Open Mathematics

Research Article

Marat V. Markin*

On the Gevrey ultradifferentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator of orders less than one

<https://doi.org/10.1515/math-2019-0001>

Received May 9, 2018; accepted December 14, 2018

Abstract: It is shown that, if all *weak solutions* of the evolution equation

$$y'(t) = Ay(t), \quad t \geq 0,$$

with a scalar type spectral operator A in a complex Banach space are Gevrey ultradifferentiable of orders less than one, then the operator A is necessarily bounded.

Keywords: Weak solution, scalar type spectral operator, Gevrey classes

MSC: Primary 34G10, 47B40, 30D15; Secondary 47B15, 47D06, 47D60, 30D60

Too much of a good thing can be wonderful.

Mae West

1 Introduction

In [1–3], found are characterizations of the strong differentiability and Gevrey ultradifferentiability of order $\beta \geq 1$, in particular *analyticity* and *entireness*, on $[0, \infty)$ and $(0, \infty)$ of all *weak solutions* of the evolution equation

$$y'(t) = Ay(t), \quad t \geq 0, \tag{1.1}$$

with a scalar type spectral operator A in a complex Banach space.

As is shown by [2, Theorem 4.1] (see also [2, Corollary 4.1]), all weak solutions of equation (1.1) can be *entire* vector functions, i.e., belong to the *first-order* Beurling type Gevrey class $\mathcal{E}^{(1)}([0, \infty), X)$ (see Preliminaries), while the operator A is *unbounded*, e.g., when A is a semibounded below *self-adjoint* operator in a complex Hilbert space (see [4, Corollary 4.1]). This remarkable fact contrasts the situation when, in (1.1), a closed densely defined linear operator A generates a C_0 -semigroup, in which case the strong differentiability of all weak solutions of (1.1) at 0 alone immediately implies *boundedness* for A (cf. [5], see also [6]).

*Corresponding Author: Marat V. Markin: Department of Mathematics, California State University, Fresno 5245 N. Backer Avenue, M/S PB 108 Fresno, CA 93740-8001, USA, E-mail: mmarkin@csufresno.edu

It remains to examine whether all weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space can belong to the Gevrey classes of orders *less than one* (not necessarily to the same one) with A remaining *unbounded*.

In this paper, developing the results of [1–3], we show that an unbounded scalar type spectral operator A in a complex Banach space cannot sustain the strong Gevrey ultradifferentiability of all weak solutions of equation (1.1) for orders less than one, i.e., that imposing on all the weak solutions along with the entireness requirement certain growth at infinity conditions (see Preliminaries) necessarily makes the operator A *bounded*. Thus, we generalize the corresponding results for equation (1.1) with a *normal operator* A in a complex Hilbert space found in [7].

Definition 1.1 (Weak Solution).

Let A be a densely defined closed linear operator in a Banach space $(X, \|\cdot\|)$. A strongly continuous vector function $y : [0, \infty) \rightarrow X$ is called a *weak solution* of equation (1.1) if, for any $g^* \in D(A^*)$,

$$\frac{d}{dt} \langle y(t), g^* \rangle = \langle y(t), A^* g^* \rangle, \quad t \geq 0,$$

where $D(\cdot)$ is the *domain* of an operator, A^* is the operator *adjoint* to A , and $\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual X^* (cf. [8]).

Remarks 1.1.

- Due to the *closedness* of A , the weak solution of (1.1) can be equivalently defined to be a strongly continuous vector function $y : [0, \infty) \rightarrow X$ such that, for all $t \geq 0$,

$$\int_0^t y(s) ds \in D(A) \text{ and } y(t) = y(0) + A \int_0^t y(s) ds$$

and is also called a *mild solution* (cf. [5, Ch. II, Definition 6.3], [9, Preliminaries]).

- Such a notion of *weak solution*, which need not be differentiable in the strong sense, generalizes that of *classical* one, strongly differentiable on $[0, \infty)$ and satisfying the equation in the traditional plug-in sense, the classical solutions being precisely the weak ones strongly differentiable on $[0, \infty)$.
- When a closed densely defined linear operator A in a complex Banach space X generates a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators (see, e.g., [5, 10]), i.e., the associated *abstract Cauchy problem* (ACP)

$$\begin{cases} y'(t) = Ay(t), & t \geq 0, \\ y(0) = f \end{cases} \quad (1.2)$$

is *well-posed* (cf. [5, Ch. II, Definition 6.8]), the weak solutions of equation (1.1) are the orbits

$$y(t) = T(t)f, \quad t \geq 0, \quad (1.3)$$

with $f \in X$ [5, Ch. II, Proposition 6.4] (see also [8, Theorem]), whereas the classical ones are those with $f \in D(A)$ (see, e.g., [5, Ch. II, Proposition 6.3]).

- In our discourse, the associated ACP need not be *well-posed*, i.e., the scalar type spectral operator A need not generate a C_0 -semigroup (cf. [6]).

2 Preliminaries

Here, for the reader's convenience, we outline certain essential preliminaries.

2.1 Scalar Type Spectral Operators

Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ with strongly σ -additive *spectral measure* (the *resolution of the identity*) $E_A(\cdot)$ assigning to each Borel set δ of the complex plane $\mathbb{C}$ a projection operator $E_A(\delta)$ on X and having the operator's *spectrum* $\sigma(A)$ as its *support* [11–13].

Observe that, in a complex finite-dimensional space, the scalar type spectral operators are all linear operators on the space, for which there exists an *eigenbasis* (see, e.g., [11–13]) and, in a complex Hilbert space, the scalar type spectral operators are precisely all those that are similar to the *normal* ones [14].

Associated with a scalar type spectral operator in a complex Banach space is the *Borel operational calculus* analogous to that for a *normal operator* in a complex Hilbert space [12, 13, 15, 16], which assigns to any Borel measurable function $F : \sigma(A) \rightarrow \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda) dE_A(\lambda)$$

defined as follows:

$$F(A)f := \lim_{n \rightarrow \infty} F_n(A)f, \quad f \in D(F(A)), \quad D(F(A)) := \left\{ f \in X \mid \lim_{n \rightarrow \infty} F_n(A)f \text{ exists} \right\},$$

where

$$F_n(\cdot) := F(\cdot) \chi_{\{\lambda \in \sigma(A) \mid |F(\lambda)| \leq n\}}(\cdot), \quad n \in \mathbb{N},$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ is the set of *natural numbers*) and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda) dE_A(\lambda), \quad n \in \mathbb{N},$$

are *bounded* scalar type spectral operators on X defined in the same manner as for a *normal operator* (see, e.g., [15, 16]).

In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+,$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of *nonnegative integers*, $A^0 := I$, I is the *identity operator* on X) and

$$e^{zA} := \int_{\sigma(A)} e^{z\lambda} dE_A(\lambda), \quad z \in \mathbb{C}. \quad (2.4)$$

The properties of the *spectral measure* and *operational calculus*, exhaustively delineated in [12, 13], underlie the entire subsequent discourse. Here, we underline a few facts of particular importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [13, 17], i.e., there is such an $M \geq 1$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \leq M. \quad (2.5)$$

Observe that the notation $\|\cdot\|$ is used here to designate the norm in the space $L(X)$ of all bounded linear operators on X . We adhere to this rather conventional economy of symbols in what follows also adopting the same notation for the norm in the dual space X^* .

For any $f \in X$ and $g^* \in X^*$, the *total variation measure* $\nu(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^* \rangle$ is a *finite* positive Borel measure with

$$\nu(f, g^*, \mathbb{C}) = \nu(f, g^*, \sigma(A)) \leq 4M\|f\|\|g^*\| \quad (2.6)$$

(see, e.g., [18, 19]).

Also (Ibid.), for a Borel measurable function $F : \mathbb{C} \rightarrow \mathbb{C}$, $f \in D(F(A))$, $g^* \in X^*$, and a Borel set $\delta \subseteq \mathbb{C}$,

$$\int_{\delta} |F(\lambda)| dv(f, g^*, \lambda) \leq 4M \|E_A(\delta)F(A)f\| \|g^*\|. \quad (2.7)$$

In particular, for $\delta = \sigma(A)$,

$$\int_{\sigma(A)} |F(\lambda)| dv(f, g^*, \lambda) \leq 4M \|F(A)f\| \|g^*\|. \quad (2.8)$$

Observe that the constant $M \geq 1$ in (2.6)–(2.8) is from (2.5).

Further, for a Borel measurable function $F : \mathbb{C} \rightarrow [0, \infty)$, a Borel set $\delta \subseteq \mathbb{C}$, a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint Borel sets in $\mathbb{C}$, and $f \in X$, $g^* \in X^*$,

$$\int_{\delta} F(\lambda) dv(E_A(\cup_{n=1}^{\infty} \Delta_n)f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) dv(E_A(\Delta_n)f, g^*, \lambda). \quad (2.9)$$

Indeed, since, for any Borel sets $\delta, \sigma \subseteq \mathbb{C}$,

$$E_A(\delta)E_A(\sigma) = E_A(\delta \cap \sigma)$$

[12, 13], for the total variation,

$$v(E_A(\delta)f, g^*, \sigma) = v(f, g^*, \delta \cap \sigma).$$

Whence, due to the nonnegativity of $F(\cdot)$ (see, e.g., [20]),

$$\begin{aligned} \int_{\delta} F(\lambda) dv(E_A(\cup_{n=1}^{\infty} \Delta_n)f, g^*, \lambda) &= \int_{\delta \cap \cup_{n=1}^{\infty} \Delta_n} F(\lambda) dv(f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) dv(f, g^*, \lambda) \\ &= \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) dv(E_A(\Delta_n)f, g^*, \lambda). \end{aligned}$$

The following statement, allowing to characterize the domains of Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures, is fundamental for our discourse.

Proposition 2.1 ([21, Proposition 3.1]).

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $F : \sigma(A) \rightarrow \mathbb{C}$ be a Borel measurable function. Then $f \in D(F(A))$ iff

- (i) for each $g^* \in X^*$, $\int_{\sigma(A)} |F(\lambda)| dv(f, g^*, \lambda) < \infty$ and
- (ii) $\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |F(\lambda)| > n\}} |F(\lambda)| dv(f, g^*, \lambda) \rightarrow 0, n \rightarrow \infty,$

where $v(f, g^*, \cdot)$ is the total variation measure of $\langle E_A(\cdot)f, g^* \rangle$.

The succeeding key theorem provides a description of the weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space.

Theorem 2.1 ([21, Theorem 4.2]).

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. A vector function $y : [0, \infty) \rightarrow X$ is a weak solution of equation (1.1) iff there is an $f \in \bigcap_{t \geq 0} D(e^{tA})$ such that

$$y(t) = e^{tA}f, \quad t \geq 0, \quad (2.10)$$

the operator exponentials understood in the sense of the Borel operational calculus (see (2.4)).

Remarks 2.1.

- Theorem 2.1 generalizing [22, Theorem 3.1], its counterpart for a normal operator A in a complex Hilbert space, in particular, implies
  - that the subspace $\bigcap_{t \geq 0} D(e^{tA})$ of all possible initial values of the weak solutions of equation (1.1) is the largest permissible for the exponential form given by (2.10), which highlights the naturalness of the notion of weak solution, and
  - that associated ACP (1.2), whenever solvable, is solvable *uniquely*.
- Observe that the initial-value subspace $\bigcap_{t \geq 0} D(e^{tA})$ of equation (1.1), containing the dense in X subspace $\bigcup_{\alpha > 0} E_A(\Delta_\alpha)X$, where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \quad \alpha > 0,$$

which coincides with the class $\mathcal{E}^{\{0\}}(A)$ of the *entire* vectors of A of *exponential type* (see below), is *dense* in X .

- When a scalar type spectral operator A in a complex Banach space generates a C_0 -semigroup $\{T(t)\}_{t \geq 0}$,

$$T(t) = e^{tA} \text{ and } D(e^{tA}) = X, \quad t \geq 0,$$

[6], and hence, Theorem 2.1 is consistent with the well-known description of the weak solutions for this setup (see (1.3)).

Subsequently, the frequent terms “*spectral measure*” and “*operational calculus*” are abbreviated to *s.m.* and *o.c.*, respectively.

2.2 Gevrey Classes of Functions**Definition 2.1** (Gevrey Classes of Functions).

Let $(X, \|\cdot\|)$ be a (real or complex) Banach space, $C^\infty(I, X)$ be the space of all X -valued functions strongly infinite differentiable on an interval $I \subseteq (-\infty, \infty)$, and $0 \leq \beta < \infty$.

The following subspaces of $C^\infty(I, X)$

$$\begin{aligned} \mathcal{E}^{\{\beta\}}(I, X) &:= \{g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \exists \alpha > 0 \exists c > 0 : \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c \alpha^n [n!]^\beta, \quad n \in \mathbb{Z}_+\}, \\ \mathcal{E}^{(\beta)}(I, X) &:= \{g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \forall \alpha > 0 \exists c > 0 : \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c \alpha^n [n!]^\beta, \quad n \in \mathbb{Z}_+\}, \end{aligned}$$

are called the β th-order Gevrey classes of strongly ultradifferentiable vector functions on I of Roumieu and Beurling type, respectively (see, e.g., [23–26]).

Remarks 2.2.

- In view of *Stirling’s formula*, the sequence $\{[n!]^\beta\}_{n=0}^\infty$ can be replaced with $\{n^{\beta n}\}_{n=0}^\infty$.
- For $0 \leq \beta < \beta' < \infty$, the inclusions

$$\mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{\{\beta\}}(I, X) \subseteq \mathcal{E}^{(\beta')}(I, X) \subseteq \mathcal{E}^{\{\beta'\}}(I, X) \subseteq C^\infty(I, X)$$

hold.

- For $1 < \beta < \infty$, the Gevrey classes are *non-quasianalytic* (see, e.g., [25]).
- For $\beta = 1$, $\mathcal{E}^{\{1\}}(I, X)$ is the class of all *analytic* on I , i.e., *analytically continuable* into complex neighborhoods of I , vector functions and $\mathcal{E}^{(1)}(I, X)$ is the class of all *entire*, i.e., allowing *entire* continuations, vector functions [27].

- For $0 \leq \beta < 1$, the Gevrey class $\mathcal{E}^{\{\beta\}}(I, X)$ ($\mathcal{E}^{(\beta)}(I, X)$) consists of all functions $g(\cdot) \in \mathcal{E}^{(1)}(I, X)$ such that, for some (any) $\gamma > 0$, there is an $M > 0$ for which

$$\|g(z)\| \leq M e^{\gamma|z|^{1/(1-\beta)}}, \quad z \in \mathbb{C}, \quad (2.11)$$

[7]. In particular, for $\beta = 0$, $\mathcal{E}^{\{0\}}(I, X)$ and $\mathcal{E}^{(0)}(I, X)$ are the classes of entire vector functions of *exponential* and *minimal exponential type*, respectively (see, e.g., [28]).

2.3 Gevrey Classes of Vectors

One can consider the Gevrey classes in a more general sense.

Definition 2.2 (Gevrey Classes of Vectors).

Let $(A, D(A))$ be a densely defined closed linear operator in a (real or complex) Banach space $(X, \|\cdot\|)$, $0 \leq \beta < \infty$, and

$$C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)$$

be the subspace of infinite differentiable vectors of A .

The following subspaces of $C^\infty(A)$

$$\begin{aligned} \mathcal{E}^{\{\beta\}}(A) &:= \left\{ x \in C^\infty(A) \mid \exists \alpha > 0 \exists c > 0 : \|A^n x\| \leq c \alpha^n [n!]^\beta, \quad n \in \mathbb{Z}_+ \right\}, \\ \mathcal{E}^{(\beta)}(A) &:= \left\{ x \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \|A^n x\| \leq c \alpha^n [n!]^\beta, \quad n \in \mathbb{Z}_+ \right\} \end{aligned}$$

are called the β th-order Gevrey classes of ultradifferentiable vectors of A of Roumieu and Beurling type, respectively (see, e.g., [29–31]).

Remarks 2.3.

- In view of Stirling's formula, the sequence $\{[n!]^\beta\}_{n=0}^\infty$ can be replaced with $\{n^{\beta n}\}_{n=0}^\infty$.
- For $0 \leq \beta < \beta' < \infty$, the inclusions

$$\mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{\{\beta\}}(A) \subseteq \mathcal{E}^{(\beta')}(A) \subseteq \mathcal{E}^{\{\beta'\}}(A) \subseteq C^\infty(A) \quad (2.12)$$

hold.

- In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the classes of *analytic* and *entire* vectors of A , respectively [32, 33] and $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ are the classes of *entire* vectors of A of *exponential* and *minimal exponential type*, respectively (see, e.g., [30, 34]).
- In view of the *closedness* of A , it is easily seen that the class $\mathcal{E}^{(1)}(A)$ forms the subspace of initial values in X generating the (classical) solutions of (1.1), which are entire vector functions represented by the power series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f, \quad t \geq 0, f \in \mathcal{E}^{(1)}(A), \quad (2.13)$$

the classes $\mathcal{E}^{\{\beta\}}(A)$ and $\mathcal{E}^{(\beta)}(A)$ with $0 \leq \beta < 1$ being the subspaces of such initial values for which the solutions satisfy growth estimate (2.11) with some (any) $\gamma > 0$ and some $M > 0$, respectively (cf. [28]).

As is shown in [29] (see also [30, 31]), if $0 < \beta < \infty$, for a *normal operator* A in a complex Hilbert space,

$$\mathcal{E}^{\{\beta\}}(A) = \bigcup_{t>0} D(e^{t|A|^{1/\beta}}) \quad \text{and} \quad \mathcal{E}^{(\beta)}(A) = \bigcap_{t>0} D(e^{t|A|^{1/\beta}}), \quad (2.14)$$

the operator exponentials $e^{t|A|^{1/\beta}}$, $t > 0$, understood in the sense of the Borel operational calculus (see, e.g., [15, 16]).

In [19, 35], descriptions (2.14) are extended to *scalar type spectral operators* in a complex Banach space, in which form they are basic for our discourse. In [35], similar nature descriptions of the classes $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ ($\beta = 0$), known for a normal operator A in a complex Hilbert space (see, e.g., [30]), are also generalized to scalar type spectral operators in a complex Banach space. In particular [35, Theorem 5.1],

$$\mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha > 0} E_A(\Delta_\alpha)X,$$

where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \quad \alpha > 0.$$

We also need the following characterization of a particular weak solution's of equation (1.1) with a scalar type spectral operator A in a complex Banach space being strongly Gevrey ultradifferentiable on a subinterval I of $[0, \infty)$ proved in [9].

Proposition 2.2 ([9, Proposition 3.1]).

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$, $0 \leq \beta < \infty$, and I be a subinterval of $[0, \infty)$. Then the restriction of a weak solution $y(\cdot)$ of equation (1.1) to I belongs to the Gevrey class $\mathcal{E}^{\{\beta\}}(I, X)$ ($\mathcal{E}^{(\beta)}(I, X)$) iff, for each $t \in I$,

$$y(t) \in \mathcal{E}^{\{\beta\}}(A) \text{ } (\mathcal{E}^{(\beta)}(A)), \text{ respectively),}$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, t \in I.$$

3 One Lemma

The following lemma generalizes [7, Lemma 4.1], its counterpart for a normal operator in a complex Hilbert space and, besides being an interesting result by itself, is necessary for proving our main statement.

Lemma 3.1. Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $0 < \beta' < \infty$. If

$$\bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{\{\beta\}}(A) = \mathcal{E}^{(\beta')}(A),$$

then the operator A is bounded.

Proof. First, observe that, in view of inclusions (2.12), for any $\beta' > 0$,

$$\bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{\{\beta\}}(A) \subseteq \mathcal{E}^{(\beta')}(A). \quad (3.15)$$

Let us prove the statement by *contrapositive* assuming A to be *unbounded*.

The operator A being scalar type spectral, this assumption implies that the spectrum $\sigma(A)$ of A is an *unbounded* set in the complex plane $\mathbb{C}$ [12, 13]. Hence, the points of the spectrum can be found in *infinitely* many semi-open annuli of the form

$$\delta_n := \{\lambda \in \mathbb{C} \mid n \leq |\lambda| < n+1\}, \quad n \in \mathbb{N},$$

i.e., there is a sequence of natural numbers $\{n(k)\}_{k=1}^\infty$ such that

$$k \leq n(k) < n(k+1), \quad k \in \mathbb{N}, \quad (3.16)$$

and, for each $k \in \mathbb{N}$, there is a

$$\lambda_k \in \delta_{n(k)} \cap \sigma(A) \neq \emptyset.$$

Setting $\varepsilon_0 := 1$, for each $k \in \mathbb{N}$, one can choose an

$$0 < \varepsilon_k < \min(1/k, \varepsilon_{k-1}) \quad (3.17)$$

such that

$$\lambda_k \in \Delta_k := \{\lambda \in \mathbb{C} \mid n(k) - \varepsilon_k < |\lambda| < n(k) + 1 - \varepsilon_k\}. \quad (3.18)$$

Since

$$n(k) < n(k+1) \text{ and } \varepsilon_{k+1} < \varepsilon_k, \quad k \in \mathbb{N},$$

we have:

$$n(k) + 1 - \varepsilon_k < n(k+1) - \varepsilon_{k+1}, \quad k \in \mathbb{N},$$

and hence, the open annuli Δ_k , $k \in \mathbb{N}$, are pairwise disjoint.

Whence, by the properties of the s.m.,

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j,$$

where 0 stands for the zero operator on X .

Observe also, that the subspaces $E_A(\Delta_k)X$, $k \in \mathbb{N}$, are nontrivial since

$$\Delta_k \cap \sigma(A) \neq \emptyset, \quad k \in \mathbb{N},$$

with Δ_k being an open set in $\mathbb{C}$.

By choosing a unit vector $e_k \in E_A(\Delta_k)X$ for each $k \in \mathbb{N}$, we obtain a sequence $\{e_n\}_{n=1}^\infty$ in X such that

$$\|e_k\| = 1, \quad k \in \mathbb{N}, \text{ and } E_A(\Delta_i)e_j = \delta_{ij}e_j, \quad i, j \in \mathbb{N}, \quad (3.19)$$

where δ_{ij} is the Kronecker delta.

As is easily seen, (3.19) implies that the vectors e_k , $k \in \mathbb{N}$, are linearly independent.

Furthermore, there is an $\varepsilon > 0$ such that

$$d_k := \text{dist}(e_k, \text{span}(\{e_i \mid i \in \mathbb{N}, i \neq k\})) \geq \varepsilon, \quad k \in \mathbb{N}. \quad (3.20)$$

Indeed, the opposite implies the existence of a subsequence $\{d_{k(m)}\}_{m=1}^\infty$ such that

$$d_{k(m)} \rightarrow 0, \quad m \rightarrow \infty.$$

Then, by selecting a vector

$$f_{k(m)} \in \text{span}(\{e_i \mid i \in \mathbb{N}, i \neq k(m)\}), \quad m \in \mathbb{N},$$

such that

$$\|e_{k(m)} - f_{k(m)}\| < d_{k(m)} + 1/m, \quad m \in \mathbb{N},$$

we arrive at

$$\begin{aligned} 1 &= \|e_{k(m)}\| \quad \text{since, by (3.19), } E_A(\Delta_{k(m)})f_{k(m)} = 0; \\ &= \|E_A(\Delta_{k(m)})(e_{k(m)} - f_{k(m)})\| \leq \|E_A(\Delta_{k(m)})\| \|e_{k(m)} - f_{k(m)}\| \quad \text{by (2.5);} \\ &\leq M \|e_{k(m)} - f_{k(m)}\| \leq M [d_{k(m)} + 1/m] \rightarrow 0, \quad m \rightarrow \infty, \end{aligned}$$

which is a contradiction proving (3.20).

As follows from the Hahn-Banach Theorem, for any $k \in \mathbb{N}$, there is an $e_k^* \in X^*$ such that

$$\|e_k^*\| = 1, \quad k \in \mathbb{N}, \text{ and } \langle e_i, e_j^* \rangle = \delta_{ij}d_i, \quad i, j \in \mathbb{N}. \quad (3.21)$$

Let

$$f := \sum_{k=1}^{\infty} n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k \in X \text{ and } h := \sum_{k=1}^{\infty} n(k)^{-\frac{1}{2}(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k \in X,$$

the elements being well defined since $\|e_k\| = 1$, $k \in \mathbb{N}$ (see (3.19)) and

$$\left\{ n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} \right\}_{k=1}^{\infty}, \left\{ n(k)^{-\frac{1}{2}(n(k)+1-\varepsilon_k)^{1/\beta'}} \right\}_{k=1}^{\infty} \in l_1$$

(l_1 is the space of absolutely summable sequences).

Indeed, in view of (3.16) and (3.17), for all $k \in \mathbb{N}$ sufficiently large so that

$$n(k) \geq 4^{\beta'},$$

we have:

$$n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} \leq n(k)^{-4} \leq k^{-4}. \quad (3.22)$$

In view of (3.19), by the properties of the s.m.,

$$E_A(\cup_{k=1}^{\infty} \Delta_k) f = f \text{ and } E_A(\Delta_k) f = n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k, \quad k \in \mathbb{N}. \quad (3.23)$$

and

$$E_A(\cup_{k=1}^{\infty} \Delta_k) h = h \text{ and } E_A(\Delta_k) h = n(k)^{-\frac{1}{2}(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k, \quad k \in \mathbb{N}. \quad (3.24)$$

For an arbitrary $t > 0$ and any $g^* \in X^*$,

$$\int_{\sigma(A)} e^{t|\lambda|^{1/\beta'}} dv(f, g^*, \lambda) \quad \text{by (3.23);}$$

$$= \int_{\sigma(A)} e^{t|\lambda|^{1/\beta'}} dv(E_A(\cup_{k=1}^{\infty} \Delta_k) f, g^*, \lambda) \quad \text{by (2.9);}$$

$$= \sum_{k=1}^{\infty} \int_{\sigma(A) \cap \Delta_k} e^{t|\lambda|^{1/\beta'}} dv(E_A(\Delta_k) f, g^*, \lambda) \quad \text{by (3.23);}$$

$$= \sum_{k=1}^{\infty} n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\sigma(A) \cap \Delta_k} e^{t|\lambda|^{1/\beta'}} dv(e_k, g^*, \lambda) \quad \text{since, by (3.18), for } \lambda \in \Delta_k, |\lambda| < n(k) + 1 - \varepsilon_k;$$

$$\leq \sum_{k=1}^{\infty} n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{t(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\sigma(A) \cap \Delta_k} 1 dv(e_k, g^*, \lambda)$$

$$= \sum_{k=1}^{\infty} e^{-\ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{t(n(k)+1-\varepsilon_k)^{1/\beta'}} v(e_k, g^*, \Delta_k)$$

$$\leq \sum_{k=1}^{\infty} e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} v(e_k, g^*, \Delta_k) \quad \text{by (2.6);}$$

$$\leq \sum_{k=1}^{\infty} e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} 4M \|e_k\| \|g^*\|$$

$$= 4M \|g^*\| \sum_{k=1}^{\infty} e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} < \infty. \quad (3.25)$$

Indeed, in view of (3.16) and (3.17), for all $k \in \mathbb{N}$ sufficiently large so that

$$\ln n(k) - t \geq 1,$$

we have:

$$e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} \leq e^{-k^{1/\beta'}}.$$

Similarly, for any $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\}} e^{s|\lambda|^{1/\beta'}} d\nu(f, g^*, \lambda) \\ & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\} \cap \Delta_k} 1 d\nu(e_k, g^*, \lambda) \\ & = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\left(\frac{1}{2} \ln n(k)-t\right)(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{-\frac{1}{2} \ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} \\ & \quad \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\} \cap \Delta_k} 1 d\nu(e_k, g^*, \lambda) \quad \text{since, by (3.16) and (3.17), there is an } L > 0 \text{ such that} \end{aligned}$$

$$e^{-\left(\frac{1}{2} \ln n(k)-t\right)(n(k)+1-\varepsilon_k)^{1/\beta'}} \leq L, \quad k \in \mathbb{N};$$

$$\begin{aligned} & \leq L \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\frac{1}{2} \ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\} \cap \Delta_k} 1 d\nu(e_k, g^*, \lambda) \\ & = L \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} n(k)^{-\frac{1}{2}(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\} \cap \Delta_k} 1 d\nu(e_k, g^*, \lambda) \end{aligned}$$

by (3.24);

$$= L \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\} \cap \Delta_k} 1 d\nu(E_A(\Delta_k)h, g^*, \lambda)$$

by (2.9);

$$= L \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\}} 1 d\nu(E_A(\cup_{k=1}^{\infty} \Delta_k)h, g^*, \lambda)$$

by (3.24);

$$= L \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n\}} 1 d\nu(h, g^*, \lambda)$$

by (2.7);

$$\begin{aligned} & \leq L \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n \right\} \right) h \right\| \|g^*\| \\ & \leq 4LM \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t|\lambda|^{1/\beta'}} > n \right\} \right) h \right\| \quad \text{by the strong continuity of the s.m.;} \\ & \rightarrow 4LM \|E_A(\emptyset)h\| = 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.26)$$

By Proposition 2.1 and (2.14), (3.25) and (3.26) jointly imply that

$$f \in \bigcap_{t>0} D(e^{t|A|^{1/\beta'}}) = \mathcal{E}^{(\beta')}(A). \quad (3.27)$$

Let

$$h^* := \sum_{k=1}^{\infty} n(k)^{-2} e_k^* \in X^*, \quad (3.28)$$

the functional being well defined since, by (3.16), $\{n(k)^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_k^*\| = 1$, $k \in \mathbb{N}$ (see (3.21)).

In view of (3.21) and (3.20), we have:

$$\langle e_k, h^* \rangle = \langle e_k, n(k)^{-2} e_k^* \rangle = d_k n(k)^{-2} \geq \varepsilon n(k)^{-2}, \quad n \in \mathbb{N}. \quad (3.29)$$

Fixing an arbitrary $0 < \beta < \beta'$, for any $t > 0$, we have:

$$\begin{aligned} & \int_{\sigma(A)} e^{t|\lambda|^{1/\beta}} d\nu(f, h^*, \lambda) && \text{by (2.9) as in (3.25);} \\ &= \sum_{k=1}^{\infty} n(k)^{-(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\sigma(A) \cap \Delta_k} e^{t|\lambda|^{1/\beta}} d\nu(e_k, h^*, \lambda) \\ & && \text{since, by (3.18) and (3.17), for } \lambda \in \Delta_k, |\lambda| > n(k) - \varepsilon_k > 0; \\ &\geq \sum_{k=1}^{\infty} e^{-\ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{t(n(k)-\varepsilon_k)^{1/\beta}} \nu(e_k, h^*, \Delta_k) && \text{by (3.17);} \\ &\geq \sum_{k=1}^{\infty} e^{t(k(n)-1)^{1/\beta} - \ln n(k)(k(n)+1)^{1/\beta'}} |\langle E_A(\Delta_k) e_k, h^* \rangle| && \text{by (3.29);} \\ &\geq \sum_{k=1}^{\infty} e^{t(k(n)-1)^{1/\beta} - \ln n(k)(k(n)+1)^{1/\beta'}} \varepsilon n(k)^{-2} \\ &= \sum_{k=1}^{\infty} \varepsilon e^{t(k(n)-1)^{1/\beta} - \ln n(k)(k(n)+1)^{1/\beta'} - 2 \ln n(k)} = \infty. \end{aligned} \quad (3.30)$$

Indeed, for $k \geq 2$,

$$\begin{aligned} t(n(k)-1)^{1/\beta} - \ln n(k)(k(n)+1)^{1/\beta'} - 2 \ln n(k) &= (n(k)-1)^{1/\beta} \left[t - \left[\frac{n(k)+1}{n(k)-1} \right]^{1/\beta'} \frac{\ln n(k)}{(n(k)-1)^{1/\beta-1/\beta'}} - \frac{2 \ln n(k)}{(n(k)-1)^{1/\beta}} \right] \\ &\rightarrow \infty, \quad k \rightarrow \infty, \end{aligned}$$

By Proposition 2.1 and (2.14), (3.30) implies that, for each $0 < \beta < \beta'$,

$$f \notin \bigcup_{t>0} D(e^{t|A|^{1/\beta}}) = \mathcal{E}^{\{\beta\}}(A).$$

Whence, in view of inclusions (2.12), we infer that

$$f \notin \bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{\{\beta\}}(A). \quad (3.31)$$

Comparing (3.27) and (3.31), we conclude that

$$\bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{\{\beta\}}(A) \neq \mathcal{E}^{(\beta')}(A),$$

which completes the proof by contrapositive. $\square$

4 Main Result

Lemma 3.1 affords a rather short proof for the following

Theorem 4.1. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. If every weak solution of equation (1.1) belongs to the β th-order Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, +\infty), X)$ with $0 \leq \beta < 1$ (each one to its own), then the operator A is bounded, and hence, all weak solutions of (1.1) are necessarily entire vector functions of exponential type.*

Proof. Let $y(\cdot)$ be an arbitrary weak solution of equation (1.1). Then, by the premise,

$$y(\cdot) \in \mathcal{E}^{\{\beta\}}([0, \infty), X)$$

with some $0 < \beta < 1$ (β depends on $y(\cdot)$).

This, by Proposition 2.2, implies that

$$y(t) \in \mathcal{E}^{\{\beta\}}(A), \quad t \geq 0.$$

In particular,

$$y(0) \in \mathcal{E}^{\{\beta\}}(A).$$

Since, by Theorem 2.1, the initial values of all weak solutions of equation (1.1) form the subspace

$$\bigcap_{t \geq 0} D(e^{tA}),$$

in view of $D(e^{0A}) = D(I) = X$, we have the inclusion

$$\bigcap_{t > 0} D(e^{tA}) = \bigcap_{t \geq 0} D(e^{tA}) \subseteq \bigcup_{0 \leq \beta < 1} \mathcal{E}^{\{\beta\}}(A). \quad (4.32)$$

Since, for an arbitrary $t > 0$ and any $f \in X, g^* \in X^*$,

$$\int_{\sigma(A)} |e^{t\lambda}| \, dv(f, h^*, \lambda) = \int_{\sigma(A)} e^{t \operatorname{Re} \lambda} \, dv(f, h^*, \lambda) \leq \int_{\sigma(A)} e^{t|\lambda|} \, dv(f, h^*, \lambda)$$

and, considering the inclusion

$$\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \subseteq \{\lambda \in \sigma(A) \mid e^{t|\lambda|} > n\}, \quad t > 0, n \in \mathbb{N},$$

for any $n \in \mathbb{N}$,

$$\begin{aligned} \int_{\{\lambda \in \sigma(A) \mid |e^{t\lambda}| > n\}} |e^{t\lambda}| \, dv(f, h^*, \lambda) &= \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} \, dv(f, h^*, \lambda) \\ &\leq \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|} > n\}} e^{t \operatorname{Re} \lambda} \, dv(f, h^*, \lambda) \leq \int_{\{\lambda \in \sigma(A) \mid e^{t|\lambda|} > n\}} e^{t|\lambda|} \, dv(f, h^*, \lambda), \end{aligned}$$

by Proposition 2.1, we infer that, for each $t > 0$,

$$D(e^{tA}) \supseteq D(e^{t|A|}),$$

which, in view of (2.14), with $\beta = 1$, implies that

$$\bigcap_{t > 0} D(e^{tA}) \supseteq \bigcap_{t > 0} D(e^{t|A|}) = \mathcal{E}^{(1)}(A), \quad (4.33)$$

the equality here being necessary and sufficient for $-A$ to be a generator of an analytic C_0 -semigroup [36] (see also [5, 10]).

Observe that inclusion (4.33) also directly follows from the fact that $\mathcal{E}^{(1)}(A)$ is the subspace of initial values in X generating the (classical) solutions of (1.1), which are entire vector functions represented by power series (2.13) (see Preliminaries).

Inclusions (4.32) and (3.15) for $\beta' = 1$ jointly with (4.33), imply the following closed chain

$$\bigcap_{t>0} D(e^{tA}) \subseteq \bigcup_{0 \leq \beta < 1} \mathcal{E}^{\{\beta\}}(A) \subseteq \mathcal{E}^{(1)}(A) \subseteq \bigcap_{t>0} D(e^{tA}),$$

which proves that

$$\bigcup_{0 \leq \beta < 1} \mathcal{E}^{\{\beta\}}(A) = \mathcal{E}^{(1)}(A).$$

Whence, by Lemma 3.1 with $\beta' = 1$, we infer that the operator A is *bounded*, which completes the proof and implies that each weak solution $y(\cdot)$ of equation (1.1) is an entire vector function of the form

$$y(z) = e^{zA}f = \sum_{n=0}^{\infty} \frac{z^n}{n!} A^n f, \quad z \in \mathbb{C}, \text{ with some } f \in X,$$

and hence, satisfying the growth condition

$$\|y(z)\| \leq \|f\| e^{\|A\||z|}, \quad z \in \mathbb{C},$$

is of *exponential type* (see Preliminaries). □

Theorem 4.1 generalizes [7, Theorem 5.1], its counterpart for a normal operator A in a complex Hilbert space and shows that, while a scalar type spectral operator A in a complex Banach space can be unbounded with all weak solutions of equation (1.1) being *entire* vector functions (see [2, Theorem 4.1] and [2, Corollary 4.1]), it cannot remain unbounded, if each weak solutions $y(\cdot)$ of (1.1), in addition to being entire, is to satisfy the growth condition

$$\|y(z)\| \leq M e^{\gamma|z|^{1/(1-\beta)}}, \quad z \in \mathbb{C},$$

with some $0 \leq \beta < 1$, $\gamma > 0$, and $M > 0$ depending on $y(\cdot)$ (see Preliminaries, cf. (2.11)).

References

- [1] Markin M.V., On the differentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator, *Int. J. Math. Math. Sci.*, 2011, 2011, Article ID 825951, 27 pp.
- [2] Markin M.V., On the Gevrey ultradifferentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator, *Methods Funct. Anal. Topology*, 2018, 24, no. 4, 349–369.
- [3] Markin M.V., On the Gevrey ultradifferentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator on the open semi-axis, *arXiv:1707.09359*.
- [4] Markin M.V., On the strong smoothness of weak solutions of an abstract evolution equation. II. Gevrey ultradifferentiability, *Appl. Anal.*, 2001, 78, no. 1–2, 97–137.
- [5] Engel K.-J., Nagel R., *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.
- [6] Markin M.V., A note on the spectral operators of scalar type and semigroups of bounded linear operators, *Int. J. Math. Math. Sci.*, 2002, 32, no. 10, 635–640.
- [7] Markin M.V., On the strong smoothness of weak solutions of an abstract evolution equation. III. Gevrey ultradifferentiability of orders less than one, *Appl. Anal.*, 2001, 78, no. 1–2, 139–152.
- [8] Ball J.M., Strongly continuous semigroups, weak solutions, and the variation of constants formula, *Proc. Amer. Math. Soc.*, 1977, 63, no. 2, 370–373.
- [9] Markin M.V., On the mean ergodicity of weak solutions of an abstract evolution equation, *Methods Funct. Anal. Topology*, 2018, 24, no. 1, 53–70.
- [10] Hille E., R.S. Phillips, *Functional Analysis and Semi-groups*, American Mathematical Society Colloquium Publications, vol. 31, Amer. Math. Soc., Providence, RI, 1957.

- [11] Dunford N., Spectral operators, *Pacific J. Math.*, 1954, 4, 321–354.
- [12] Dunford N., A survey of the theory of spectral operators, *Bull. Amer. Math. Soc.*, 1958, 64, 217–274.
- [13] Dunford N., Schwartz J.T. with the assistance of Bade W.G. and Bartle R.G., *Linear Operators. Part III: Spectral Operators*, Interscience Publishers, New York, 1971.
- [14] Wermer J., Commuting spectral measures on Hilbert space, *Pacific J. Math.* 1954, 4, no. 3, 355–361.
- [15] Dunford N., Schwartz J.T. with the assistance of Bade W.G. and Bartle R.G., *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*, Interscience Publishers, New York, 1963.
- [16] Plesner A.I., *Spectral Theory of Linear Operators*, Nauka, Moscow, 1965 (Russian).
- [17] Dunford N., Schwartz J.T. with the assistance of Bade W.G. and Bartle R.G., *Linear Operators. Part I: General Theory*, Interscience Publishers, New York, 1958.
- [18] Markin M.V., On scalar type spectral operators, infinite differentiable and Gevrey ultradifferentiable C_0 -semigroups, *Int. J. Math. Math. Sci.*, 2004, 2004, no. 45, 2401–2422.
- [19] Markin M.V., On the Carleman classes of vectors of a scalar type spectral operator, *Ibid.*, 2004, 2004, no. 60, 3219–3235.
- [20] Halmos P.R., *Measure Theory*, Graduate Texts in Mathematics, vol. 18, Springer-Verlag, New York, 1974.
- [21] Markin M.V., On an abstract evolution equation with a spectral operator of scalar type, *Int. J. Math. Math. Sci.*, 2002, 32, no. 9, 555–563.
- [22] Markin M.V., On the strong smoothness of weak solutions of an abstract evolution equation. I. Differentiability, *Appl. Anal.*, 1999, 73, no. 3-4, 573–606.
- [23] Gevrey M., Sur la nature analytique des solutions des équations aux dérivées partielles, *Ann. Éc. Norm. Sup. Paris*, 1918, 35, 129–196.
- [24] Komatsu H., Ultradistributions and hyperfunctions. Hyperfunctions and pseudo-differential equations, *Lecture Notes in Math.*, vol. 287, 164–179, Springer, Berlin, 1973.
- [25] Komatsu H., Ultradistributions. I. Structure theorems and a characterization, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 1973, 20, 25–105.
- [26] Komatsu H., Microlocal analysis in Gevrey classes and in complex domains, *Lecture Notes in Math.*, vol. 1495, 161–236, Springer, Berlin, 1991.
- [27] Mandelbrojt S., *Séries de Fourier et Classes Quasi-Analytiques de Fonctions*, Gauthier-Villars, Paris, 1935.
- [28] Levin B.Ja., *Distribution of Zeros of Entire Functions*, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, RI, 1980.
- [29] Gorbachuk V.I., Spaces of infinitely differentiable vectors of a nonnegative self-adjoint operator, *Ukrainian Math. J.*, 1983, 35, 531–534.
- [30] Gorbachuk V.I., Knyazyuk A.V., Boundary values of solutions of operator-differential equations, *Russ. Math. Surveys*, 1989, 44, 67–111.
- [31] Gorbachuk V.I., Gorbachuk M.L., *Boundary Value Problems for Operator Differential Equations, Mathematics and Its Applications (Soviet Series)*, vol. 48, Kluwer Academic Publishers Group, Dordrecht, 1991.
- [32] Nelson E., Analytic vectors, *Ann. of Math. (2)*, 1959, 70, no. 3, 572–615.
- [33] Goodman R., Analytic and entire vectors for representations of Lie groups, *Trans. Amer. Math. Soc.*, 1969, 143, 55–76.
- [34] Radyno, Ya.V., The space of vectors of exponential type, *Dokl. Akad. Nauk BSSR*, 1983, 27, no. 9, 791–793 (Russian with English summary).
- [35] Markin M.V., On the Carleman ultradifferentiable vectors of a scalar type spectral operator, *Methods Funct. Anal. Topology*, 2015, 21, no. 4, 361–369.
- [36] Markin M.V., A characterization of the generators of analytic C_0 -semigroups in the class of scalar type spectral operators, *Abstr. Appl. Anal.*, 2004, 2004, no. 12, 1007–1018.