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Periodic and subharmonic solutions for a $2n$ th-order p -Laplacian difference equation containing both advances and retardations

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Abstract: We consider a $2n$ th-order nonlinear difference equation containing both many advances and retardations with p -Laplacian. Using the critical point theory, we obtain some new explicit criteria for the existence and multiplicity of periodic and subharmonic solutions. Our results generalize and improve some known related ones.

Keywords: Periodic and subharmonic solution, $2n$ th-order, Nonlinear difference equation, p -Laplacian, Critical point theory

MSC: 39A23

1 Introduction

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers, respectively. For $a \leq b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \dots, b\}$.

Consider the following $2n$ th-order nonlinear difference equation

$$\Delta^n (r_{k-n} \varphi_p(\Delta^n u_{k-n})) = (-1)^n f(k, u_{k+\tau}, \dots, u_{k+1}, u_k, u_{k-1}, \dots, u_{k-\tau}), \quad k \in \mathbf{Z}, \quad (1.1)$$

where $\tau, n \in \mathbf{Z}(1)$, Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$, $\Delta^2 u_k = \Delta(\Delta u_k)$, $r_k > 0$ is real valued for each $k \in \mathbf{Z}$, $\{r_k\}$ and $f(k, v_\tau, \dots, v_{-\tau})$ are T -periodic in k for a given positive integer T , and $f \in C(\mathbf{Z} \times \mathbf{R}^{2\tau+1}, \mathbf{R})$. $\varphi_p(s)$ is the p -Laplacian operator given by $\varphi_p(s) = |s|^{p-2}s$ ($1 < p < \infty$). For any integer $m \geq 2$, a solution to Eq. (1.1) is called a m th-order subharmonic solution if it is a mT -periodic solution.

Earlier, the main methods were all kinds of fixed point theorems in cones for the study of periodic solutions and boundary value problems of difference equations. It was not until 2003 that the critical point theory was used to establish sufficient conditions for the existence of periodic solutions. Guo and Yu [1, 2] first established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations by using the critical point theory.

In 2007, Cai and Yu [3] obtained some criteria for the existence of periodic solutions of a $2n$ th-order difference equation

$$\Delta^n (r_{k-n} \Delta^n u_{k-n}) + f(k, u_k) = 0, \quad n \in \mathbf{Z}(3), \quad k \in \mathbf{Z}, \quad (1.2)$$

where f grows superlinearly at both 0 and ∞ . Using Linking Theorem and Saddle Point Theorem, Zhou [4] improved the results of [3] and extended f in Eq. (1.2) into sublinear or asymptotically linear.

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In fact, there are some papers which studied the periodic solutions of difference equations involving both advances and retardations which have important background and meaning in the field of cybernetics and biological mathematics. Chen and Fang [5] in 2007 considered the following second-order nonlinear difference equation containing both advance and retardation with p -Laplacian

$$\Delta(\varphi_p(\Delta x_{n-1})) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \quad n \in \mathbf{Z}. \quad (1.3)$$

In 2014, Lin and Zhou [6] obtained some new sufficient conditions on the existence and multiplicity of periodic solutions of the following ϕ -Laplacian difference equation

$$\Delta^n(r_{k-n}\phi(\Delta^n u_{k-1})) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \quad k \in \mathbf{Z}. \quad (1.4)$$

In the past literature, when it comes to a special Eq. (1.1) that $\tau = 1$, many excellent works have been done (e.g. see [5–9]). Using the critical point theory, they obtained some sufficient conditions on the existence and multiplicity of periodic solutions in the special case of Eq. (1.1).

Nevertheless, to the best of our knowledge, the results on periodic solutions of nonlinear difference equations containing both many advances and retardations with p -Laplacian are very scarce. To fill this gap, this paper gives some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions to Eq. (1.1).

We mention that, in recent years, the critical point theory is also used on the study of homoclinic solutions [10–17] and boundary value problems [18–23] for difference equations.

Let

$$\underline{r} = \min_{k \in \mathbf{Z}(1, T)} \{r_k\}, \quad \bar{r} = \max_{k \in \mathbf{Z}(1, T)} \{r_k\}.$$

Our main result is as follows.

Theorem 1.1. Assume that the following hypotheses hold:

(T_1) there exists a functional $F(k, v_0, v_1, \dots, v_\tau) \in C^1(\mathbf{Z} \times \mathbf{R}^{\tau+1}, \mathbf{R})$ with $F(k, v_0, v_1, \dots, v_\tau) \geq 0$ and it satisfies

$$F(k + T, v_0, v_1, \dots, v_\tau) = F(k, v_0, v_1, \dots, v_\tau), \quad k \in \mathbf{Z},$$

$$\sum_{j=k}^{k+\tau} \frac{\partial F(j, v_j, v_{j-1}, \dots, v_{j-\tau})}{\partial v_k} = f(k, v_{k+\tau}, \dots, v_{k+1}, v_k, v_{k-1}, \dots, v_{k-\tau});$$

(T_2) there exists a constant $\alpha \in \left(0, \frac{\underline{r}}{(\tau+1)^{p/2}p} (mT)^{\frac{|2-p|}{2}} \left(2 \sin \frac{\pi}{mT}\right)^{np}\right)$ such that

$$\limsup_{\delta_1 \rightarrow 0} \frac{F(k, v_0, v_1, \dots, v_\tau)}{\delta_1^p} \leq \alpha, \quad \text{for } k \in \mathbf{Z} \text{ and } \delta_1 = \sqrt{\sum_{j=0}^{\tau} v_j^2};$$

(T_3) there exists a constant $\beta \in \left(\frac{\bar{r}}{(\tau+1)^{p/2}p} (mT)^{\frac{|2-p|}{2}} \left(2 \cos \frac{1-(-1)^{mT}}{2mT} \pi\right)^{np}, +\infty\right)$ such that

$$\liminf_{\delta_1 \rightarrow \infty} \frac{F(k, v_0, v_1, \dots, v_\tau)}{\delta_1^p} \geq \beta, \quad \text{for } k \in \mathbf{Z} \text{ and } \delta_1 = \sqrt{\sum_{j=0}^{\tau} v_j^2}.$$

Then for any given positive integer m , Eq. (1.1) possesses at least two mT -periodic nontrivial solutions.

It is worth pointing out that our sufficient conditions are based on the limit superior and limit inferior which are more applicable. Moreover, we also extend the conclusions to a more general form. As far as we know, in most of the previous results (e.g. see [7]), the values of c_1 and c_2 cannot be determined, that is not operable, but we present their specific values. In fact, if $\tau = 1$, the assumptions in Theorem 1.1 are more explicit and easier to verify than those in Theorem 1.1 in [7]. For the sake of clarity, we put the remaining results at the end of the article. Our results complement the existing ones. See Remarks 4.6 and 4.7 for details.

The outline of this paper is as follows. In Section 2 we establish the variational framework associated with Eq. (1.1) and transfer the problem of the existence of periodic solutions of Eq. (1.1) into that of the existence of critical points of the corresponding functional. In Section 3, some related fundamental results are recalled for convenience, and some lemmas are proven. Then, we complete the proof of our main result by using Linking Theorem in Section 4. Finally, in Section 5, we illustrate our results with an example.

2 Variational structure

This section is to establish the corresponding variational framework for Eq. (1.1) and cite some basic conclusions for the forthcoming discussion.

Let S be the set of all two-side sequences, that is

$$S = \{\{u_k\} | u_k \in \mathbf{R}, k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space. For any given positive integers m and T , we define the subspace E_{mT} of S as

$$E_{mT} = \{u \in S | u_{k+mT} = u_k, k \in \mathbf{Z}\}.$$

It is trivial to show that, E_{mT} is isomorphic to $\mathbf{R}^{mT}$ and can be endowed with the inner product

$$\langle u, v \rangle = \sum_{j=1}^{mT} u_j v_j, \quad \forall u, v \in E_{mT},$$

and corresponding norm

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2 \right)^{\frac{1}{2}}, \quad \forall u \in E_{mT}.$$

On the other hand, we define the norm $\|\cdot\|_p$ on E_{mT} as follows:

$$\|u\|_p = \left(\sum_{j=1}^{mT} |u_j|^p \right)^{\frac{1}{p}},$$

for all $u \in E_{mT}$. Similarly to the derivation of [6], by Hölder inequality and Jensen inequality, we have

$$\begin{cases} \|u\| \leq \|u\|_p \leq (mT)^{\frac{2-p}{2p}} \|u\|, & 1 < p < 2, \\ (mT)^{\frac{2-p}{2p}} \|u\| \leq \|u\|_p \leq \|u\|, & 2 \leq p. \end{cases}$$

Let

$$c_1(p) = \begin{cases} 1, & 1 < p < 2, \\ (mT)^{\frac{2-p}{2p}}, & 2 \leq p, \end{cases} \quad c_2(p) = \begin{cases} (mT)^{\frac{2-p}{2p}}, & 1 < p < 2, \\ 1, & 2 \leq p, \end{cases}$$

then

$$c_1(p)\|u\| \leq \|u\|_p \leq c_2(p)\|u\|, \quad \forall u \in E_{mT}, \quad (2.1)$$

and

$$\frac{c_1(p)}{c_2(p)} = (mT)^{\frac{-|2-p|}{2p}}.$$

Define the functional J on E_{mT} as

$$J(u) = \frac{1}{p} \sum_{k=1}^{mT} r_{k-1} |\Delta^n u_{k-1}|^p - \sum_{k=1}^{mT} F(k, u_k, u_{k-1}, \dots, u_{k-\tau}), \quad u \in E_{mT}.$$

Clearly, $J \in C^1(E_{mT}, \mathbf{R})$ and by the fact that $u_0 = u_{mT}$, $u_1 = u_{mT+1}$, after a careful computation, we can find

$$\frac{\partial J}{\partial u_k} = (-1)^n \Delta^n (r_{k-n} \varphi_p(\Delta^n u_{k-n})) - f(k, u_{k+\tau}, \dots, u_{k+1}, u_k, u_{k-1}, \dots, u_{k-\tau}), \quad \forall k \in \mathbf{Z}(1, mT).$$

Thus, u is a critical point of J on E_{mT} if and only if Eq. (1.1) holds.

So, we reduce the existence of periodic solutions of Eq. (1.1) to that of the critical points of the functional J on E_{mT} . Indeed, $u \in E_{mT}$ can be identified with $u = (u_1, u_2, \dots, u_{mT})^*$, where $*$ denotes the transpose of the vector.

Let P be the corresponding $mT \times mT$ matrix to the quadratic form $\sum_{k=1}^{mT} (\Delta u_k)^2$ with $u_{k+mT} = u_k$ for $k \in \mathbf{Z}$, which is defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By the matrix theory, we obtain that the eigenvalues of P are

$$\lambda_j = 4 \sin^2 \frac{j\pi}{mT}, \quad j = 0, 1, 2, \dots, mT - 1.$$

This implies $\lambda_0 = 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\dots$, $\lambda_{mT-1} > 0$. Therefore,

$$\begin{cases} \underline{\lambda} = \min\{\lambda_1, \lambda_2, \dots, \lambda_{mT-1}\} = 4 \sin^2 \frac{\pi}{mT}, \\ \bar{\lambda} = \max\{\lambda_1, \lambda_2, \dots, \lambda_{mT-1}\} = 4 \cos^2 \frac{1-(-1)^{mT}}{2mT} \pi. \end{cases}$$

Let

$$W = \ker P = \{u \in E_{mT} | Pu = 0, u \in \mathbf{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} | u = \{c\}, c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W , i.e., $E_{mT} = V \oplus W$.

3 Some results and lemmas

For the reader's convenience, we give some basic notations and some known results about the critical point theory.

Definition 3.1. Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E . If any sequence $\{u^{(i)}\} \subset E$ for which $\{J(u^{(i)})\}$ is bounded and $J'(u^{(i)}) \rightarrow 0 (i \rightarrow \infty)$ possesses a convergent subsequence, then we say J satisfies the Palais-Smale condition (P.S. condition for short).

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 3.2 (Linking Theorem [24]). Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, \mathbf{R})$ satisfies the P.S. condition and

(J₁) There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;

(J₂) There exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se | 0 < s < R_0\}$.

Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$, where id denotes the identity operator.

Then we prove some lemmas which are useful in the proof of Theorem 1.1. First, similarly to the derivation of [3], we can find the following lemma.

Lemma 3.3. Let $x = (\Delta^{n-1}u_1, \Delta^{n-1}u_2, \dots, \Delta^{n-1}u_{mT})^*$. For any $u \in E_{mT}$, one has

$$\underline{\lambda}^{\frac{(n-1)p}{2}} \|u\|^p \leq \|x\|^p \leq \bar{\lambda}^{\frac{(n-1)p}{2}} \|u\|^p.$$

Let $G(u) = \frac{1}{p} \sum_{k=1}^{mT} r_{k-1} |\Delta^n u_{k-1}|^p$, it follows from Lemma 3.3 that

$$G(u) \leq \frac{\bar{r}}{p} \left[\left(\sum_{k=1}^{mT} (\Delta^n u_{k-1})^p \right)^{\frac{1}{p}} \right]^p \leq \frac{\bar{r}}{p} \left[c_2(p) \left(\sum_{k=1}^{mT} (\Delta^n u_{k-1})^2 \right)^{\frac{1}{2}} \right]^p \leq \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{p}{2}} \|x\|^p \leq \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} \|u\|^p$$

and

$$G(u) \geq \frac{r}{p} \left[\left(\sum_{k=1}^{mT} (\Delta^n u_{k-1})^p \right)^{\frac{1}{p}} \right]^p \geq \frac{r}{p} \left[c_1(p) \left(\sum_{k=1}^{mT} (\Delta^n u_{k-1})^2 \right)^{\frac{1}{2}} \right]^p \geq \frac{r}{p} c_1^p(p) \underline{\lambda}^{\frac{p}{2}} \|x\|^p \geq \frac{r}{p} c_1^p(p) \underline{\lambda}^{\frac{np}{2}} \|u\|^p.$$

Lemma 3.4. Assume that (T_1) and (T_3) hold. Then the functional J is bounded from above on E_{mT} .

Proof. By (T_3) , there exist constants $\zeta > 0$ and $\beta' \in \left(\frac{\bar{r}}{(\tau+1)^{p/2} p} (mT)^{\frac{|2-p|}{2}} \left(2 \cos \frac{1-(-1)^{mT}}{2mT} \pi \right)^{np}, \beta \right)$ such that

$$F(k, v_0, v_1, \dots, v_\tau) \geq \beta' \left(\sqrt{\sum_{j=0}^{\tau} v_j^2} \right)^p - \zeta, \quad \forall (k, v_0, v_1, \dots, v_\tau) \in \mathbf{Z} \times \mathbf{R}^{\tau+1}.$$

For any $u \in E_{mT}$, by Lemma 3.3 we have

$$\begin{aligned} J(u) &= \frac{1}{p} \sum_{k=1}^{mT} r_{k-1} |\Delta^n u_{k-1}|^p - \sum_{k=1}^{mT} F(k, u_k, u_{k-1}, \dots, u_{k-\tau}) \\ &\leq \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} \|u\|^p - \sum_{k=1}^{mT} \left[\beta' \left(\sqrt{\sum_{i=-\tau}^0 u_{k+i}^2} \right)^p - \zeta \right] \\ &\leq \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} \|u\|^p - \beta' c_1^p(p) \left[\sum_{k=1}^{mT} \left(\sum_{i=-\tau}^0 u_{k+i}^2 \right) \right]^{\frac{p}{2}} + mT\zeta \\ &= \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} \|u\|^p - (\tau+1)^{\frac{p}{2}} \beta' c_1^p(p) \|u\|^p + mT\zeta \\ &= \left(\frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} - (\tau+1)^{\frac{p}{2}} \beta' c_1^p(p) \right) \|u\|^p + mT\zeta \\ &\leq mT\zeta. \end{aligned}$$

The proof of Lemma 3.4 is complete. □

Remark 3.5. The case $mT = 1$ is trivial. For the case $mT = 2$, P has a different form, namely,

$$P = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

However, in this special case, the argument need not to be changed and we omit it.

Lemma 3.6. Assume that (T_1) and (T_3) hold. Then the functional J satisfies the P.S. condition.

Proof. Let $\{J(u^{(i)})\}$ be a bounded sequence from the lower bound, i.e., there exists a positive constant M such that

$$-M \leq J(u^{(i)}), \quad \forall i \in \mathbf{N}.$$

By the proof of Lemma 3.4, it is easy to see that

$$-M \leq J(u^{(i)}) \leq \left(\frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} - (\tau+1)^{\frac{p}{2}} \beta' c_1^p(p) \right) \|u^{(i)}\|^p + mT\zeta, \quad \forall i \in \mathbf{N}.$$

Therefore,

$$\left((\tau+1)^{\frac{p}{2}} \beta' c_1^p(p) - \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} \right) \|u^{(i)}\|^p \leq M + mT\zeta.$$

Since $\beta' > \frac{\bar{r}}{(\tau+1)^{p/2} p} \left(\frac{c_2(p)}{c_1(p)} \right)^p \bar{\lambda}^{\frac{np}{2}}$, it is not difficult to know that $\{u^{(i)}\}$ is a bounded sequence on E_{mT} . As a consequence, $\{u^{(i)}\}$ possesses a convergence subsequence and J satisfies the P.S. condition. $\square$

4 Proof of the main result

In this section, we shall prove our main results by using Linking Theorem.

Assumptions (T_1) and (T_2) imply that $F(k, 0, \dots, 0) = 0$ and $f(k, 0, \dots, 0) = 0$ for $k \in \mathbf{Z}$. Then $u = 0$ is a trivial mT -periodic solution of Eq. (1.1). It suffices to prove that J has at least two nontrivial critical points on E_{mT} .

Firstly, we show the existence of one nontrivial critical point. By Lemma 3.4, J is bounded from above on E_{mT} . The proof of it implies $\lim_{\|u\| \rightarrow +\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. Let us define $c_0 = \sup_{u \in E_{mT}} J(u)$. There exists $\bar{u} \in E_{mT}$ such that $J(\bar{u}) = c_0$ by the continuity of $J(u)$. Clearly, $\bar{u}$ is a critical point of J .

We claim that $c_0 > 0$, which implies that $\bar{u}$ is a nontrivial critical point of J . Indeed, by (T_2) , there exist two positive constants δ and $\alpha' \in \left(\alpha, \frac{r}{(\tau+1)^{p/2} p} (mT)^{\frac{-|2-p|}{2}} \left(2 \sin \frac{\pi}{mT} \right)^{np} \right)$ such that

$$F(k, u_0, u_1, \dots, u_\tau) \leq \alpha' \left(\sqrt{\sum_{j=0}^{\tau} u_j^2} \right)^p, \quad \text{for } k \in \mathbf{Z} \text{ and } \sum_{j=0}^{\tau} u_j^2 \leq \delta^2.$$

For any $u \in V$ with $\|u\| \leq \delta$, we have

$$J(u) \geq \left(\frac{r}{p} c_1^p(p) \bar{\lambda}^{\frac{np}{2}} - (\tau+1)^{\frac{p}{2}} \alpha' c_2^p(p) \right) \|u\|^p. \quad (4.1)$$

Taking $\sigma = \left(\frac{r}{p} c_1^p(p) \bar{\lambda}^{\frac{np}{2}} - (\tau+1)^{\frac{p}{2}} \alpha' c_2^p(p) \right) \delta^p$, then by Eq. (4.1),

$$J(u) \geq \sigma, \quad \forall u \in V \cap \partial B_\delta.$$

This implies that $c_0 = \sup_{u \in E_{mT}} J(u) \geq \sigma > 0$. Hence the claim is proved. At the same time, we have proved that there exist constants $\sigma > 0$ and $\delta > 0$ such that $J|_{\partial B_\delta \cap V} \geq \sigma$. In the other word, J satisfies the condition (J_1) of Linking Theorem.

In the remaining of the proof, we shall use Lemma 3.2 to obtain another nontrivial critical point. We have known that J satisfies the P.S. condition on E_{mT} . In the following, we shall verify the condition (J_2) .

Take $e \in \partial B_1 \cap V$. For any $z \in W$ and $s \in \mathbf{R}$, we denote $u = se + z$. Then we have

$$\begin{aligned} J(u) &= \frac{1}{p} \sum_{k=1}^{mT} r_{k-1} |\Delta^n u_{k-1}|^p - \sum_{k=1}^{mT} F(k, u_k, u_{k-1}, \dots, u_{k-\tau}) \\ &\leq \frac{\bar{r}}{p} s^p \sum_{k=1}^{mT} |\Delta^n e_k|^p - \sum_{k=1}^{mT} F(k, se_k + z_k, se_{k-1} + z_{k-1}, \dots, se_{k-\tau} + z_{k-\tau}) \\ &\leq \frac{\bar{r}}{p} s^p c_2^p(p) \left[\sum_{k=1}^{mT} (\Delta^n e_k)^2 \right]^{\frac{p}{2}} - \beta' c_1^p(p) \left\{ \sum_{k=1}^{mT} \left[\sum_{i=-\tau}^0 (se_{k+i} + z_{k+i})^2 \right] \right\}^{\frac{p}{2}} + mT\zeta \\ &\leq \frac{\bar{r}}{p} s^p c_2^p(p) \bar{\lambda}^{\frac{np}{2}} - \beta' c_1^p(p) \left[(\tau+1) \sum_{k=1}^{mT} (se_k + z_k)^2 \right]^{\frac{p}{2}} + mT\zeta \end{aligned}$$

$$\leq \frac{\bar{r}}{p} c_2^p(p) \bar{\lambda}^{\frac{np}{2}} s^p - (\tau + 1)^{\frac{p}{2}} \beta' c_1^p(p) \left(s^2 + \|z\|^2 \right)^{\frac{p}{2}} + mT\zeta.$$

Noting that for all $u \in W$, $\sum_{k=1}^{mT} r_{k-1} |\Delta^n u_{k-1}|^p = 0$, we have

$$J(u) = \frac{1}{p} \sum_{k=1}^{mT} r_{k-1} |\Delta^n u_{k-1}|^p - \sum_{k=1}^{mT} F(k, u_k, u_{k-1}, \dots, u_{k-\tau}) = - \sum_{k=1}^{mT} F(k, u_k, u_{k-1}, \dots, u_{k-\tau}) \leq 0.$$

Thus, there exists a positive constant $R_1 > \delta$ such that $J(u) \leq 0$ for any $u \in \partial Q$, where $Q = (\bar{B}_{R_1} \cap W) \oplus \{se | 0 < s < R_1\}$. This verifies the condition (J_2) of Lemma 3.2. Therefore, J possesses a critical value $c \geq \sigma > 0$ with

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)) \quad \text{and} \quad \Gamma = \{h \in C(\bar{Q}, E_{mT}) \mid h|_{\partial Q} = id\}.$$

Let $\tilde{u} \in E_{mT}$ be a critical point associated to the critical value c of J , i.e., $J(\tilde{u}) = c$. If $\tilde{u} \neq \bar{u}$, then we are done. Otherwise, if $\tilde{u} = \bar{u}$, it follows that

$$c_0 = \sup_{u \in E_{mT}} J(u) = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)).$$

In particular, choosing $h = id$, we have $\sup_{u \in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap V$ is arbitrary, we can take $-e \in \partial B_1 \cap V$. Similarly, there exists a positive number $R_2 > \delta$, so that $J(u) \leq 0$ for any $u \in \partial Q_1$, where $Q_1 = (\bar{B}_{R_2} \cap W) \oplus \{-se | 0 < s < R_2\}$.

Using Linking Theorem again, J possesses a critical value $c' \geq \sigma > 0$, where

$$c' = \inf_{h \in \Gamma_1} \sup_{u \in Q_1} J(h(u)) \quad \text{and} \quad \Gamma_1 = \{h \in C(\bar{Q}_1, E_{mT}) \mid h|_{\partial Q_1} = id\}.$$

We claim that $c' \neq c_0$. Otherwise, suppose that $c' = c_0$, then $\sup_{u \in Q_1} J(u) = c_0$. Due to the facts $J|_{\partial Q} \leq 0$ and $J|_{\partial Q_1} \leq 0$, J attains its maximum at some points in the interior of sets Q and Q_1 . However, $Q \cap Q_1 \subset W$ and $J(u) \leq 0$ for any $u \in W$. This implies that $c_0 \leq 0$, which contradicts $c_0 > 0$. This proves the claim and hence the proof is complete. $\square$

According to Theorem 1.1, it is easy to obtain the following theorems and corollaries.

Theorem 4.1. Assume that (T_1) and the following conditions hold:

(T_4) there exist constants $\rho_1 > 0$, $\alpha \in \left(0, \frac{\bar{r}}{(\tau+1)^{p/2}p} (mT)^{\frac{|2-p|}{2}} \left(2 \sin \frac{\pi}{mT}\right)^{np}\right)$ such that

$$F(k, v_0, v_1, \dots, v_\tau) \leq \alpha \left(\sqrt{\sum_{j=0}^{\tau} v_j^2} \right)^p, \quad \text{for } k \in \mathbf{Z} \text{ and } \sum_{j=0}^{\tau} v_j^2 \leq \rho_1^2;$$

(T_5) there exist constants $\rho_2 > 0$, $\zeta_1 > 0$, $\beta \in \left(\frac{\bar{r}}{(\tau+1)^{p/2}p} (mT)^{\frac{|2-p|}{2}} \left(2 \cos \frac{1-(-1)^{mT}}{2mT} \pi\right)^{np}, +\infty\right)$ such that

$$F(k, v_0, v_1, \dots, v_\tau) \geq \beta \left(\sqrt{\sum_{j=0}^{\tau} v_j^2} \right)^p - \zeta_1, \quad \text{for } k \in \mathbf{Z} \text{ and } \sum_{j=0}^{\tau} v_j^2 \geq \rho_2^2.$$

Then for any given positive integer m , Eq. (1.1) has at least two mT -periodic nontrivial solutions.

Theorem 4.2. Assume that (T_1) and the following conditions hold:

(T_6)

$$\lim_{\delta_1 \rightarrow 0} \frac{F(k, v_0, v_1, \dots, v_\tau)}{\delta_1^p} = 0, \quad \delta_1 = \sqrt{\sum_{j=0}^{\tau} v_j^2}, \quad \forall (k, v_0, v_1, \dots, v_\tau) \in \mathbf{Z} \times \mathbf{R}^{\tau+1};$$

(T_7) there exist constants $\rho_3 > 0$ and $p' > p$ such that for $k \in \mathbf{Z}$ and $\sum_{j=0}^{\tau} v_j^2 \geq \rho_3^2$,

$$0 < p' F(k, v_0, v_1, \dots, v_{\tau}) \leq \sum_{j=0}^{\tau} \frac{\partial F(k, v_0, v_1, \dots, v_{\tau})}{\partial v_j} v_j.$$

Then for any given positive integer m , Eq. (1.1) has at least two mT -periodic nontrivial solutions.

If $\tau = 0$ and $f(k, u_k) = q_k g(u_k)$, Eq. (1.1) reduces to the following $2n$ th-order nonlinear equation with p -Laplacian,

$$\Delta^n (r_{k-n} \varphi_p(\Delta^n u_{k-n})) = (-1)^n q_k g(u_k), \quad k \in \mathbf{Z}, \quad (4.2)$$

where $g \in C(\mathbf{R}, \mathbf{R})$, $q_{k+T} = q_k > 0$, for all $k \in \mathbf{Z}$. Then, we have the following results.

Corollary 4.3. Assume that the following hypotheses hold:

(G_1) there exists a functional $G(v) \in C^1(\mathbf{R}, \mathbf{R})$ with $G(v) \geq 0$ and it satisfies

$$G'(v) = g(v);$$

(G_2) there exists a constant $\alpha \in \left(0, \frac{\tau}{(\tau+1)^{p/2} p} (mT)^{\frac{-|2-p|}{2}} \left(2 \sin \frac{\pi}{mT}\right)^{np}\right)$ such that

$$\limsup_{|v| \rightarrow 0} \frac{G(v)}{|v|^p} \leq \alpha;$$

(G_3) there exists a constant $\beta \in \left(\frac{\tau}{(\tau+1)^{p/2} p} (mT)^{\frac{|2-p|}{2}} \left(2 \cos \frac{1-(-1)^{mT}}{2mT} \pi\right)^{np}, +\infty\right)$ such that

$$\liminf_{|v| \rightarrow \infty} \frac{G(v)}{|v|^p} \geq \beta.$$

Then for any given positive integer m , Eq. (4.2) has at least two mT -periodic nontrivial solutions.

Corollary 4.4. Assume that (G_1) and the following conditions hold:

(G_4) there exist constants $\rho'_1 > 0$, $\alpha \in \left(0, \frac{\tau}{p} (mT)^{\frac{-|2-p|}{2}} \left(2 \sin \frac{\pi}{mT}\right)^{np}\right)$ such that

$$G(v) \leq \alpha |v|^p, \quad \text{for } |v| \leq \rho'_1;$$

(G_5) there exist constants $\rho'_2 > 0$, $\zeta'_1 > 0$, $\beta \in \left(\frac{\tau}{p} (mT)^{\frac{|2-p|}{2}} \left(2 \cos \frac{1-(-1)^{mT}}{2mT} \pi\right)^{np}, +\infty\right)$ such that

$$G(v) \geq \beta |v|^p - \zeta'_1, \quad \text{for } |v| \geq \rho'_2.$$

Then for any given positive integer m , Eq. (4.2) has at least two mT -periodic nontrivial solutions.

Corollary 4.5. Assume that (G_1) and the following conditions hold:

(G_6)

$$\lim_{|v| \rightarrow 0} \frac{G(v)}{|v|^p} = 0, \quad \forall v \in \mathbf{R};$$

(G_7) there exist constants $\rho'_3 > 0$ and $\tilde{p}' > p$ such that for $k \in \mathbf{Z}$ and $|v| \geq \rho'_3$,

$$0 < \tilde{p}' G(v) \leq v g(v).$$

Then for any given positive integer m , Eq. (4.2) has at least two mT -periodic nontrivial solutions.

Remark 4.6. If $\tau = 1$, $r_k \equiv 1$ and $n = 1$, Theorem 1.1 reduces to Theorem 3.1 in [5].

Remark 4.7. If $\tau = 0$ and $p = 2$, Theorem 4.2 reduces to Theorem 1.1, Corollary 4.5 reduces to Corollary 1.1 in [3].

5 Example

Finally, as an application of Theorem 1.1, we give an example to illustrate our main results.

Example 5.1. For given $n \in \mathbb{Z}(1)$, consider the following difference equation

$$\Delta^n (r_{k-n} \varphi_p (\Delta^n u_{k-n})) = (-1)^n \mu u_k \sum_{j=k}^{k+\tau} \left[2 + \sin \left(\frac{j\pi}{T} \right) \right] \left(\sum_{i=-\tau}^0 u_{i+j}^2 \right)^{\frac{\mu}{2}-1}, \quad k \in \mathbb{Z} \quad (5.1)$$

where $\{r_k\}_{k \in \mathbb{Z}}$ is a real sequence and $r_{k+T} = r_k > 0$, $1 < p < \infty$, $\mu > p$, T is a given positive integer. Here

$$f(k, v_{k+\tau}, \dots, v_{k+1}, v_k, v_{k-1}, \dots, v_{k-\tau}) = \mu v_k \sum_{j=k}^{k+\tau} \left[2 + \sin \left(\frac{j\pi}{T} \right) \right] \left(\sum_{i=-\tau}^0 v_{i+j}^2 \right)^{\frac{\mu}{2}-1}$$

and

$$F(k, v_0, v_{-1}, \dots, v_{-\tau}) = \left[2 + \sin \left(\frac{k\pi}{T} \right) \right] \left(\sum_{i=-\tau}^0 v_i^2 \right)^{\frac{\mu}{2}}.$$

Then

$$\sum_{j=k}^{k+\tau} \frac{\partial F(j, v_j, v_{j-1}, \dots, v_{j-\tau})}{\partial v_k} = \mu v_k \sum_{j=k}^{k+\tau} \left[2 + \sin \left(\frac{j\pi}{T} \right) \right] \left(\sum_{i=-\tau}^0 v_{i+j}^2 \right)^{\frac{\mu}{2}-1}.$$

It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer m , Eq. (5.1) has at least two mT -periodic nontrivial solutions.

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References

- [1] Guo Z. M., Yu J. S., Existence of periodic and subharmonic solutions for second-order superlinear difference equations, *Sci. China Math.*, 2003, 46, 506–515
- [2] Guo Z. M., Yu J. S., The existence of periodic and subharmonic solutions of subquadratic second order difference equations, *J. London Math. Soc.*, 2003, 68, 419–430
- [3] Cai X. C., Yu J. S., Existence of periodic solutions for a $2n$ th-order nonlinear difference equation, *J. Math. Anal. Appl.*, 2007, 329, 870–878
- [4] Zhou Z., Yu J. S., Chen Y. M., Periodic solutions for a $2n$ th-order nonlinear difference equation, *Sci. China Math.*, 2010, 53, 41–50
- [5] Chen P., Fang H., Existence of periodic and subharmonic solutions for second-order p -Laplacian difference equations, *Adv. Difference Equ.*, 2007, 2007, 9 pages
- [6] Lin G. H., Zhou Z., Periodic and subharmonic solutions for a $2n$ th-order difference equation containing both advance and retardation with ϕ -Laplacian, *Adv. Difference Equ.*, 2014, 74
- [7] Deng X. Q., Liu X., Zhang Y. B., Shi H. P., Periodic and subharmonic solutions for a $2n$ th-order difference equation involving p -Laplacian, *Indag. Math.*, 2013, 24, 613–625
- [8] Liu X., Zhang Y. B., Shi H. P., Deng X. Q., Periodic and subharmonic solutions for fourth-order nonlinear difference equations, *Appl. Math. Comput.*, 2014, 236, 613–620
- [9] Shi H. P., Zhang Y. B., Existence of periodic solutions for a $2n$ th-order nonlinear difference equation, *Taiwanese J. Math.*, 2016, 20, 143–160
- [10] Zhou Z., Yu J. S., On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems, *J. Differential Equations*, 2010, 249, 1199–1212
- [11] Zhou Z., Yu J. S., Chen Y. M., Homoclinic solutions in periodic difference equations with saturable nonlinearity, *Sci. China Math.*, 2011, 54, 83–93
- [12] Chen P., Tang X. H., Existence of homoclinic orbits for $2n$ th-order nonlinear difference equations containing both many advances and retardations, *J. Math. Anal. Appl.*, 2011, 381, 485–505

- [13] Mai A., Zhou Z., Discrete solitons for periodic discrete nonlinear Schrödinger equations, *Appl. Math. Comput.*, 2013, 222, 34–41
- [14] Zhou Z., Ma D. F., Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials, *Sci. China Math.*, 2015, 58, 781–790
- [15] Zhang Q. Q., Homoclinic orbits for discrete Hamiltonian systems with indefinite linear part, *Comm. Pure Appl. Anal.*, 2017, 14, 1929–1940
- [16] Lin G. H., Zhou Z., Homoclinic solutions of discrete ϕ -Laplacian equations with mixed nonlinearities, *Comm. Pure Appl. Anal.*, 2018, 17, 1723–1747
- [17] Zhang Q. Q., Homoclinic orbits for discrete Hamiltonian systems with local super-quadratic conditions, *Comm. Pure Appl. Anal.*, 2019, 18, 1, 425–434
- [18] Yu J. S., Guo Z. M., On boundary value problems for a discrete generalized Emden-Fowler equation, *J. Differential Equations*, 2006, 231, 18–31
- [19] Zhang Q. Q., Boundary value problems for forth order nonlinear p -Laplacian difference equations, *J. Appl. Math.*, 2014, 2014, Article ID 343129, 6 pages
- [20] Bonanno G., Candito P., D'Agui G., Variational methods on finite dimensional Banach spaces and discrete problems, *Adv. Nonlinear Stud.*, 2014, 14, 4, 915–939
- [21] Zhou Z., Su M. T., Boundary value problems for $2n$ th-order ϕ_c -Laplacian difference equations containing both advance and retardation, *Appl. Math. lett.*, 2015, 41, 7–11
- [22] D'Agui G., Mawhin J., Sciammetta A., Positive solutions for a discrete two point nonlinear boundary value problem with p -Laplacian, *J. Math. Anal. Appl.*, 2017, 447, 1, 383–397
- [23] Balanov Z., Garcia-Azpeitia C., Krawcewicz W., On variational and topological methods in nonlinear difference equations, *Commun. Pure Appl. Anal.*, 2018, 17, 6, 2813–2844
- [24] Rabinowitz P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math., 1986, 65