples increases. The estimates from the dynamic D-vine copula, in this case, are superior to the estimates from the static D-vine, both in terms of the mean errors and the root mean squared errors. When the samples are drawn from the static D-vine copula, both models have similar performance in terms of the mean errors, with unbiased estimates. The dynamic D-vine copula fails only in terms of the root mean squared errors, when the data comes from the static model, what suggests that its estimates display higher variability in this case.

In an empirical study, we model the dependence among the returns of DAX, CAC 40, FTSE 100, S&P 500 and IBOVESPA indexes, using both the dynamic D-vine copula model and a static D-vine copula. We consider two distinct periods, one from January 03, 2003 to December 28, 2007 and another one from January 02, 2008 to May 04, 2012, which we call non-crisis and crisis period, respectively. Our findings illustrate that time variation is present in the dependence structure of multivariate financial returns. In particular, time-varying pair-copulas are selected in the first level of the estimated dynamic D-vine copulas. They provide accurate description of variations in the unconditional dependencies all over the non-crisis and crisis periods, as well as from one period to the other. Overall, both estimated static and dynamic D-vine copulas capture stronger dependence during the crisis period. It is worth noticing, though, that the dynamic model indicates the occurrence of a sharp decrease in dependence between the indexes FTSE and CAC in the beginning of 2011, and also between CAC and DAX during mid-2011 and in the beginning of 2008, suggesting the absence of contagion in these cases. The estimated dynamic D-vine copulas give insightful information about the joint dependence among the above-mentioned indexes: there is evidence of joint upper and lower tail dependence, with some degree of flexibility, in both periods. The estimated static D-vines, on the other hand, suggest that the range of upper and lower tail dependencies of the margins is less flexible. In an out-of-sample exercise, the estimated models are used to forecast one-day VaR for an equally weighted portfolio of the investigated indexes in the period from January 02, 2008 to August 19, 2008 (150 days) and also from May 08, 2012 to December 28, 2012 (149 days). Both testing periods belong to the crisis period. Based on the results of the superior predictive ability (SPA) test of Hansen (2005), the dynamic D-vine copula model outperforms the static D-vine copula in terms of predictive accuracy.

Further research is to be done on improving the dynamic D-vine copula model by extending it to the general case of regular vine copulas and investigating it more closely in higher-dimensional applications. Additionally, in future, we can assume that the pair-copulas dependence parameters not only follow an ARMA(1, m) process, but they are also influenced by a Markov chain, since we found evidence of change in the degree of dependence among the returns from the non-crisis to the crisis period.

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Appendix

A Copula Functions

Normal copula: the Normal copula, extracted from the bivariate Normal distribution, is defined as follows:

$$C_N(u_1, u_2 | \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{r^2 - 2\rho rs + s^2}{2(1-\rho^2)}\right\} dr ds, \rho \in (-1, 1),$$

where the dependence parameter, ρ , is the linear correlation coefficient. Its dynamic equation may be written as¹⁷

$$\rho_t = \Lambda \left(\omega_N + \beta_N \rho_{t-1} + \alpha_N \cdot \frac{1}{m} \sum_{j=1}^m \Phi^{-1}(u_{1,t-j}) \cdot \Phi^{-1}(u_{2,t-j}) \right).$$

The Normal copula is symmetric and has no tail dependence, that is, $\lambda_L = \lambda_U = 0$. The Kendall's tau may be computed based on the correlation coefficient as $\tau = (2/\pi) \arcsin \rho$.

Student- t copula: it is associated with the bivariate Student- t distribution and has the following functional form:

$$C_T(u_1, u_2 | \rho, \nu) = \int_{-\infty}^{t_v^{-1}(u_1)} \int_{-\infty}^{t_v^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{r^2 - 2\rho rs + s^2}{\nu(1-\rho^2)} \right)^{-\frac{\nu+2}{2}} dr ds,$$

where the parameters ρ and ν are the linear correlation coefficient and the degrees of freedom, respectively. In addition, their evolution equations are given by

$$\rho_t = \Lambda \left(\omega_{1T} + \beta_{1T} \rho_{t-1} + \alpha_{1T} \cdot \frac{1}{m} \sum_{j=1}^m T_v^{-1}(u_{1,t-j}) \cdot T_v^{-1}(u_{2,t-j}) \right)$$

and

$$\nu_t = \tilde{\Lambda} \left(\omega_{2T} + \beta_{2T} \nu_{t-1} + \alpha_{2T} \cdot \frac{1}{m} \sum_{j=1}^m T_v^{-1}(u_{1,t-j}) \cdot T_v^{-1}(u_{2,t-j}) \right).$$

The Student- t copula has symmetrical tail dependence, with $\lambda_L = \lambda_U = 2T_{\nu+1}(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}})$, where $T_{\nu+1}$ is the Student- t c.d.f. with $(\nu+1)$ degrees of freedom. The Kendall's tau is given by $\tau = (2/\pi) \arcsin \rho$.

Gumbel copula: it has the form of

$$C_G(u_1, u_2 | \theta) = \exp \left(- \left((-\log u_1)^\theta + (-\log u_2)^\theta \right)^{1/\theta} \right), \theta \in [1, \infty).$$

The dynamics is given by the following equation governing the dependence parameter evolution:

$$\theta_t = \Lambda \left(\omega_G + \beta_G \theta_{t-1} + \alpha_G \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right).$$

The Gumbel copula exhibits only upper tail dependence, with $\lambda_U = 2 - 2^{1/\theta}$. It can be shown that the Kendall's tau is given by $\tau = 1 - \theta^{-1}$.

Rotated-Gumbel copula: or Survival Gumbel copula, which is the complement ("Probability of survival") of the Gumbel copula. It has the following form:

$$C_{RG}(u_1, u_2 | \theta) = u_1 + u_2 - 1 + C_G(1 - u_1, 1 - u_2 | \theta),$$

where C_G corresponds to the Gumbel copula. The dependence parameter, θ , follows the process

$$\theta_t = \Lambda \left(\omega_{RG} + \beta_{RG} \theta_{t-1} + \alpha_{RG} \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right).$$

The Rotated-Gumbel copula has only lower tail dependence, given by $\lambda_L = 2 - 2^{1/\theta}$, and the Kendall's tau may be computed as $\tau = 1 - \theta^{-1}$.

Clayton copula: or Kimeldorf-Sampson copula, has the following distribution function:

$$C_C(u_1, u_2 | \delta) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-1/\delta}, \delta \in (0, \infty).$$

The evolution equation of the dependence parameter is

$$\delta_t = \Lambda \left(\omega_C + \beta_C \delta_{t-1} + \alpha_C \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right).$$

This copula exhibits only lower tail dependence, $\lambda_L = 2^{-1/\delta}$. The Kendall's tau has the form $\tau = \delta/(\delta + 2)$.

Symmetrized Joe-Clayton copula: this copula was defined by Patton (2006) and takes the form of

$$C_{SJC}(u_1, u_2 | \lambda_U, \lambda_L) = 0.5 \cdot (C_{JC}(u_1, u_2 | \lambda_U, \lambda_L) + C_{JC}(1 - u_1, 1 - u_2 | \lambda_U, \lambda_L) + u_1 + u_2 - 1),$$

where C_{JC} is the Joe-Clayton copula, also called BB7 copula (Joe 1997), given by

$$C_{JC}(u_1, u_2 | \lambda_U, \lambda_L) = 1 - \left(1 - \left\{ [1 - (1 - u_1)^\kappa]^{-\gamma} + [1 - (1 - u_2)^\kappa]^{-\gamma} - 1 \right\}^{-1/\gamma} \right)^{-1/\kappa},$$

with $\kappa = 1/\log_2(2 - \lambda_U)$, $\gamma = -1/\log_2(\lambda_L)$ and $\lambda_U, \lambda_L \in (0, 1)$.

The SJC copula has upper and lower tail dependence and its dependence parameters are the upper and lower tail dependence parameters, λ_U and λ_L , respectively. Furthermore, λ_U and λ_L range freely and are not dependent on each other. Since this copula nests symmetry as a special case, it is a more interesting specification than the BB7 copula. The evolution equations for the parameters λ_U and λ_L are

$$\lambda_{Ut} = \Lambda \left(\omega_U + \beta_U \lambda_{Ut-1} + \alpha_U \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right)$$

and

$$\lambda_{Lt} = \Lambda \left(\omega_L + \beta_L \lambda_{Lt-1} + \alpha_L \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right).$$

The Kendall's tau, in this case, has no closed form, so it has to be computed numerically.

BB1 copula (Joe 1997): it has the following functional form:

$$C_{bb1}(u_1, u_2 | \kappa, \gamma) = \{1 + [(u_1^{-\kappa} - 1)^\gamma + (u_2^{-\kappa} - 1)^\gamma]^{1/\gamma}\}^{-1/\kappa}, \quad \kappa \in (0, \infty), \gamma \in [1, \infty).$$

The dynamic equations of the dependence parameters are

$$\kappa_t = \Lambda \left(\omega_{bb1} + \beta_{bb1} \kappa_{t-1} + \alpha_{bb1} \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right)$$

$$\gamma_t = \tilde{\Lambda} \left(\omega_{2bb1} + \beta_{2bb1} \gamma_{t-1} + \alpha_{2bb1} \cdot \frac{1}{m} \sum_{j=1}^m |u_{1,t-j} - u_{2,t-j}| \right).$$

The BB1 copula has upper and lower tail dependence given by $\lambda_U = 2 - 2^{1/\gamma}$ and $\lambda_L = 2^{-1/\gamma\kappa}$, respectively. The Kendall's tau may be calculated based on κ and γ as $\tau = 1 - (2/(\gamma(\kappa + 2)))$.

Notes

- 1 Here we use the notation of Haff (2013).
- 2 For more details on these and other copula-based measures of dependence, see Chapter 2 of Joe (1997) and Chapter 5 of Nelsen (2006).
- 3 Extensions of the GARCH model, such as EGARCH, TARCH, among others, are also fitted to data in order to find out the best model for the marginals.
- 4 Let $c_{1,\dots,m}$ be the copula density, then reflection symmetry implies $c(u_1, \dots, u_m) = c(1 - u_1, \dots, 1 - u_m)$.
- 5 Their functional forms as well as the evolution equations of their dependence parameters following Patton (2006) are described in Appendix A.
- 6 The algorithm uses the conditional inversion method described in e.g. Embrechts et al. (2003). Given the D-vine structure, all the conditional distribution functions involved are of the form eq. (9), so, in order to be computed, only the first partial derivative of a bivariate copula is required. A numerical inversion is necessary for the Rotated-Gumbel and BB1 copulas.
- 7 To obtain the estimates of the vine copula parameters in the former case, we adopt the sequential estimation procedure. For the second model, as it is usual in the literature of static vine copulas, we first estimate the parameters of the D-vine copula using the sequential estimation procedure and, then, we maximize the copula log-likelihood over all dependence parameters, using as starting values the parameters obtained from the stepwise procedure.
- 8 The conditional variance, h_t , of an EGARCH(m,o,n) process can be modeled as follows:

$$\ln(h_t) = \alpha_0 + \sum_{i=1}^m \alpha_i (|\varepsilon_{t-i}| - \mathbb{E}|\varepsilon_{t-i}|) + \sum_{k=1}^o \gamma_k \varepsilon_{t-k} + \sum_{j=1}^n \beta_j \ln(h_{t-j}).$$

For lack of space, we do not present here the other asymmetric specifications. For a survey on GARCH models, see Bollerslev (2008).

9 Although not before also testing for the symmetric Student- t and Normal distributions.

10 All marginals were estimated using the Oxford MFE Toolbox by Kevin Shepard.

11 Recall that, for each pair of transformed data, we fit both a static and a dynamic version of the copulas listed in Section 3.2, whose functional forms as well as the evolution equations of their dependence parameters following Patton (2006) are described in Appendix A. Using the AIC, we choose the best of all of them. Regarding the dynamics, for two-parameter copulas, it may happen that only one of the estimated parameters displays time variation. It may also happen that not all estimated coefficients of the evolution equation are statistically significant. In this case, the copula is re-estimated omitting the non-significant coefficient.

12 Heinen and Valdesogo (2009, 2011) and So and Yeung (2014) also find evidence of time variation especially in level 1.

13 For each pair of transformed data, we estimate a static version of the copulas listed in Section 3.2 and choose the best of them based on the AIC criterion.

14 We re-estimate the parameters of the ARMA-GARCH specifications in a recursive scheme, using an expanding window up to $T + t - 1$, and use these estimates to obtain one-step ahead forecasts of the conditional mean and variance in $T + t$.

15 Note that, since the VaR is a negative value, to compute the loss function, it will be calculated here as minus the $(100\alpha$ -th percentile) of the c.d.f. of the returns.

16 To compute Hansen's consistent p-value, we use the "bsds" function from the Oxford MFE Toolbox by Kevin Shepard, along with the Matlab code "opt_block_length_REV_dec07" compiled by Andrew Patton to implement the automatic optimum block length selection in accordance with Politis and White (2004). For the stationary block bootstrap, we use 10,000 re-samples.

17 $\Lambda(\cdot)$ and $\tilde{\Lambda}(\cdot)$, which appear hereafter, are logistic transformations to keep the parameters in their intervals.

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