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Research Article

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Some identities of degenerate special polynomials

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Abstract: In this paper, by considering higher-order degenerate Bernoulli and Euler polynomials which were introduced by Carlitz, we investigate some properties of mixed-type of those polynomials. In particular, we give some identities of mixed-type degenerate special polynomials which are derived from the fermionic integrals on $\mathbb{Z}_p$ and the bosonic integrals on $\mathbb{Z}_p$.

Keywords: Mixed-type degenerate special polynomial, Fermionic integral, Bosonic integral

MSC: 11B68, 11S80

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\, \mu_{-1}\left(x + p^N \mathbb{Z}_p\right) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^N. \tag{1}$$

Then, by (1), we get

$$I_{-1}(f_1) = -I(f) + 2f(0), \quad \text{where } f_1(x) = f(x+1), \quad (\text{see [9–13]}). \tag{2}$$

From (2), we can derive the following integral equation:

$$I_{-1}(f_n) = (-1)^n I(f) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{3}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

The bosonic integral on $\mathbb{Z}_p$(or p-adic invariant integral on $\mathbb{Z}_p$) is defined by Volkenborn as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\, \mu_0\left(x + p^N \mathbb{Z}_p\right) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{4}$$

(see [11, 22, 23]). Then, by (4), we get

$$I_0(f_1) - I_0(f) = f'(0), \quad \text{where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}, \tag{5}$$

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and

$$I_0(f_n) - I_0(f) = \sum_{l=0}^{n-1} f'(l), \quad \text{where } f_n(x) = f(x+n), \quad (n \in \mathbb{N}). \tag{6}$$

For $r \in \mathbb{N}$, the higher-order Bernoulli polynomials are given by the generating function as

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [1–24]}). \tag{7}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the higher-order Bernoulli numbers. For $r = 1$, $B_n(x) = B_n^{(1)}(x)$ are called the ordinary Bernoulli polynomials and $B_n = B_n^{(1)}(0)$ are called the ordinary Bernoulli numbers. As is well known, the higher-order Euler polynomials are also defined by the generating function

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [9, 10, 13]}). \tag{8}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the higher-order Euler numbers. For $r = 1$, $E_n(x) = E_n^{(1)}(x)$ are called the ordinary Euler polynomials and $E_n = E_n^{(1)}(0)$ are called the ordinary Euler numbers. By (7) and (8), we get

$$B_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_l^{(r)} x^{n-l}, \quad E_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(r)} x^l, \quad (\text{see [18]}). \tag{9}$$

The Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad (n \geq 0), \tag{10}$$

and the Stirling numbers of the first kind are given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \quad (n \geq 0), \quad (\text{see [18]}). \tag{11}$$

Now, we consider the analogue of $(x)_n$ as follows:

$$(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda) = \sum_{l=0}^{n} S_1(n,l \mid \lambda) x^l, \quad \text{where } \lambda \neq 0. \tag{12}$$

Note that $\lim_{\lambda \to 1} S_1(n,l \mid \lambda) = S_1(n,l)$, $(n \geq 0)$.

For $\lambda \neq 0$, Carlitz considered the degenerate Bernoulli polynomials given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}, \quad (\text{see [3, 9]}). \tag{13}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called the degenerate Bernoulli numbers. Note that $\lim_{\lambda \to 0} \beta_n(\lambda, x) = B_n(x)$ and $\lim_{\lambda \to \infty} \lambda^{-n} \beta_n(\lambda, \lambda x) = b_n(x)$, where $b_n(x)$ are the Bernoulli polynomials of the second kind given by the generating function

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \quad (\text{see [18]}). \tag{14}$$

The degenerate Euler polynomials are also defined by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda, x) \frac{t^n}{n!}, \quad (\text{see [3, 9]}). \tag{15}$$

When $x = 0$, $\mathcal{E}_n(\lambda) = \mathcal{E}_n(\lambda, 0)$ are called the degenerate Euler numbers. Note that $\lim_{\lambda \to 0} \mathcal{E}_n(\lambda, x) = E_n(x)$.

From (13) and (15), we note that

$$\beta_n(\lambda, x) = \sum_{l=0}^{n} \binom{n}{l} \beta_l(\lambda)(x)_{n-l,\lambda}, \quad (n \geq 0), \tag{16}$$

and

$$\mathcal{E}_n(\lambda, x) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l(\lambda)(x)_{n-l,\lambda}, \quad (n \geq 0). \tag{17}$$

For $r \in \mathbb{N}$, the higher-order degenerate Bernoulli polynomials and the higher-order degenerate Euler polynomials are introduced by Carlitz and are respectively given by

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(\lambda, x) \frac{t^n}{n!}, \tag{18}$$

and

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(\lambda, x) \frac{t^n}{n!}, \quad (\text{see [3, 9]}). \tag{19}$$

When $x = 0$, $\beta_n^{(r)}(\lambda) = \beta_n^{(r)}(\lambda, 0)$ and $\mathcal{E}_n^{(r)}(\lambda) = \mathcal{E}_n^{(r)}(\lambda, 0)$ are respectively called the higher-order degenerate Bernoulli numbers and the higher-order degenerate Euler numbers.

From (18) and (19), we have

$$\beta_n^{(r)}(\lambda, x) = \sum_{l=0}^{n} \binom{n}{l} \beta_l^{(r)}(\lambda)(x)_{n-l,\lambda}, \quad \mathcal{E}_n^{(r)}(\lambda, x) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l^{(r)}(\lambda)(x)_{n-l,\lambda}. \tag{20}$$

The Bernoulli polynomials of the second kind with order r and the Daehee polynomials of order r are respectively defined by the generating functions

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [18]}). \tag{21}$$

and

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [10, 18]}). \tag{22}$$

When $x = 0$, $b_n^{(r)} = b_n^{(r)}(0)$ and $D_n^{(r)} = D_n^{(r)}(0)$ are called the higher-order Bernoulli numbers of the second kind and the higher-order Daehee numbers.

In this paper, by considering higher-order degenerate Bernoulli and Euler polynomials which were introduced by Carlitz, we investigate some properties of mixed-type of those polynomials. In particular, we give some identities of mixed-type degenerate special polynomials which are derived from the fermionic integrals on $\mathbb{Z}_p$ and the bosonic integrals on $\mathbb{Z}_p$(also called p-adic invariant integrals on $\mathbb{Z}_p$).

2 Some identities of mixed-type degenerate polynomials

For $\lambda \neq 0$ and $t \in \mathbb{C}_p$ with $|t|_p < |\lambda|_p^{-1} p^{-\frac{1}{p-1}}$, we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r \left(\frac{\log(1+\lambda t)}{\lambda t}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\
&= \left(\sum_{l=0}^{\infty} \beta_l^{(r)}(\lambda, x) \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} D_m^{(r)} \lambda^m \frac{t^m}{m!}\right) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \beta_l^{(r)}(\lambda, x) D_{n-l}^{(r)} \lambda^{n-l} \frac{n!}{l!(n-l)!}\right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \lambda^{n-l} \beta_l^{(r)}(\lambda, x) D_{n-l}^{(r)}\right) \frac{t^n}{n!}.
\end{aligned} \tag{23}$$

It is not difficult to show that

$$\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^r = r!\sum_{l=0}^{\infty}\frac{S_1(l+r,r)\,l!\lambda^l}{(l+r)!}\frac{t^l}{l!} = \sum_{l=0}^{\infty}\frac{S_1(l+r,r)}{\binom{l+r}{l}}\lambda^l\frac{t^l}{l!}. \tag{24}$$

Thus, by (24), we get

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{S_1(l+r,r)}{\binom{l+r}{r}}\binom{n}{l}\lambda^l\beta_{n-l}^{(r)}(\lambda,x)\right)\frac{t^n}{n!}. \tag{25}$$

The left hand side of (23) is given by

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}}\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ &=\sum_{n=0}^{\infty}\lambda^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{x+x_1+\cdots+x_r}{\lambda}\right)_n d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!} \\ &=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{n,\lambda}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!}.\end{aligned} \tag{26}$$

Therefore, by (23), (25) and (26), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{n,\lambda}\,d\mu_0(x_1)\cdots d\mu_0(x_r) &= \sum_{l=0}^{n}\binom{n}{l}\lambda^{n-l}\beta_l^{(r)}(\lambda,x)\,D_{n-l}^{(r)} \\ &= \sum_{l=0}^{n}\binom{n}{l}\frac{S_1(l+r,r)}{\binom{l+r}{r}}\lambda^l\beta_{n-l}^{(r)}(\lambda,x).\end{aligned}$$

From (12), we note that

$$(x_1+\cdots+x_r+x)_{n,\lambda} = \sum_{l=0}^{n}S_1(n,l\mid\lambda)(x_1+\cdots+x_r+x)^l. \tag{27}$$

By (2), we easily get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r+x)t}d\mu_0(x_1)\cdots d\mu_0(x_r) = \left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}. \tag{28}$$

Thus, by (28), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n\,d\mu_0(x_1)\cdots d\mu_0(x_r) = B_n^{(r)}(x), \quad (n\geq 0). \tag{29}$$

Therefore, by (29), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}S_1(n,l\mid\lambda)B_l^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}\frac{S_1(l+r,r)}{\binom{l+r}{r}}\lambda^l\beta_{n-l}^{(r)}(\lambda,x).$$

From (1), (2) and (3), we can derive the following equation:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \tag{30}$$

$$= \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} \right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(\lambda, x) \frac{t^n}{n!}.$$

Thus, by (30), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + x_1 + \cdots + x_r)_{n,\lambda} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \mathcal{E}_n^{(r)}(\lambda, x), \quad (n \geq 0). \tag{31}$$

Now, we observe that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{32}$$

Thus, by (32), we see

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x), \quad (n \geq 0). \tag{33}$$

From (12), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_{n,\lambda} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l=0}^{n} S_1(n, l \mid \lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + x_1 + \cdots + x_r)^l \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{l=0}^{n} S_1(n, l \mid \lambda) E_l^{(r)}(x). \end{aligned} \tag{34}$$

Therefore, by (31) and (34), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\mathcal{E}_n^{(r)}(\lambda, x) = \sum_{l=0}^{n} S_1(n, l \mid \lambda) E_l^{(r)}(x).$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t} - 1\right)$ in (13), we get

$$\begin{aligned} \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda, x) \frac{1}{\lambda^m} \left(e^{\lambda t} - 1\right)^m \frac{1}{m!} &= \left(\frac{\frac{1}{\lambda}\left(e^{\lambda t} - 1\right)}{e^t - 1} \right)^r e^{xt} = \left(\frac{e^{\lambda t} - 1}{\lambda t} \right)^r \left(\frac{t}{e^t - 1} \right)^r e^{xt} \\ = \left(\sum_{l=0}^{\infty} \frac{S_2(l+r, r)}{\binom{l+r}{r}} \lambda^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!} \right) &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{S_2(l+r, r)}{\binom{l+r}{r}} \binom{n}{l} \lambda^l B_{n-l}^{(r)}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda, x) \frac{1}{\lambda^m} \left(e^{\lambda t} - 1\right)^m \frac{1}{m!} &= \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda, x) \frac{1}{\lambda^m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \beta_m^{(r)}(\lambda, x) S_2(n, m) \lambda^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{36}$$

Therefore, by (35) and (36), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r, r) \lambda^l B_{n-l}^{(r)}(x) = \sum_{l=0}^{n} \beta_l^{(r)}(\lambda, x) S_2(n, l) \lambda^{n-l}.$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (19), we get

$$\sum_{m=0}^{\infty}\mathcal{E}_m^{(r)}(\lambda,x)\lambda^{-m}\left(e^{\lambda t}-1\right)^m\frac{1}{m!}=\left(\frac{2}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}. \tag{37}$$

We observe that

$$\begin{aligned}\sum_{m=0}^{\infty}\mathcal{E}_m^{(r)}(\lambda,x)\lambda^{-m}\left(e^{\lambda t}-1\right)^m\frac{1}{m!}&=\sum_{m=0}^{\infty}\mathcal{E}_m^{(r)}(\lambda,x)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\mathcal{E}_m^{(r)}(\lambda,x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{38}$$

Thus, by (37) and (38), we get

$$E_n^{(r)}(x)=\sum_{m=0}^{n}\mathcal{E}_m^{(r)}(\lambda,x)\lambda^{n-m}S_2(n,m),\quad (n,\geq 0). \tag{39}$$

On the other hand,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_n^{(r)}(\lambda,x)\frac{t^n}{n!}&=\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\left(\frac{2}{e^{\frac{1}{\lambda}\log(1+\lambda t)}+1}\right)^r e^{\frac{x}{\lambda}\log(1+\lambda t)}\\&=\sum_{m=0}^{\infty}E_m^{(r)}(x)\frac{\lambda^{-m}}{m!}(\log(1+\lambda t))^m=\sum_{m=0}^{\infty}E_m^{(r)}(x)\frac{1}{\lambda^m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}E_m^{(r)}(x)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{40}$$

By comparing the coefficients on both sides of (40), we get

$$\mathcal{E}_n^{(r)}(\lambda,x)=\sum_{m=0}^{n}E_m^{(r)}(x)\lambda^{n-m}S_1(n,m). \tag{41}$$

Therefore, by (39) and (41), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$E_n^{(r)}(x)=\sum_{m=0}^{n}\mathcal{E}_m^{(r)}(\lambda,x)\lambda^{n-m}S_2(n,m),$$

and

$$\mathcal{E}_n^{(r)}(\lambda,x)=\sum_{m=0}^{n}E_n^{(r)}(x)\lambda^{n-m}S_1(n,m).$$

From (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n^{(r)}(\lambda,x)\frac{t^n}{n!}&=\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}\\&=\left(\frac{t}{e^{\frac{1}{\lambda}\log(1+\lambda t)}-1}\right)^r e^{\frac{x}{\lambda}\log(1+\lambda t)}\\&=\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{e^{\frac{1}{\lambda}\log(1+\lambda t)}-1}\right)^r e^{\frac{x}{\lambda}\log(1+\lambda t)}\\&=\left(\sum_{l=0}^{\infty}b_l^{(r)}\lambda^l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}B_m^{(r)}(x)\frac{1}{\lambda^m}\frac{(\log(1+\lambda t))^m}{m!}\right)\end{aligned} \tag{42}$$

$$
\begin{aligned}
&=\left(\sum_{l=0}^{\infty} b_l^{(r)} \lambda^l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{1}{\lambda^m} \sum_{j=m}^{\infty} S_1(j, m) \lambda^j \frac{t^j}{j!}\right) \\
&=\left(\sum_{l=0}^{\infty} b_l^{(r)} \lambda^l \frac{t^l}{l!}\right)\left(\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j} B_m^{(r)}(x) \lambda^{j-m} S_1(j, m)\right) \frac{t^j}{j!}\right) \\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left(\sum_{m=0}^{j} B_m^{(r)}(x) \lambda^{j-m} S_1(j, m)\right) \frac{b_{n-j}^{(r)} \lambda^{n-j} n!}{j!(n-j)!}\right) \frac{t^n}{n!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} \sum_{m=0}^{j} B_m^{(r)} \lambda^{n-m} S_1(j, m) b_{n-j}^{(r)}\right) \frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on the both sides of (42), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$
\beta_n^{(r)}(\lambda, x)=\sum_{j=0}^{n}\binom{n}{j} \sum_{m=0}^{j} B_m^{(r)}(x) \lambda^{n-m} S_1(j, m) b_{n-j}^{(r)}.
$$

For $r, s \in \mathbb{N}$, let us consider the degenerate Bernoulli-Euler mixed-type polynomials of order (r, s) as follows:

$$
\beta \mathcal{E}_n^{(r, s)}(\lambda, x)=\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \mathcal{E}_n^{(s)}\left(\lambda, x+y_1+\cdots+y_r\right) d \mu_0\left(y_1\right) \cdots d \mu_0\left(y_r\right), \quad(n \geq 0). \tag{43}
$$

From (43), (20) and Theorem 2.1, we have

$$
\begin{aligned}
\beta \mathcal{E}_n^{(r, s)}(\lambda, x) &=\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \mathcal{E}_n^{(s)}\left(\lambda, x+y_1+\cdots+y_r\right) d \mu_0\left(y_1\right) \cdots d \mu_0\left(y_r\right) \qquad (44)\\
&=\sum_{l=0}^{n}\binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}\left(x+y_1+\cdots+y_r\right)_{n-l, \lambda} d \mu_0\left(y_1\right) \cdots d \mu_0\left(y_r\right) \\
&=\sum_{l=0}^{n}\binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l}\binom{n-l}{m} \frac{S_1(m+r, r)}{\binom{m+r}{r}} \lambda^m \beta_{n-l-m}^{(r)}(\lambda, x).
\end{aligned}
$$

Therefore, by (44), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$
\beta \mathcal{E}_n^{(r, s)}(\lambda, x)=\sum_{l=0}^{n}\binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l}\binom{n-l}{m} \frac{S_1(m+r, r)}{\binom{m+r}{r}} \lambda^m \beta_{n-l-m}^{(r)}(\lambda, x).
$$

Remark. *The generating function of $\beta \mathcal{E}_n^{(r, s)}(\lambda, x)$ is given by*

$$
\begin{aligned}
\sum_{n=0}^{\infty} \beta \mathcal{E}_n^{(r, s)}(\lambda, x) \frac{t^n}{n!} &=\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \mathcal{E}_n^{(s)}\left(\lambda, x+y_1, \ldots, y_r\right) \frac{t^n}{n!} d \mu_0\left(y_1\right) \cdots d \mu_0\left(y_r\right) \qquad (45)\\
&=\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y_1+\cdots+y_r}{\lambda}} d \mu_0\left(y_1\right) \cdots d \mu_0\left(y_r\right) \\
&=\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^s\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\left(\frac{\log (1+\lambda t)}{\lambda t}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}.
\end{aligned}
$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ *in* (45), *we get*

$$\sum_{n=0}^{\infty}\beta\mathcal{E}_{n}^{(r,s)}(\lambda,x)\frac{\lambda^{-n}}{n!}\left(e^{\lambda t}-1\right)^{n}=\left(\frac{2}{e^{t}+1}\right)^{s}\left(\frac{t}{e^{t}-1}\right)^{r}e^{xt}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_{l}^{(s)}B_{n-l}^{(r)}(x)\right)\frac{t^{n}}{n!}. \tag{46}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty}\beta\mathcal{E}_{m}^{(r,s)}(\lambda,x)\frac{\lambda^{-m}}{m!}\left(e^{\lambda t}-1\right)^{m}&=\sum_{m=0}^{\infty}\beta\mathcal{E}_{m}^{(r,s)}(\lambda,x)\lambda^{-m}\sum_{n=m}^{\infty}S_{2}(n,m)\frac{\lambda^{n}t^{n}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\beta\mathcal{E}_{m}^{(r,s)}(\lambda,x)S_{2}(n,m)\lambda^{n-m}\right)\frac{t^{n}}{n!}.\end{aligned} \tag{47}$$

Therefore, by (46) and (47), we obtain the following theorem.

Theorem 2.8. *For* $n\geq 0$*, we have*

$$\sum_{m=0}^{n}\beta\mathcal{E}_{m}^{(r,s)}(\lambda,x)S_{2}(n,m)\lambda^{n-m}=\sum_{m=0}^{n}\binom{n}{m}E_{m}^{(s)}B_{n-m}^{(r)}(x).$$

For $r,s\in\mathbb{N}$, let us define the higher-order degenerate Euler-Bernoulli mixed-type polynomials as follows:

$$\mathcal{E}\beta_{n}^{(r,s)}(\lambda,x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\beta_{n}^{(s)}(\lambda,y_{1}+\cdots+y_{r}+x)\,d\mu_{-1}(y_{1})\cdots d\mu_{-1}(y_{r}). \tag{48}$$

Thus, by (20) and (48), we get

$$\begin{aligned}\mathcal{E}\beta_{n}^{(r,s)}(\lambda,x)&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l}^{(s)}(\lambda)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(y_{1}+\cdots+y_{r}+x)_{n-l,\lambda}\,d\mu_{-1}(y_{1})\cdots d\mu_{-1}(y_{r})\\&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l}^{(s)}(\lambda)\,\mathcal{E}_{n-l}^{(r)}(\lambda,x).\end{aligned} \tag{49}$$

The generating function of $\mathcal{E}\beta_{n}^{(r,s)}(\lambda,x)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}\beta_{n}^{(r,s)}(\lambda,x)\frac{t^{n}}{n!}&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\beta_{n}^{(s)}(\lambda,y_{1}+\cdots+y_{r}+x)\frac{t^{n}}{n!}d\mu_{-1}(y_{1})\cdots d\mu_{-1}(y_{r})\\&=\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{s}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{y_{1}+\cdots+y_{r}+x}{\lambda}}\,d\mu_{-1}(y_{1})\cdots d\mu_{-1}(y_{r})\\&=\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{s}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}.\end{aligned} \tag{50}$$

From (12), we note that

$$\begin{aligned}&\mathcal{E}\beta_{n}^{(r,s)}(\lambda,x)\\&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l}^{(s)}(\lambda)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(y_{1}+\cdots+y_{r}+x)_{n-l,\lambda}\,d\mu_{-1}(y_{1})\cdots d\mu_{-1}(y_{r})\\&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l}^{(s)}(\lambda)\sum_{m=0}^{n-l}S_{1}(n-l,m\mid\lambda)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(y_{1}+\cdots+y_{r}+x)^{m}\,d\mu_{-1}(y_{1})\cdots d\mu_{-1}(y_{R})\\&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l}^{(s)}(\lambda)\sum_{m=0}^{n-l}S_{1}(n-l,m\mid\lambda)\,E_{m}^{(r)}(x).\end{aligned} \tag{51}$$

Therefore, by (49) and (51), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$\mathcal{E}\beta_n^{(r,s)}(\lambda, x) = \sum_{l=0}^{n} \binom{n}{l} \beta_l^{(s)}(\lambda)\, \mathcal{E}_{n-l}^{(r)}(\lambda, x) = \sum_{l=0}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \beta_l^{(s)}(\lambda)\, S_1(n-l, m \mid \lambda)\, E_m^{(r)}(x).$$

Remark. *From* (43)*, we have*

$$\begin{aligned}
\beta\mathcal{E}_n^{(r,s)}(\lambda, x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \mathcal{E}_n^{(s)}(\lambda, x + y_1 + \cdots + y_r)\, d\mu_0(y_1) \cdots d\mu_0(y_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)_{n-l,\lambda}\, d\mu_0(y_1) \cdots d\mu_0(y_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l} S_1(n-l, m \mid \lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)^m\, d\mu_0(y_1) \cdots d\mu_0(y_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l} S_1(n-l, m \mid \lambda)\, B_m^{(r)}(x).
\end{aligned}$$

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