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θ -type Calderón-Zygmund Operators and Commutators in Variable Exponents Herz space

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Abstract: The aim of this paper is to deal with the boundedness of the θ -type Calderón-Zygmund operators and their commutators on Herz spaces with two variable exponents $p(\cdot)$, $q(\cdot)$. It is proved that the θ -type Calderón-Zygmund operators are bounded on the homogeneous Herz space with variable exponents $\dot{K}_{p(\cdot)}^{\alpha, q(\cdot)}(\mathbb{R}^n)$. Furthermore, the boundedness of the corresponding commutators generated by BMO function and Lipschitz function is also obtained respectively.

Keywords: θ -type Calderón-Zygmund operator; commutator; Herz spaces with variable exponent; BMO space

MSC: 42B20, 42B25, 42B35

1 Introduction and Main Results

The theory of Calderón-Zygmund operators, which has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, has been extensively studied (see [1–6], for instance). In 1985, Yabuta introduced certain θ -type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators (see [7]). Following the terminology of Yabuta, we recall the so-called θ -type Calderón-Zygmund operators. Let θ be a non-negative and non-decreasing function on $\mathbb{R}^+ = (0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} dt < \infty. \quad (1.1)$$

A measurable function $K(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ is said to be a θ -type Calderón-Zygmund kernel if it satisfies

$$|K(x, y)| \leq C|x - y|^{-n}, \quad x \neq y, \quad (1.2)$$

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C\theta\left(\frac{|x - x'|}{|x - y|}\right)|x - y|^{-n}, \quad \text{as } |x - y| \geq 2|x - x'|, \quad x \neq y. \quad (1.3)$$

Definition 1.1^[7] Let T_θ be a linear operator from $\mathcal{S}$ into its dual $\mathcal{S}'$. One can say that T_θ is a θ -type Calderón-Zygmund operator if it satisfies the following conditions:

- (1) T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;

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(2) there is a θ -type Calderón-Zygmund kernel $K(x, y)$ such that

$$T_{\theta}f(x) := \int_{\mathbb{R}^n} K(x, y)f(y)dy, \text{ as } f \in C_c^{\infty}(\mathbb{R}^n) \text{ and } x \notin \text{supp}f. \quad (1.4)$$

It is easy to see that the classical Calderón-Zygmund operator with standard kernel is a special case of θ -type operator T_{θ} as $\theta(t) = t^{\delta}$ with $0 < \delta \leq 1$. Given a locally integrable function b , the commutator generated by T_{θ} and b is defined by

$$[b, T_{\theta}]f(x) = b(x)T_{\theta}f(x) - T_{\theta}(b \cdot f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]K(x, y)f(y)dy. \quad (1.5)$$

Such type of operators are extensively applied in PDE with non-smooth area. Many authors concentrate on the boundedness of these operators on various function spaces, we refer the reader to see [8–14] for its developments and applications. In [11], Quek-Yang established the boundedness of T_{θ} on spaces such as weighted Lebesgue spaces, weighted weak Lebesgue spaces, weighted Hardy spaces and weighted weak Hardy spaces. Ri-Zhang obtained the boundedness of T_{θ} on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces in [12, 13]. Wang proved the boundedness of T_{θ} and $[b, T_{\theta}]$ on the generalized weighted Morrey spaces in [14]. Inspired by the results mentioned previously, a natural and interesting problem is to consider whether or not the θ -type Calderón-Zygmund operators T_{θ} and their commutators $[b, T_{\theta}]$ are bounded on Herz space with variable exponents. The purpose of this paper is to give a positive answer.

The spaces with variable exponent have been widely studied in recent ten years. The results show that they are not only the generalized forms of the classical function spaces with invariable exponent, but also there are some new breakthroughs in the research techniques. These new real variable methods help people further understand the function spaces. Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one class of important function spaces due to the fundamental paper [15] by Kováčik Rákosník. The theory of the function spaces with variable exponent have been applied in fluid dynamics, elasticity dynamics, calculus of variations and differential equations with non-standard growth conditions (for example, see [16–20]). In [21], authors proved the extrapolation theorem which leads the boundedness of some classical operators including the commutators on $L^{p(\cdot)}(\mathbb{R}^n)$. Karlovich and Lerner also obtained the boundedness of the singular integral commutators in [22]. The boundedness of some typical operators is being studied with keen interest on spaces with variable exponent (see [23–26]). Recently, Tao and Yang established the boundedness of θ -Type C-Z operators and their commutators generated respectively by BMO functions, Lipschitz functions and Besov functions on variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see [27]), at the same time, the boundedness of singular integrals with variable kernel and fractional differentiations is also obtained.

It is well known that Herz spaces play an important role in harmonic analysis. After the Herz spaces with one exponent $p(\cdot)$ were introduced in [28], the boundedness of some operators and some characterizations of these spaces were studied widely (see [29–32]). In this paper, we will study the boundedness of the θ -type Calderón-Zygmund operators T_{θ} and their commutators $[b, T_{\theta}]$ on Herz spaces with two variable exponents $p(\cdot), q(\cdot)$. In order to do this, we need to recall some notations and definitions.

Denote $\mathcal{P}(\mathbb{R}^n)$ to be the set of the all measurable functions $p(x)$ with $p_- =: \text{ess inf}_{x \in \mathbb{R}^n} p(x) > 1$ and $p_+ =: \text{ess sup}_{x \in \mathbb{R}^n} p(x) < \infty$ and $\mathcal{B}(\mathbb{R}^n)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, $\mathcal{P}^0(\mathbb{R}^n)$ the set of all measurable functions $p(x)$ with $p_- > 0$ and $p_+ < \infty$.

Given a function $p(x) \in \mathcal{P}(\mathbb{R}^n)$, the space $L^{p(x)}(\mathbb{R}^n)$ is now defined by^[15]

$$\|f\|_{L^{p(x)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \leq 1 \right\}.$$

Given a function $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, the space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : |f|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n) \text{ for some } p_0 \text{ with } 0 < p_0 < p_- \text{ and } q(x) = \frac{p(x)}{p_0} \right\}.$$

It is easy to see that the above Luxemburg-Nakano quasi-norm is also equivalent with^[21]

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \| |f|^{p_0} \|_{L^{q(\cdot)}(\mathbb{R}^n)}^{1/p_0},$$

where $q(x) = \frac{p(x)}{p_0}$ and for some p_0 with $0 < p_0 < p_-$.

Before recalling the Herz spaces with two variable exponents, we will introduce the following function space, which is named as the mixed Lebesgue sequence space (see [33]).

Definition 1.2^[24, 33] Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. The mixed Lebesgue sequence space with variable exponent $l^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{f_j\}_{j=0}^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$\|\{f_j\}_{j=0}^\infty\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \mu > 0 : \Lambda_{l^{q(\cdot)}(L^{p(\cdot)})} \left(\{f_j/\mu\}_{j=0}^\infty \right) \leq 1 \right\} < \infty,$$

where

$$\Lambda_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=0}^\infty) = \sum_{j=0}^\infty \inf \left\{ \mu_j > 0 : \int_{\mathbb{R}^n} \left(|f_j(x)|/\mu_j^{\frac{1}{q(x)}} \right)^{p(x)} dx \leq 1 \right\}.$$

Noting $q_+ < \infty$, we see that

$$\Lambda_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=0}^\infty) = \sum_{j=0}^\infty \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Let $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\chi_k = \chi_{C_k}$.

Definition 1.3^[24] Let $\alpha \in \mathbb{R}$, $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}_{p(\cdot)}^{\alpha, q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}_{p(\cdot)}^{\alpha, q(\cdot)}(\mathbb{R}^n) = \{f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q(\cdot)}(\mathbb{R}^n)} = \|\{2^{k\alpha} |f \chi_k|\}_{k=-\infty}^\infty\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \eta > 0, \sum_{k=-\infty}^\infty \left\| \left(2^{k\alpha} |f \chi_k|/\eta \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

Remark 1.1 (1) It is easy to see that $\dot{K}_{p(\cdot)}^{0, p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ and if $p(x) = p_0$ and $q(x) = q_0$ are constants, then $\dot{K}_{p(\cdot)}^{\alpha, q(\cdot)}(\mathbb{R}^n) = \dot{K}_{p_0}^{\alpha, q_0}(\mathbb{R}^n)$ is just the usual Herz spaces (see [34]).

(2) If $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ satisfying $(q_1)_+ \leq (q_2)_-$, then (see [24])

$$\dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n) \subset \dot{K}_{p(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n).$$

Definition 1.5^[35] When $0 < \beta \leq 1$, the Homogeneous Lipschitz space $\text{Lip}_\beta(\mathbb{R}^n)$ is the space of functions such that

$$\|f\|_{\text{Lip}_\beta} = \sup_{x, h \in \mathbb{R}^n; h \neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

Our main results in this paper are formulated as follows.

Theorem 1.1 Suppose that T_θ is a θ -type Calderón-Zygmund operators with θ satisfies (1.1), $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$ and $p(\cdot)/q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $-n\delta_{12} < \alpha < n\delta_{11}$, where δ_{11}, δ_{12} are the constants in Lemma 2.2, then the operator T_θ is bounded from $\dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)$.

Theorem 1.2 Let $b \in \text{BMO}$, $m \in \mathbb{N}$ and $p(\cdot), q_1(\cdot), q_2(\cdot)$ and α are the same as in Theorem 1.1. Suppose that $[b, T_\theta]$ is defined by (1.5) with θ satisfying

$$\int_0^1 \frac{\theta(t)}{t} |\log t| dt < \infty. \quad (1.6)$$

Then the commutator $[b, T_\theta]$ is bounded from $\dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)$.

Theorem 1.3 Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ ($0 < \beta < 1$) and $[b, T_\theta]$ be defined by (1.5) with θ satisfying (1.6). Suppose that $q_1(\cdot), q_2(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is such that $(p_1)_+ < \frac{n}{\beta}$, $1/p_1(x) - 1/p_2(x) = \beta/n$, $(1 - \beta/n)p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $p_2(\cdot)/q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $-n\delta_{12} < \alpha < n\delta_{11}$, then the commutator $[b, T_\theta]$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_2(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)$.

We make some conventions. In what follows, C always denotes a positive constant which is independent of the main parameters involved but whose value may differ in different occurrences, $|E|$ denotes the Lebesgue measure of $E \in \mathbb{R}^n$. Given a function f , we denote the mean value of f on E by $f_E =: \frac{1}{|E|} \int_E f(x) dx$. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely, $1/p(x) + 1/p'(x) = 1$ holds.

2 Preliminary Lemmas

Before proving the main results, we need the following lemmas.

Lemma 2.1^[15] (Generalized Hölder's Inequality) Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

(1) For any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $C_p = 1 + 1/p^- - 1/p^+$.

(2) For any $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'_2(\cdot)}(\mathbb{R}^n)$, when $1/p(x) = 1/p_1(x) + 1/p_2(x)$, we have

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_{p_1, p_2} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p'_2(\cdot)}},$$

where $C_{p_1, p_2} = (1 + 1/p_{1-} - 1/p_{1+})^{1/p_-}$.

Lemma 2.2^[22] If $p_i \in \mathcal{B}(\mathbb{R}^n)$ ($i = 1, 2$), then there exist constants $p_{i,1}, p_{i,2}, C > 0$, such that for all balls $B \in \mathbb{R}^n$ and all measurable subsets $S \in B$,

$$\frac{\| \chi_B \|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\| \chi_S \|_{L^{p_i(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|}, \quad \frac{\| \chi_S \|_{L^{p'_i(\cdot)}(\mathbb{R}^n)}}{\| \chi_B \|_{L^{p'_i(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|} \right)^{\delta_{i,1}}, \quad \frac{\| \chi_S \|_{L^{p'_i(\cdot)}(\mathbb{R}^n)}}{\| \chi_B \|_{L^{p'_i(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|} \right)^{\delta_{i,2}}.$$

Lemma 2.3^[30] If $p_i(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all balls $B \in \mathbb{R}^n$,

$$\frac{1}{|B|} \| \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| \chi_B \|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.4^[24] Let $p(\cdot), q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)$, then

$$\min(\|f\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)}^{q_+}, \|f\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)}^{q_-}) \leq \| |f|^{q(\cdot)} \|_{L^{p(\cdot)}} \leq \max(\|f\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)}^{q_+}, \|f\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)}^{q_-}).$$

Lemma 2.5^[30] Let $b \in \text{BMO}$ and m be a positive integer. There exists a constant $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$,

$$(1) C^{-1} \|b\|_*^m \leq \sup_B \frac{1}{\| \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_*^m;$$

$$(2) \|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C(k - j)^m \|b\|_*^m \| \chi_{B_k} \|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.6^[36] If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exist constants $0 < \delta < 1, C > 0$, such that for all $Y \in \mathcal{T}$, all nonnegative numbers t_Q and all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $f_Q' = 0$ ($Q \in Y$),

$$\left\| \sum_{Q \in Y} t_Q \left| \frac{f}{f_Q} \right|^\delta \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \sum_{Q \in Y} t_Q \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\mathcal{T}$ denotes all families of disjoint and open cube in $\mathbb{R}^n$.

Lemma 2.7^[24] Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ and m be a positive integer. There exists a constant $C > 0$, such that for any $k, j \in \mathbb{Z}$ with $k > j$

- (1) $C^{-1} \|b\|_{\text{Lip}_\beta}^m \leq \frac{1}{|B|^{m\beta/n}} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_\beta}^m$;
- (2) $\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C |B_k|^{m\beta/n} \|b\|_{\text{Lip}_\beta}^m \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 2.8^[37] Let $f \in \text{Lip}_\beta(\mathbb{R}^n)$, $0 < \beta < 1$, $1 \leq p < \infty$, $B_1 \subset B_2$. We have

- (1) $\|f\|_{\text{Lip}_\beta} \approx \sup_B \frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right)^{1/p}$;
- (2) $|f_{B_1} - f_{B_2}| \leq C \|f\|_{\text{Lip}_\beta} |B_2|^{\beta/n}$.

Lemma 2.9^[24] Let $k \in \mathbb{N}$, $a_k \geq 0$, $1 \leq p_k < \infty$. Then

$$\sum_{k=0}^{\infty} a_k^{p_k} \leq \left(\sum_{k=0}^{\infty} a_k \right)^{p_*}, \quad \text{where } p_* = \begin{cases} \min_{k \in \mathbb{N}} p_k, & \sum_{k=0}^{\infty} a_k \leq 1; \\ \max_{k \in \mathbb{N}} p_k, & \sum_{k=0}^{\infty} a_k > 1. \end{cases}$$

Lemma 2.10^[27] Suppose that T_θ is a θ -type Calderón-Zygmund operator and θ satisfies (1.1). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant C independent of f such that

$$\|T_\theta(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.11^[27] Let $b \in \text{BMO}(\mathbb{R}^n)$. Suppose that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and θ satisfies (1.6), then there exists a constant C independent of f such that

$$\|[b, T_\theta](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_* \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.12^[27] Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$ for $0 < \beta < 1$. Suppose that θ satisfies (1.1) and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $p_+ < \frac{n}{\beta}$. Define $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\beta}{n}.$$

If $q(\cdot)(1 - \frac{\beta}{n}) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant C independent of f such that

$$\|[b, T_\theta](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

3 Proof of Main Theorems

Proof of Theorem 1.1. Let $f \in \dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_j(x).$$

Due to $p(\cdot)/q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, which implies $\inf(p(\cdot)/q_2(\cdot)) \geq 1$, then we have

$$\|T_\theta(f)\|_{\dot{K}_{p(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |T_\theta(f) \chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1 \right\}.$$

Since

$$\left\| \left(\frac{2^{k\alpha} |T_\theta(f) \chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k|}{\eta_{11}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$+ C \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\eta_{12}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} + C \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} T_\theta(f_j) \chi_k|}{\eta_{13}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},$$

where

$$\eta_{11} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p(\cdot)})}, \quad \eta_{12} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p(\cdot)})},$$

$$\eta_{13} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k+3}^{\infty} T_\theta(f_j) \chi_k \right| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p(\cdot)})} \quad \text{and} \quad \eta = \eta_{11} + \eta_{12} + \eta_{13}.$$

Hence,

$$\|T_\theta(f)\|_{\dot{K}_{p(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} \leq C(\eta_{11} + \eta_{12} + \eta_{13}).$$

This implies that, in order to prove our theorem, we only need to show $\eta_{11}, \eta_{12}, \eta_{13} \leq C\|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}$. For simplicity, we denote $\lambda_0 = \|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}$.

First, we consider η_{12} . By Lemma 2.4 and using Aoki-Rolewicz's theorem, we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\lambda_0} \right\|_{L^{p(\cdot)}}^{(q_2^1)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} \left\| \frac{2^{k\alpha} T_\theta(f_j) \chi_k}{\lambda_0} \right\|_{L^{p(\cdot)}} \right)^{(q_2^1)_k}, \end{aligned}$$

where

$$(q_2^1)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

By using Lemma 2.10, it follows

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} \left\| \frac{2^{k\alpha} |f_j|}{\lambda_0} \right\|_{L^{p(\cdot)}} \right)^{(q_2^1)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |f \chi_k|}{\lambda_0} \right\|_{L^{p(\cdot)}}^{(q_2^1)_k}. \end{aligned}$$

Since $f \in \dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$, we can easily see $\left\| \frac{2^{k\alpha} |f \chi_k|}{\lambda_0} \right\|_{L^{p(\cdot)}} \leq 1$ and $\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |f \chi_k|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \leq 1$. Hence, by Lemma 2.4 and Lemma 2.9, we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |f \chi_k|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1^1)_+}} \\ &\leq C \left\{ \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |f \chi_k|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \right\}^{q^*} \\ &\leq C. \end{aligned}$$

Here $(q_1)_+ \leq (q_2)_- \leq (q_2^1)_k$ and $q^* = \min_{k \in \mathbb{Z}} \frac{(q_2^1)_k}{(q_1)_+}$. Consequently, we have $\eta_{12} \leq C\lambda_0$.

Now let us turn to estimate η_{11} .

$$|T_\theta(f_j)(x)| = \left| \int_{\mathbb{R}^n} K(x, y) f_j(y) dy \right| \leq C \int_{\mathbb{R}^n} \frac{1}{|x - y|^n} |f_j(y)| dy.$$

Let $x \in C_k$, $y \in C_j$, $j \leq k - 3$, then we have $|x - y| \geq |x| - |y| \geq \frac{3}{8}|x|$. Hence, we have

$$|T_\theta(f_j)(x)| \leq C|x|^{-n} \|f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, By Lemmas 2.1-2.4 and the fact that $\left\| \left(\frac{2^{ja} |f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}} \leq 1$, it follows that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} |x|^{-n} \|f_j\|_{L^1(\mathbb{R}^n)} \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ &\leq C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{ka} |\sum_{j=-\infty}^{k-3} |x|^{-n} \|f_j\|_{L^1(\mathbb{R}^n)} \chi_k|}{\lambda_0} \right\|_{L^{p(\cdot)}}^{(q_2^2)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{f_j}{\lambda_0} \right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^2)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{f_j}{\lambda_0} \right\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} |B_k| \right)^{(q_2^2)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-3} 2^{(k-j)(\alpha-n\delta_{11})} \left\| \left(\frac{2^{ja} |f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}} \right\}^{(q_2^2)_k}, \end{aligned}$$

where

$$(q_2^2)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

If $(q_1)_+ < 1$, then by Lemma 2.9 and the fact $(q_1)_+ \leq (q_2)_- \leq (q_2^1)_k$, we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} &\leq C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-3} 2^{(k-j)(\alpha-n\delta_{11})} \left\| \left(\frac{2^{ja} |f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{(q_2^2)_k}{(q_1)_+}} \right\} \\ &\leq C \left\{ \sum_{j=-\infty}^{\infty} \left\| \left(\frac{2^{ja} |f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{q^*} \sum_{k=j+3}^{\infty} 2^{(k-j)(\alpha-n\delta_{11})} \right\} \leq C, \end{aligned}$$

where $q^* = \min_{k \in \mathbb{Z}} \frac{(q_2^2)_k}{(q_1)_+}$.

If $(q_1)_+ \geq 1$, then $1 \leq (q_1)_+ \leq (q_2)_- \leq (q_2^1)_k$. Thus, for $\alpha < n\delta_{11}$, by applying Hölder's inequality and Lemma 2.9, we get

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-3} 2^{(k-j)(\alpha-n\delta_{11})(q_1)_+/2} \left\| \left(\frac{2^{ja} |f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{(q_2^2)_k}{(q_1)_+}} \right\}$$

$$\begin{aligned} & \times \left(\sum_{j=-\infty}^{k-3} 2^{(k-j)(\alpha-n\delta_{11})((q_1)_+)' / 2} \right)^{\frac{(q_2^3)_k}{((q_1)_+)'}} \\ & \leq C \left\{ \sum_{j=-\infty}^{\infty} \left\| \left(\frac{|2^{ja} f \chi_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}(\mathbb{R}^n)} \sum_{k=j+3}^{\infty} 2^{(k-j)(\alpha-n\delta_{11})(q_1)_+ / 2} \right\}^{q^*}, \end{aligned}$$

where $q^* = \min_{k \in \mathbb{Z}} \frac{(q_2^3)_k}{(q_1)_+}$. This implies that $\eta_{21} \leq C\lambda_0$.

Finally, we are going to estimate η_{23} . Let $x \in C_k$, $y \in C_j$, $j \geq k+3$, then we have $|x-y| \geq |y|-|x| \geq \frac{3}{8}|y|$. Hence, we have

$$|T_\theta(f_j)(x)| \leq C2^{-jn} \|f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, By Lemmas 2.1-2.4 and the fact that $\left\| \left(\frac{2^{ja} |f \chi_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}} \leq 1$, it follows that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{ka} |\sum_{j=k+3}^{\infty} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} & \leq C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{ka} |\sum_{j=k+3}^{\infty} 2^{-jn} \|f_j\|_{L^1(\mathbb{R}^n)} \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ & \leq C \sum_{k=-\infty}^{\infty} \left(2^{ka} \sum_{j=k+3}^{\infty} 2^{-jn} \left\| \frac{f_j}{\lambda_0} \right\|_{L^p(\mathbb{R}^n)} \| \chi_j \|_{L^{p'(\cdot)}} \| \chi_k \|_{L^{p(\cdot)}} \right)^{(q_2^3)_k} \\ & \leq C \sum_{k=-\infty}^{\infty} \left(2^{ka} \sum_{j=k+3}^{\infty} 2^{-jn} \left\| \frac{f \chi_j}{\lambda_0} \right\|_{L^p(\mathbb{R}^n)} \frac{\| \chi_{B_k} \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\| \chi_{B_j} \|_{L^{p(\cdot)}(\mathbb{R}^n)}} |B_j| \right)^{(q_2^3)_k} \\ & \leq C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=k+3}^{\infty} 2^{(k-j)(\alpha+n\delta_{12})} \left\| \left(\frac{|2^{ja} f \chi_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}} \right\}^{(q_2^3)_k}, \end{aligned}$$

where

$$(q_2^3)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{ka} |\sum_{j=k+3}^{\infty} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{ka} |\sum_{j=k+3}^{\infty} T_\theta(f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Noticing that $(q_2)_- \geq (q_1)_+$ and $\alpha > -n\delta_{12}$, by a similar argument about η_{11} , we have $\eta_{13} \leq C\lambda_0$.

Combing the estimates of η_{11} , η_{12} and η_{13} , we finish the proof of **Theorem 1.1**. $\square$

Proof of Theorem 1.2. Let $b \in \text{BMO}(\mathbb{R}^n)$, $f \in \dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$. As in the proof of Theorem 1.1, we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_j(x).$$

Since

$$\begin{aligned} \left\| \left(\frac{2^{ka} |[b, T_\theta](f) \chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} & \leq C \left\| \left(\frac{2^{ka} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k|}{\eta_{21}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ & + C \left\| \left(\frac{2^{ka} |\sum_{j=k-2}^{k+2} [b, T_\theta](f_j) \chi_k|}{\eta_{22}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ & + C \left\| \left(\frac{2^{ka} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j) \chi_k|}{\eta_{23}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}, \end{aligned}$$

where

$$\eta_{21} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k \right| \right\}_{k=-\infty}^\infty \right\|_{l^{q_2(\cdot)}(L^{p(\cdot)})}, \quad \eta_{22} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k-2}^{k+2} [b, T_\theta](f_j) \chi_k \right| \right\}_{k=-\infty}^\infty \right\|_{l^{q_2(\cdot)}(L^{p(\cdot)})},$$

$$\eta_{23} = \left\| \left\{ 2^{k\alpha} \left| \sum_{j=k+3}^\infty [b, T_\theta](f_j) \chi_k \right| \right\}_{k=-\infty}^\infty \right\|_{l^{q_2(\cdot)}(L^{p(\cdot)})} \quad \text{and} \quad \eta = \eta_{21} + \eta_{22} + \eta_{23}.$$

Hence, it is enough to prove $\eta_{21}, \eta_{22}, \eta_{23} \leq C \|b\|_* \lambda_0$, where $\lambda_0 = \|f\|_{\dot{K}_{p(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}$.

We first estimate η_{22} . Noticing that $[b, T_\theta]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see Lemma 2.11), as the same argument about in η_{12} in the proof of Theorem 1.1, we immediately get

$$\sum_{k=-\infty}^\infty \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k-2}^{k+2} [b, T_\theta](f_j) \chi_k \right|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C.$$

That is to say, $\eta_{22} \leq C \|b\|_* \lambda_0$.

Next, we estimate η_{21} . Let $x \in C_k$, $j \leq k-3$, $\text{supp} f_j \in C_j$. By the estimation of $T_\theta(f_j)$ in the proof of Theorem 1.1, we have

$$|T_\theta(f_j)(x)| \leq C |x|^{-n} \|f_j\|_{L^1(\mathbb{R}^n)}.$$

Then, it follows that

$$|[b, T_\theta]f_j(x)| = |T_\theta[(b(x) - b)f_j](x)| \leq C |x|^{-n} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, using Lemma 2.4 and Aoki-Rolewicz's theorem, we obtain

$$\begin{aligned} & \sum_{k=-\infty}^\infty \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k \right|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ & \leq C \sum_{k=-\infty}^\infty \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} |x|^{-n} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)} \chi_k \right|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\ & \leq C \sum_{k=-\infty}^\infty \left\| \frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} |x|^{-n} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)} \chi_k \right|}{\lambda_0 \|b\|_*} \right\|_{L^{p(\cdot)}}^{(q_2^2)_k} \\ & \leq C \sum_{k=-\infty}^\infty \left(2^{k(a-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|(b - b_j)f_j|}{\lambda_0 \|b\|_*} \right\|_{L^1(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k} \\ & \leq C \sum_{k=-\infty}^\infty \left(2^{k(a-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{1}{\|b\|_*} \|(b - b_j)\chi_{B_j}\|_{L^{p'(\cdot)}} \|\chi_k\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k}, \end{aligned}$$

where $b_j = b_{C_j} = \frac{1}{|C_j|} \int_{C_j} b(y) dy$ and

$$(q_2^2)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k \right|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k \right|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Applying the Hölder's inequality and Lemma 2.5 with the case $m = 1$, we know that

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_{\theta}](f_j) \chi_k|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
& \leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^p(\mathbb{R}_n)} \frac{1}{\|b\|_*} \|(b-b_j)\chi_{B_j}\|_{L^{p'(\cdot)}} \|\chi_k\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k} \\
& \leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^p(\mathbb{R}_n)} \|\chi_j\|_{L^{p'(\cdot)}} (k-j) \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k} \\
& \leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} (k-j) |B_k| \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^p(\mathbb{R}_n)} \right)^{(q_2^2)_k}.
\end{aligned}$$

Furthermore, by the same argument as η_{11} in the proof of Theorem 1.1, we have

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_{\theta}](f_j) \chi_k|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
& \leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} (k-j) 2^{(k-j)(\alpha-n\delta_{11})} \left\| \left(\frac{|f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^p(\mathbb{R}_n)}^{\frac{1}{(q_1)_+}} \right)^{(q_2^2)_k} \\
& \leq C \begin{cases} \left\{ \sum_{j=-\infty}^{\infty} \left\| \left(\frac{|2^{ja} f_{\chi_j}|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}} \sum_{k=j+3}^{\infty} (k-j) 2^{(k-j)(\alpha-n\delta_{11})} \right\}^{q^*}, & (p)_+ \leq 1, \\ \left\{ \sum_{j=-\infty}^{\infty} \left\| \left(\frac{|2^{ja} f_{\chi_j}|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)q_1(\cdot)}} \sum_{k=j+3}^{\infty} (k-j) 2^{(k-j)(\alpha-n\delta_{11})(q_1)_+/2} \right\}^{q^*}, & (p)_+ > 1. \end{cases} \\
& \leq C,
\end{aligned}$$

where $q^* = \min_{k \in \mathbb{Z}} \frac{(q_2^2)_k}{(q_1)_+}$. Therefore, $\eta_{21} \leq C \|b\|_* \lambda_0$.

Finally, we estimate η_{23} . Let $x \in C_k$, $y \in C_j$, $j \geq k+3$, $\text{supp} f_j \in C_j$. By the estimation of the $T_{\theta}(f_j)$ in the proof of Theorem 1.1, we have

$$|T_{\theta}(f_j)(x)| \leq C 2^{-jn} \|f_j\|_{L^1(\mathbb{R}^n)}.$$

From this, it follows

$$|[b, T_{\theta}]f_j(x)| = |T_{\theta}[(b(x) - b)f_j](x)| \leq C 2^{-jn} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)}.$$

Thus, when $\alpha > -n\delta_{12}$, as in the similar way to estimate η_{21} before, we obtain

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_{\theta}](f_j) \chi_k|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
& \leq C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} 2^{-jn} \|(b(\cdot) - b)f_j\|_{L^1(\mathbb{R}^n)} \chi_k|}{\lambda_0 \|b\|_*} \right\|_{L^{p(\cdot)}}^{(q_2^2)_k} \\
& \leq C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=k+3}^{\infty} 2^{jn} \left\| \frac{|(b(\cdot) - b)f_j|}{\lambda_0 \|b\|_*} \right\|_{L^1(\mathbb{R}_n)} \|\chi_k\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=-\infty}^{k-3} 2^{jn} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^p(\mathbb{R}^n)} \frac{1}{\|b\|_*} \|(b-b_k)\chi_{B_j}\|_{L^{p'(\cdot)}} \|\chi_k\|_{L^{p(\cdot)}} \right)^{(q_2^3)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=k+3}^{\infty} 2^{jn(k-j)|B_j|} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^p(\mathbb{R}^n)} \right)^{(q_2^3)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=k+3}^{\infty} (j-k) 2^{(k-j)(\alpha+n\delta_{12})} \left\| \left(\frac{|f\chi_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}} \right\}^{(q_2^3)_k} \leq C,
\end{aligned}$$

where

$$(q_2^3)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j)\chi_k|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta]\chi_k|}{\lambda_0 \|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Hence,

$$\eta_{23} \leq C \|b\|_* \lambda_0.$$

Combining the estimates of η_{21} , η_{22} and η_{23} , we finish the proof of **Theorem 1.2**. $\square$

Proof of Theorem 1.3. Let $b \in \text{Lip}_\beta(\mathbb{R}^n)$, $0 < \beta < 1$, $f \in \dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \triangleq \sum_{j=-\infty}^{\infty} f_j(x).$$

We use similar notations as in the proof of Theorem 1.2. Let

$$\begin{aligned}
\eta_{31} &= \left\| \{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j)\chi_k|\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}, \quad \eta_{32} = \left\| \{2^{k\alpha} |\sum_{j=k-2}^{k+2} [b, T_\theta](f_j)\chi_k|\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}, \\
\eta_{33} &= \left\| \left\{ 2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j)\chi_k| \right\}_{k=-\infty}^{\infty} \right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})} \quad \text{and} \quad \eta = \eta_{31} + \eta_{32} + \eta_{33}.
\end{aligned}$$

Similarly, we have

$$\|[b, T_\theta](f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} \leq C\eta = C(\eta_{31} + \eta_{32} + \eta_{33}).$$

We are now going to estimate η_{31} , η_{32} , η_{33} . For simplicity, we also denote $\lambda_0 = \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}$.

We first estimate η_{32} . Noticing that $[b, T_\theta]$ is bounded $L^{p_1(\cdot)}(\mathbb{R}^n)$ to $L^{p_2(\cdot)}(\mathbb{R}^n)$ (Lemma 2.12), as the same argument about in η_{12} in the proof of Theorem 1.1, we immediately get

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} [b, T_\theta](f_j)\chi_k|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq C.$$

Thus, $\eta_{32} \leq C \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \lambda_0$.

Now let us turn to estimate η_{31} . Let $x \in C_k$, $j \leq k-3$, $\text{supp } f_j \in C_j$. By the estimation of η_{21} in the proof of Theorem 1.2 and the generalized Hölder's inequality, we have

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j)\chi_k|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\begin{aligned}
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|(b-b_j)f_j|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| \chi_k \|_{L^{p_2(\cdot)}(\cdot)} \right)^{(q_2^2)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{f_j}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\| (b-b_j) \chi_{B_k} \|_{L^{p_2(\cdot)}(\cdot)}}{\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{(q_2^2)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{1}{\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \| (b-b_j) \chi_{B_j} \|_{L^{p_1'(\cdot)}(\cdot)} \| \chi_k \|_{L^{p_2(\cdot)}(\cdot)} \right)^{(q_2^2)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| \chi_{B_k} \|_{L^{p_2(\cdot)}(\cdot)} \frac{\| (b-b_j) \chi_{B_j} \|_{L^{p_1'(\cdot)}(\cdot)}}{\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{(q_2^2)_k},
\end{aligned}$$

where

$$(q_2^2)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}(\cdot)} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}(\cdot)} > 1. \end{cases}$$

As $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}$, we use the following fact (see [38])

$$C_1 |B|^{\beta/n} \| \chi_B \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq \| \chi_B \|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \leq C_2 |B|^{\beta/n} \| \chi_B \|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Together with Lemma 2.2-2.4 and Lemma 2.7, we obtain

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}(\cdot)} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{|B_j|^{\beta/n}}{|B_k|^{\beta/n}} \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\cdot)} \| \chi_k \|_{L^{p_1(\cdot)}(\cdot)} \right)^{(q_2^2)_k} \\
&\leq \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\cdot)} \| \chi_k \|_{L^{p_1(\cdot)}(\cdot)} \right)^{(q_2^2)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=-\infty}^{k-3} |B_k| \frac{\| \chi_{B_j} \|_{L^{p_1'(\cdot)}(\cdot)}}{\| \chi_{B_k} \|_{L^{p_1'(\cdot)}(\cdot)}} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^2)_k} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} 2^{(k-j)(\alpha-n\delta_{11})} \left\| \left(\frac{|f_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1^*)_+}} \right)^{(q_2^2)_k}.
\end{aligned}$$

Furthermore, by the same argument as η_{11} in the proof of Theorem 1.1, we get

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=-\infty}^{k-3} [b, T_\theta](f_j) \chi_k|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}(\cdot)} \leq C.$$

This implies that

$$\eta_{31} \leq C \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \lambda_0.$$

Finally, we estimate η_{33} . Let $x \in C_k$, $y \in C_j$, $j \geq k+3$, $\text{supp} f_j \in C_j$. By the estimation of the η_{23} in the proof of Theorem 1.2 and the generalized Hölder's inequality, we have

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j) \chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}(\cdot)}$$

$$\begin{aligned} &\leq C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=k+3}^{\infty} 2^{-jn} \left\| \frac{|(b-b_k)f_j|}{\lambda_0 \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right\|_{L^1(\mathbb{R}^n)} \| \chi_k \|_{L^{p_2(\cdot)}} \right)^{(q_2^3)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=k+3}^{\infty} 2^{-jn} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^1(\mathbb{R}^n)} \frac{\|(b-b_k)\chi_{B_k}\|_{L^{p_2(\cdot)}}}{\|b\|_{\text{Lip}_\beta(\mathbb{R}^n)}} \right)^{(q_2^3)_k}, \end{aligned}$$

where

$$(q_2^3)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j)\chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j)\chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Observe that $1/p_1(x) - 1/p_2(x) = \beta/n$, which implies $1/p_2'(x) - 1/p_1'(x) = \beta/n$. Hence, when $\alpha > -n\delta_{22}$, as argued about η_{31} before, one has

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} |\sum_{j=k+3}^{\infty} [b, T_\theta](f_j)\chi_k|}{\lambda_0} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(2^{k(\alpha-n)} \sum_{j=k+3}^{\infty} 2^{-jn} |B_k| \frac{\|\chi_{B_k}\|_{L^{p_2(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_2(\cdot)}}} \left\| \frac{|f_j|}{\lambda_0} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^3)_k} \\ &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+3}^{\infty} 2^{(k-j)(\alpha+n\delta_{22})} \left\| \left(\frac{|f\chi_j|}{\lambda_0} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1^+)_+}} \right)^{(q_2^3)_k} \leq C. \end{aligned}$$

Therefore,

$$\eta_{33} \leq C \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \lambda_0.$$

Summing up the estimates of η_{31} , η_{32} , and η_{33} , it follows that

$$\|[b, T_\theta](f)\|_{\dot{K}_{p_2(\cdot)}^{\alpha, q_2(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_\beta(\mathbb{R}^n)} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha, q_1(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.3. $\square$

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