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Positive solutions of the discrete Dirichlet problem involving the mean curvature operator

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Abstract: In this paper, by using critical point theory, we obtain some sufficient conditions on the existence of infinitely many positive solutions of the discrete Dirichlet problem involving the mean curvature operator. We show that the suitable oscillating behavior of the nonlinear term near at the origin and at infinity will lead to the existence of a sequence of pairwise distinct nontrivial positive solutions. We also give two examples to illustrate our main results.

Keywords: discrete Dirichlet problem, mean curvature operator, infinitely many positive solutions, critical point theory

MSC: 39A11

1 Introduction

Denote Z and R the sets of integers and real numbers, respectively. For $a, b \in Z$, define $Z(a) = \{a, a+1, \dots\}$, and $Z(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$.

In this paper, we consider the following Dirichlet problem of the second order nonlinear difference equation

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) + q_k \phi_c(u_k) = \lambda f(k, u_k), & k \in Z(1, T), \\ u_0 = u_{T+1} = 0, \end{cases} \quad (1.1)$$

where T is a given positive integer, Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$, $\Delta^2 u_k = \Delta(\Delta u_k)$, $q_k \geq 0$ for all $k \in Z(1, T)$, ϕ_c is the mean curvature operator defined by $\phi_c(s) = \frac{s}{\sqrt{1+s^2}}$ [1], λ is a real positive parameter, and $f(k, \cdot) \in C(R, R)$ for each $k \in Z(1, T)$.

When $q_k \equiv 0$, problem (1.1) may be regarded as the discrete analog of the following one-dimensional prescribed curvature problem

$$\begin{cases} -(\phi_c(u'))' = \lambda f(t, u), & t \in R, \\ u(0) = u(1) = 0. \end{cases} \quad (1.2)$$

In 2007, based on the variational methods and a regularization of the action functional associated with the curvature problem, Bonheure etc. in [2] obtained the existence and multiplicity of positive solutions of (1.2) according to the behaviour near at the origin and at infinity of the potential $\int_0^u f(t, s)ds$. In [3], Bonanno, Livrea and Mawhin obtained an explicit interval Λ of positive parameters, such that, for every $\lambda \in \Lambda$, problem

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(1.2) admits at least one non trivial nonnegative solution u_λ . For the corresponding case of higher dimensions to problem (1.2), we refer to [4].

Difference equations arise in various research fields. Many authors have discussed the existence and multiplicity of solutions for difference problems by using fixed point theory, the method of upper and lower solution techniques, Rabinowitz's global bifurcation theorem etc., see [5–7]. Since 2003, critical point theory has been employed to study difference equations [8], by which various results have been obtained. See, for example, [9–18]. In recent years, variational methods have been used to study the boundary value problems of difference equations [19–23]. For example, in [21], the authors considered the discrete Dirichlet problem

$$\begin{cases} -\Delta^2 u_{k-1} = \lambda f(k, u_k), & k \in Z(1, T), \\ u_0 = u_{T+1} = 0. \end{cases} \quad (1.3)$$

By using critical point theory, the authors obtained the existence of at least two positive solutions for (1.3). In [22], the authors extended the results of [21] to the following discrete boundary value problem with p -Laplacian

$$\begin{cases} -\Delta (\phi_p(\Delta u_{k-1})) + q_k \phi_p(u_k) = \lambda f(k, u_k), & k \in Z(1, T), \\ u_0 = u_{T+1} = 0. \end{cases} \quad (1.4)$$

While the existence results of infinitely many solutions of (1.4) were also established in [20]. Very recently, the authors in [23] considered the existence of positive solutions of (1.1) for the special case $q_k \equiv 0$ according to the behavior of f at infinity.

Compared with differential equations, there is less work on the boundary value problems of difference equations involving the mean curvature operator. In this paper, we will consider the existence of infinitely many positive solutions of (1.1) by means of a critical point result in [24], see also [25]. The results show that the suitable oscillating behavior of the nonlinear term f near at the origin and at infinity will lead to the existence of a sequence of pairwise distinct nontrivial positive solutions for problem (1.1). We refer the reader to monographs [26, 27] for the general background on difference equations.

This paper is organized as follows. In section 2, the variational framework associated with (1.1) is established, and the abstract critical point theorem is recalled. In section 3, our main results are presented. In particular, we establish a strong maximum principle and obtain the existence of infinitely many positive solutions for (1.1) according to the oscillating behavior of f near at the origin and at infinity, respectively. Finally, in section 4, we present two examples to illustrate our main results.

2 Preliminaries

In this section, we first establish the variational framework associated with (1.1). We consider the T -dimensional Banach space $S = \{u : Z(0, T+1) \rightarrow \mathbb{R} : u_0 = u_{T+1} = 0\}$ endowed with the norm

$$\|u\| := \left(\sum_{k=0}^T (\Delta u_k)^2 \right)^{\frac{1}{2}}.$$

Define

$$\Phi(u) = \sum_{k=0}^T \left(\sqrt{1 + (\Delta u_k)^2} - 1 \right) + \sum_{k=1}^T q_k \left(\sqrt{1 + u_k^2} - 1 \right), \quad \Psi(u) = \sum_{k=1}^T F(k, u_k), \quad (2.1)$$

for every $u \in S$, where $F(k, u) = \int_0^u f(k, \tau) d\tau$ is the primary function of $f(k, u)$ with $F(k, 0) = 0$ for each $k \in Z(1, T)$. Let

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u),$$

for any $u \in S$. It is clear that Φ and Ψ are two functionals of class $C^1(S, \mathbb{R})$ whose Gâteaux derivatives at the point $u \in S$ are given by

$$\Phi'(u)(v) = \sum_{k=0}^T \phi_c(\Delta u_k) \Delta v_k + \sum_{k=1}^T q_k \phi_c(u_k) v_k,$$

and

$$\Psi'(u)(v) = \sum_{k=1}^T f(k, u_k) v_k,$$

for all $u, v \in S$. Since

$$\begin{aligned} \sum_{k=0}^T \phi_c(\Delta u_k) \Delta v_k &= \sum_{k=0}^T \phi_c(\Delta u_k) v_{k+1} - \sum_{k=0}^T \phi_c(\Delta u_k) v_k \\ &= \sum_{k=1}^T \phi_c(\Delta u_{k-1}) v_k - \sum_{k=1}^T \phi_c(\Delta u_k) v_k \\ &= - \sum_{k=1}^T \Delta(\phi_c(\Delta u_{k-1})) v_k, \end{aligned} \quad (2.2)$$

then,

$$[\Phi'(u) - \lambda \Psi'(u)](v) = \sum_{k=1}^T [-\Delta(\phi_c(\Delta u_{k-1})) + q_k \phi_c(u_k) - \lambda f(k, u_k)] v_k.$$

Consequently, the critical points of I_λ in S are exactly the solutions of the problem (1.1).

Assume that X is a reflexive real Banach space and $I_\lambda : X \rightarrow \mathbb{R}$ is a function which satisfies the following structure hypothesis:

(Λ) Assuming that $\Phi, \Psi : X \rightarrow \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\| \rightarrow +\infty} \Phi(u) = +\infty$.

Let $I_\lambda(u) := \Phi(u) - \lambda \Psi(u)$ for each $u \in X$, where λ is a real positive parameter.

If $\inf_X \Phi < r$, let

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) \right) - \Psi(u)}{r - \Phi(u)},$$

and

$$\delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \varphi(r), \quad \gamma := \liminf_{r \rightarrow +\infty} \varphi(r).$$

Clearly, $\delta \geq 0$ and $\gamma \geq 0$. When $\delta = 0$ (or $\gamma = 0$), in the sequel, we agree to read $\frac{1}{\delta}$ (or $\frac{1}{\gamma}$) as $+\infty$.

The following lemma comes from Theorem 7.4 of [24] and will be used to investigate problem (1.1).

Lemma 2.1. Assume that the condition (Λ) holds. We have

(a) If $\delta < +\infty$, then, for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either

(a₁) there is a global minimum of Φ which is a local minimum of I_λ , or

(a₂) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ , with $\lim_{n \rightarrow +\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ .

(b) If $\gamma < +\infty$, then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either

(b₁) I_λ possesses a global minimum, or

(b₂) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty$.

3 Main results

Put

$$B := \limsup_{t \rightarrow 0^+} \frac{\sum_{k=1}^T F(k, t)}{t^2}, \quad D := \limsup_{t \rightarrow +\infty} \frac{\sum_{k=1}^T F(k, t)}{t}, \quad q_* := \min\{q_k : k \in Z(1, T)\}, \quad Q := \sum_{k=1}^T q_k.$$

First, if considering the oscillating behavior of f near at the origin, we have

Theorem 3.1. Assume that there exist two real sequences $\{a_n\}$ and $\{b_n\}$, with $b_n > 0$ and $\lim_{n \rightarrow +\infty} b_n = 0$, such that

$$(2 + Q) \left(\sqrt{1 + a_n^2} - 1 \right) < (1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right), \quad (3.1)$$

for $n \in Z(1)$, and

$$A := \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^T \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^T F(k, a_n)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right) - (2 + Q) \left(\sqrt{1 + a_n^2} - 1 \right)} < \frac{2B}{2 + Q}. \quad (3.2)$$

Then, for each $\lambda \in (\frac{2+Q}{2B}, \frac{1}{A})$, problem (1.1) admits a sequence of nontrivial solutions which converges to zero.

Proof. To prove Theorem 3.1, we will need to use Lemma 2.1. Firstly, (A) is clearly satisfied. Put

$$r_n = (1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right).$$

Fix $u \in S$ such that $\Phi(u) = \Phi_1(u) + \Phi_2(u) < r_n$, where

$$\Phi_1(u) = \sum_{k=0}^T \left(\sqrt{1 + (\Delta u_k)^2} - 1 \right), \quad \Phi_2(u) = \sum_{k=1}^T q_k \left(\sqrt{1 + u_k^2} - 1 \right).$$

Put

$$v_k = \sqrt{1 + (\Delta u_k)^2} - 1$$

for each $k \in Z(0, T)$. Then $\sum_{k=0}^T v_k = \Phi_1(u)$ and

$$\sum_{k=0}^T (\Delta u_k)^2 = \sum_{k=0}^T (v_k^2 + 2v_k) \leq \left(\sum_{k=0}^T v_k \right)^2 + 2 \sum_{k=0}^T v_k = \Phi_1^2(u) + 2\Phi_1(u),$$

which implies that

$$\Phi_1(u) \geq \sqrt{\|u\|^2 + 1} - 1.$$

Noticing that

$$\|u\|_\infty \leq \frac{\sqrt{T+1}}{2} \|u\|,$$

by Lemma 2.2 in [28], where

$$\|u\|_\infty := \max\{|u_k| : k \in Z(1, T)\},$$

for $u \in S$. Thus, we have

$$\Phi_1(u) \geq \sqrt{\frac{4}{T+1} \|u\|_\infty^2 + 1} - 1.$$

It is clear that

$$\Phi_2(u) \geq q_* \left(\sqrt{\|u\|_\infty^2 + 1} - 1 \right).$$

Therefore

$$(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} \|u\|_\infty^2} - 1 \right) \leq \Phi_1(u) + \Phi_2(u) < r_n,$$

which implies that

$$\|u\|_\infty^2 < \max \left\{ \frac{T+1}{4}, 1 \right\} \left[\left(\frac{r_n}{1 + q_*} \right)^2 + \frac{2r_n}{1 + q_*} \right] = b_n^2.$$

By the definition of φ , we have

$$\varphi(r_n) \leq \inf_{u \in \Phi^{-1}(-\infty, r_n)} \frac{\sum_{k=1}^T \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^T F(k, u_k)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right) - \Phi(u)}.$$

For each $n \in Z(1)$, let $w_n \in S$ given by $(w_n)_k = a_n$ for each $k \in Z(1, T)$, and $(w_n)_0 = (w_n)_{T+1} = 0$. Then, by using (3.1),

$$\Phi(w_n) = (2 + Q)(\sqrt{1 + a_n^2} - 1) < r_n.$$

Thus,

$$\begin{aligned} \varphi(r_n) &\leq \frac{\sum_{k=1}^T \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^T F(k, (w_n)_k)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right) - \Phi(w_n)} \\ &= \frac{\sum_{k=1}^T \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^T F(k, a_n)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right) - (2 + Q)(\sqrt{1 + a_n^2} - 1)}. \end{aligned} \quad (3.3)$$

Therefore, by (3.2), we know that $\gamma \leq \liminf_{n \rightarrow +\infty} \varphi(r_n) \leq A < +\infty$.

Clearly, $u \equiv 0$ is a global minimum of Φ . In order to get the conclusion (a_2) , we need to prove that $u \equiv 0$ is not a local minimum of I_λ . To prove this, we consider two cases: $B = +\infty$ and $B < +\infty$. If $B = +\infty$, let $\{c_n\}$ be a sequence of positive numbers, with $\lim_{n \rightarrow +\infty} c_n = 0$, such that

$$\sum_{k=1}^T F(k, c_n) \geq \frac{(2 + Q)c_n^2}{\lambda}, \text{ for } n \in Z(1).$$

Defining a sequence $\{\omega_n\}$ in S by $(\omega_n)_k = c_n$ for each $k \in Z(1, T)$ and $(\omega_n)_0 = (\omega_n)_{T+1} = 0$, we have

$$I_\lambda(\omega_n) = (2 + Q) \left(\sqrt{1 + c_n^2} - 1 \right) - \lambda \sum_{k=1}^T F(k, c_n) \leq \frac{2 + Q}{2} c_n^2 - (2 + Q)c_n^2 = -\frac{2 + Q}{2} c_n^2 < 0.$$

If $B < +\infty$, since $\lambda > \frac{2+Q}{2B}$, we can choose $\epsilon_0 > 0$ such that

$$2 + Q - 2\lambda(B - \epsilon_0) < 0.$$

Then we can find a sequence of positive numbers $\{c_n\}$ satisfying $\lim_{n \rightarrow +\infty} c_n = 0$ and

$$(B - \epsilon_0)c_n^2 \leq \sum_{k=1}^T F(k, c_n) \leq (B + \epsilon_0)c_n^2.$$

Let the sequence $\{\omega_n\}$ in S be the same as the case where $B = +\infty$, we get

$$I_\lambda(\omega_n) = (2 + Q) \left(\sqrt{1 + c_n^2} - 1 \right) - \lambda \sum_{k=1}^T F(k, c_n) \leq \frac{2 + Q - 2\lambda(B - \epsilon_0)}{2} c_n^2 < 0.$$

Since $I_\lambda(0) = 0$, by combining the above two cases, we see that $u \equiv 0$ is not a local minimum of I_λ and by Lemma 2.1, we conclude Theorem 3.1 holds.

Now, let

$$A_* = \liminf_{t \rightarrow 0^+} \frac{2 \max \left\{ \frac{T+1}{4}, 1 \right\} \sum_{k=1}^T \max_{|s| \leq t} F(k, s)}{(1 + q_*)t^2}.$$

Then there exists a sequence $\{b_n\}$ of positive numbers with $\lim_{n \rightarrow +\infty} b_n = 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^T \max_{|t| \leq b_n} F(k, t)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} b_n^2} - 1 \right)} = A_*.$$

Taking $a_n = 0$ for all $n \in Z(1)$, then by Theorem 3.1, we have the following corollary.

Corollary 3.1. Assume that

$$A_* < \frac{2B}{2 + Q}, \quad (3.4)$$

then, for each $\lambda \in (\frac{2+Q}{2B}, \frac{1}{A_*})$, problem (1.1) admits a sequence of nontrivial solutions which converges to zero.

Now, considering the oscillating behavior of f at infinity, we have

Theorem 3.2. Assume that there exist two real sequences $\{c_n\}$, $\{d_n\}$, and $\lim_{n \rightarrow +\infty} d_n = +\infty$, such that

$$(2 + Q) \left(\sqrt{1 + c_n^2} - 1 \right) < (1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} d_n^2} - 1 \right), \quad (3.5)$$

for $n \in Z(1)$, and

$$C := \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^T \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^T F(k, c_n)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} d_n^2} - 1 \right) - (2 + Q) \left(\sqrt{1 + c_n^2} - 1 \right)} < \frac{D}{2 + Q}. \quad (3.6)$$

Then, for each $\lambda \in (\frac{2+Q}{D}, \frac{1}{C})$, problem (1.1) admits an unbounded sequence of solutions.

The proof of Theorem 3.2 is similar to that of Theorem 3.1, we omit it.

Let

$$C_* = \liminf_{t \rightarrow +\infty} \frac{\sqrt{\max \left\{ \frac{T+1}{4}, 1 \right\}} \sum_{k=1}^T \max_{|s| \leq t} F(k, s)}{(1 + q_*)t}.$$

Then there exists a sequence $\{d_n\}$ of positive numbers with $\lim_{n \rightarrow +\infty} d_n = +\infty$ such that

$$\liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^T \max_{|t| \leq d_n} F(k, t)}{(1 + q_*) \left(\sqrt{1 + \min \left\{ \frac{4}{T+1}, 1 \right\} d_n^2} - 1 \right)} = C_*.$$

Taking $c_n = 0$ for each $n \in Z(1)$, by Theorem 3.2, we have the following corollary.

Corollary 3.2. Assume that

$$C_* < \frac{D}{2 + Q}, \quad (3.7)$$

then, for each $\lambda \in (\frac{2+Q}{D}, \frac{1}{C_*})$, problem (1.1) admits an unbounded sequence of solutions.

In order to obtain the positive solutions of (1.1), we need to establish the following strong maximum principle.

Theorem 3.3. Let $u \in S$ be such that either

$$u_k > 0 \quad \text{or} \quad -\Delta(\phi_c(\Delta u_{k-1})) + q_k \phi_c(u_k) \geq 0, \quad (3.8)$$

for all $k \in Z(1, T)$. Then, either $u_k > 0$ for all $k \in Z(1, T)$ or $u \equiv 0$.

Proof. Assume that

$$u_m = \min\{u_k : k \in N(1, T)\},$$

for some $m \in Z(1, T)$.

If $u_m > 0$, we see that $u_k > 0$ for all $k \in Z(1, T)$ and we complete the proof.

If $u_m \leq 0$, then $u_m = \min\{u_k : k \in N(0, T+1)\}$. Since $\Delta u_{m-1} = u_m - u_{m-1} \leq 0$ and $\Delta u_m = u_{m+1} - u_m \geq 0$, and $\phi_c(s)$ is increasing in s with $\phi_c(0) = 0$. We have

$$\phi_c(\Delta u_m) \geq 0 \geq \phi_c(\Delta u_{m-1}). \quad (3.9)$$

On the other hand, by (3.8), we see that $-\Delta(\phi_c(\Delta u_{m-1})) \geq -q_m \phi_c(u_m) \geq 0$, which implies

$$\phi_c(\Delta u_m) \leq \phi_c(\Delta u_{m-1}). \quad (3.10)$$

By combining (3.9) with (3.10), we get $\phi_c(\Delta u_m) = 0 = \phi_c(\Delta u_{m-1})$. That is $u_{m+1} = u_{m-1} = u_m$. If $m+1 = T+1$, we get $u_m = 0$. Otherwise, $m+1 \in N(1, T)$. In this case, we replace m by $m+1$ and have $u_{m+2} = u_{m+1}$. We get $u_m = u_{m+1} = u_{m+2} = \dots = u_{T+1} = 0$ by continuing this process $T+1-m$ times. Similarly, we have $u_m = u_{m-1} = u_{m-2} = \dots = u_0 = 0$. Thus $u \equiv 0$ and we complete the proof.

Now, considering the existence of positive solutions of problem (1.1), we have

Corollary 3.3. Assume that $f(k, 0) \geq 0$ for all $k \in Z(1, T)$, and

$$\bar{A} := \liminf_{t \rightarrow 0} \frac{2 \max\{\frac{T+1}{4}, 1\} \sum_{k=1}^T \max_{0 \leq s \leq t} \int_0^s f(k, \tau) d\tau}{(1+q_*)t^2} < \frac{2B}{2+Q}, \quad (3.11)$$

then, for each $\lambda \in (\frac{2+Q}{2B}, \frac{1}{\bar{A}})$, problem (1.1) admits a sequence of positive solutions which converges to zero.

Proof. Let

$$f^*(k, t) = \begin{cases} f(k, t), & \text{if } t > 0, \\ f(k, 0), & \text{if } t \leq 0. \end{cases} \quad (3.12)$$

Since $f(k, 0) \geq 0$, we see that

$$\max_{0 \leq |s| \leq t} \int_0^s f^*(k, \tau) d\tau = \max_{0 \leq s \leq t} \int_0^s f(k, \tau) d\tau,$$

for all $t \geq 0$. According to Corollary 3.1, we see that problem (1.1) with f replaced by f^* admits a sequence of nontrivial solutions which converges to zero for each $\lambda \in (\frac{2+Q}{2B}, \frac{1}{\bar{A}})$. And by Theorem 3.3, all these solutions are positive.

Arguing as in Corollary 3.3, we have

Corollary 3.4. Assume that $f(k, 0) \geq 0$ for all $k \in Z(1, T)$, and

$$\bar{C} := \liminf_{t \rightarrow +\infty} \frac{\sqrt{\max\{\frac{T+1}{4}, 1\}} \sum_{k=1}^T \max_{0 \leq s \leq t} \int_0^s f(k, \tau) d\tau}{(1+q_*)t} < \frac{D}{2+Q}, \quad (3.13)$$

then, for each $\lambda \in (\frac{2+Q}{D}, \frac{1}{\bar{C}})$, problem (1.1) admits an unbounded sequence of positive solutions.

4 Examples

In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the boundary value problem (1.1) with

$$f(k, u) = f(u) = \begin{cases} u(2 + 2\varepsilon + 2\cos(\varepsilon \ln |u|) - \varepsilon \sin(\varepsilon \ln |u|)), & u \neq 0, \\ 0, & u = 0, \end{cases} \quad (4.1)$$

for each $k \in Z(1, T)$. Then,

$$F(k, u) = F(u) = \int_0^u f(\tau) d\tau = u^2(1 + \varepsilon + \cos(\varepsilon \ln u)), \text{ for } u > 0.$$

Since $f(u) \geq 0$ for $u \geq 0$, we see that $F(u)$ is increasing in $u \in [0, +\infty)$. Let

$$\alpha_n = \exp\left(-\frac{2n\pi}{\varepsilon}\right), \quad \beta_n = \exp\left(-\frac{2n\pi + \pi}{\varepsilon}\right).$$

Then $\lim_{n \rightarrow +\infty} \alpha_n = 0 = \lim_{n \rightarrow +\infty} \beta_n$, and

$$\frac{F(\alpha_n)}{\alpha_n^2} = 2 + \varepsilon, \quad \frac{\max_{0 \leq s \leq \beta_n} F(s)}{\beta_n^2} = \frac{F(\beta_n)}{\beta_n^2} = \varepsilon,$$

which implies that

$$\bar{A} \leq \frac{2T \max\left\{\frac{T+1}{4}, 1\right\}}{1 + q_*} \varepsilon, \quad B \geq 2 + \varepsilon.$$

Let ε be small enough, such that

$$\frac{2T \max\left\{\frac{T+1}{4}, 1\right\}}{1 + q_*} \varepsilon < \frac{2(2 + \varepsilon)}{2 + Q}.$$

Then (3.11) holds. By Corollary 3.3, for each $\lambda \in \left(\frac{2+Q}{2(2+\varepsilon)}, \frac{(1+q_*) \min\left\{\frac{4}{T+1}, 1\right\}}{2T\varepsilon}\right)$, problem (1.1) admits a sequence of positive solutions which converges to zero.

Example 4.2 Consider the boundary value problem (1.1) with

$$f(k, u) = f(u) = 1 + \varepsilon + \cos(\varepsilon \ln(|u| + 1)) - \varepsilon \sin(\varepsilon \ln(|u| + 1)), \quad (4.2)$$

for each $k \in Z(1, T)$. Then,

$$F(k, u) = F(u) = \int_0^u f(\tau) d\tau = (1 + u)(1 + \varepsilon + \cos(\varepsilon \ln(u + 1))) - 2 - \varepsilon,$$

for $u \geq 0$. Since $f(u) \geq 0$ for $u \geq 0$, we see that $F(u)$ is increasing in $u \in [0, +\infty)$. Let

$$\gamma_n = \exp\left(\frac{2n\pi}{\varepsilon}\right) - 1, \quad \eta_n = \exp\left(\frac{2n\pi + \pi}{\varepsilon}\right) - 1.$$

Then $\lim_{n \rightarrow +\infty} \gamma_n = +\infty = \lim_{n \rightarrow +\infty} \eta_n$, and

$$\lim_{n \rightarrow +\infty} \frac{F(\gamma_n)}{\gamma_n} = 2 + \varepsilon, \quad \lim_{n \rightarrow +\infty} \frac{\max_{0 \leq s \leq \eta_n} F(s)}{\eta_n} = \lim_{n \rightarrow +\infty} \frac{F(\eta_n)}{\eta_n} = \varepsilon,$$

which implies that

$$\bar{C} \leq \frac{T \sqrt{\max\left\{\frac{T+1}{4}, 1\right\}}}{1 + q_*} \varepsilon, \quad D \geq 2 + \varepsilon.$$

Let ε be small enough, such that

$$\frac{T \sqrt{\max\left\{\frac{T+1}{4}, 1\right\}}}{1 + q_*} \varepsilon < \frac{2 + \varepsilon}{2 + Q}.$$

Then (3.13) holds. By Corollary 3.4, for each $\lambda \in \left(\frac{2+Q}{2+\varepsilon}, \frac{(1+q_*) \sqrt{\min\left\{\frac{4}{T+1}, 1\right\}}}{T\varepsilon}\right)$, problem (1.1) admits an unbounded sequence of positive solutions.

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