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L-fuzzy ideals and *L*-fuzzy subalgebras of Novikov algebras

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Abstract: In this paper, we apply the concept of fuzzy sets to Novikov algebras, and introduce the concepts of *L*-fuzzy ideals and *L*-fuzzy subalgebras. We get a sufficient and necessary condition such that an *L*-fuzzy subspace is an *L*-fuzzy ideal. Moreover, we show that the quotient algebra A/μ of the *L*-fuzzy ideal μ is isomorphic to the algebra A/A_μ of the non-fuzzy ideal A_μ . Finally, we discuss the algebraic properties of surjective homomorphic image and preimage of an *L*-fuzzy ideal.

Keywords: fuzzy set, fuzzy algebra, fuzzy subalgebra, fuzzy ideal

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1 Introduction

Since Rosenfeld [1] introduced fuzzy sets in the realm of the group theory, many researchers are engaged in extending the concepts and results of abstract algebra to the boarder framework of the fuzzy set. Liu [2] defined the concepts of fuzzy rings and fuzzy ideals in a ring. Katsaras and Liu [3] introduced the concept of a fuzzy subspace of a vector space. In [4, 5] Nanda used fuzzy sets to develop the theory of fuzzy fields. Negoita and Ralescu [6] introduced the notion of fuzzy modules, etc. However, not all results can be extended to the fuzzy set [7–11] and fuzzification develops slowly in algebra theory. On the other hand, algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, information sciences, coding theory and so on. This provides sufficient motivations for us to review various concepts and results from the realm of abstract algebra to a broader framework of a fuzzy set.

In this paper, the concept of a fuzzy subspace is extended to a Novikov algebra. In section 2, we define *L*-fuzzy ideals and *L*-fuzzy subalgebras of Novikov algebras, and discuss some fundamental properties. In section 3, we show that addition, product and intersection of *L*-fuzzy ideals are *L*-fuzzy ideals [resp. *L*-fuzzy subalgebras], but the union of *L*-fuzzy ideals may not be an *L*-fuzzy ideal. In section 4, we show that the quotient algebra A/μ of the *L*-fuzzy ideal μ is isomorphic to the algebra A/A_μ of the non-fuzzy ideal A_μ . In section 5, we show that if $f : A_1 \rightarrow A_2$ is an *L*-fuzzy Novikov algebra homomorphism, then the preimage of an *L*-fuzzy ideal is an *L*-fuzzy ideal [resp. *L*-fuzzy subalgebra]. When f is surjective, a homomorphic image is an *L*-fuzzy ideal. Moreover, the addition, product and intersection of *L*-fuzzy ideals in A_1 are preserved by f .

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2 Preliminaries

Let X be any set and L be a non-trivial complete distributive lattice (in particular L could be $[0, 1]$). Then an L -fuzzy set μ in X is characterised by a map $\mu : X \rightarrow L$. L^X will be denoted as all the L -fuzzy subsets in X , it can be given whatever operations L has, and these operations in L^X will obey any law valid in L which extends point by point [12].

A pre-Lie algebra A is a vector space with a binary operation $(x, y) \rightarrow x \cdot y$ satisfying

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z)$$

for all $x, y, z \in A$. The algebra is called Novikov algebra, if

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y$$

is satisfied.

Throughout this paper A will be denote as a Novikov algebra over a field F , unless explicitly stated otherwise.

Definition 2.1. [3] Let V be a vector space over a field F . An L -fuzzy subspace is an L -fuzzy subset $\mu : V \rightarrow L$, satisfying

- (1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,
- (2) $\mu(kx) \geq \mu(x)$,
- (3) $\mu(0) = 1$

for all $k \in F, x, y \in V$.

Lemma 2.2. [3] Let V be a vector space over a field F , an L -fuzzy subset $\mu : V \rightarrow L$ is an L -fuzzy subspace if and only if

- (1) $\mu(kx + ly) \geq \mu(x) \wedge \mu(y)$,
- (2) $\mu(0) = 1$

for all $k, l \in F, x, y \in V$.

Definition 2.3. Let A be a Novikov algebra over a field F with a bilinear product $(x, y) \rightarrow x \cdot y$. An L -fuzzy subspace $\mu : A \rightarrow L$ is called an L -fuzzy subalgebra of A , if the inequation

$$\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$$

is satisfied for all $x, y \in A$.

An L -fuzzy subspace $\mu : A \rightarrow L$ is called an L -fuzzy ideal of A , if the inequation

$$\mu(x \cdot y) \geq \mu(x) \vee \mu(y)$$

is satisfied for all $x, y \in A$.

Remark 2.4. L -Fuzzy subalgebras and L -Fuzzy ideals of a Novikov algebra A are L -fuzzy subspaces of A , then for all $k \in F, x \in V$ we have $\mu(kx) = \mu x$.

Example 2.5. Let $(A, *)$ be a commutative associative algebra, and D be its derivation. Then the new product

$$x \cdot y = x * Dy + a * x * y \text{ for all } x, y \in A,$$

makes $(A, \cdot)$ become a Novikov algebra for $a = 0$ by Gelfand and Doffman [13], for $a \in F$ by Filippov [14] and for a fixed element $a \in A$ by Xu [15].

If μ is an L -fuzzy subspace of $(A, *)$, then μ is an L -fuzzy subalgebra under the conditions of Gelfand [13] and Filippov [14], but μ may not be an L -fuzzy subalgebra under the condition of $a \in A$ by Xu [15].

Let $x, y \in A$. If $a \in F$, then

$$\begin{aligned}\mu(x \cdot y) &= \mu(x * Dy + a * x * y) \\ &\geq \mu(x) \wedge \sup\{\mu(t) : t \in D^{-1}(y)\} \wedge \mu(x) \wedge \mu(y) \\ &= \mu(x) \wedge \mu(y).\end{aligned}$$

Thus μ is an L -fuzzy subalgebra of A .

If $a \in A$, then

$$\begin{aligned}\mu(x \cdot y) &\geq \mu(x) \wedge \sup\{\mu(t) : t \in D^{-1}(y)\} \wedge \mu(a) \wedge \mu(x) \wedge \mu(y) \\ &= \mu(a) \wedge \mu(x) \wedge \mu(y).\end{aligned}$$

Thus μ is not necessarily an L -fuzzy subalgebra of A .

The addition and the multiplication of A are extended by means of Zadeh's extension principle [16], to two operations on L^A denoted by $\oplus$ and $\otimes$ as follows:

- (1) $(\mu \oplus \rho)(x) = \sup\{\mu(a) \wedge \rho(b) : a + b = x\},$
- (2) $(\mu \otimes \rho)(x) = \sup\{\mu(a) \wedge \rho(b) : a \cdot b = x\}$

for all $\mu, \rho \in L^A$, $x, a, b \in A$.

The scalar multiplication kx for $k \in F$ and $x \in A$ is extended to an action of the field F on L^A denoted by $\odot$ as follows:

$$(k \odot \mu)(x) = \begin{cases} \mu(k^{-1}x) & \text{if } k \neq 0, \\ 1 & \text{if } k = 0, x = 0, \\ 0 & \text{if } k = 0, x \neq 0. \end{cases}$$

3 L -fuzzy ideals and subalgebras

Theorem 3.1. (1) Let μ be an L -fuzzy ideal and ρ be an L -fuzzy subalgebra of A . Then $\mu \oplus \rho$ is also an L -fuzzy subalgebra of A .

(2) Let $\{\mu_i : i \in I\}$ be a set of L -fuzzy subalgebras of A . Then the intersection $\bigcap_{i \in I} \mu_i$ of A is also an L -fuzzy subalgebra of A .

Proof. (1) $\mu \oplus \rho$ is an L -fuzzy subspace of A by Proposition 3.3 of [3]. Let $x, y \in A$. Then

$$\begin{aligned}(\mu \oplus \rho)(x \cdot y) &= \sup\{\mu(x_1 \cdot y) \wedge \rho(x_2 \cdot y) : x_1 + x_2 = x\} \\ &\geq \sup\{(\mu(x_1) \vee \mu(y)) \wedge (\rho(x_2) \wedge \rho(y)) : x_1 + x_2 = x\} \\ &= \sup\{(\mu(x_1) \wedge \rho(x_2) \wedge \rho(y)) \vee (\mu(y) \wedge \rho(x_2) \wedge \rho(y)) : x_1 + x_2 = x\} \\ &\geq \sup\{(\mu(x_1) \wedge \rho(x_2) \wedge \rho(y)) : x_1 + x_2 = x\} \\ &= (\mu \oplus \rho)(x) \wedge (\mu \oplus \rho)(y) \\ &\geq (\mu \oplus \rho)(x) \wedge (\mu \oplus \rho)(y) \text{ for } x_1, x_2 \in A.\end{aligned}$$

Thus $\mu \oplus \rho$ is an L -fuzzy subalgebra of A .

(2) $\bigcap_{i \in I} \mu_i$ is an L -fuzzy subspace of A by Proposition 3.4 of [3]. Let $x, y \in A$. Then

$$\begin{aligned}\left(\bigcap_{i \in I} \mu_i\right)(x \cdot y) &= \inf_{i \in I} \{\mu_i(x \cdot y)\} \geq \inf_{i \in I} \{\mu_i(x) \wedge \mu_i(y)\} \\ &= \left(\bigcap_{i \in I} \mu_i\right)(x) \wedge \left(\bigcap_{i \in I} \mu_i\right)(y).\end{aligned}$$

Thus $\bigcap_{i \in I} \mu_i$ is an L-fuzzy subalgebra of A. □

Theorem 3.2. (1) Let μ, ρ be L-fuzzy ideals of A. Then $\mu \oplus \rho$ is also an L-fuzzy ideal of A.

(2) Let $\{\mu_i : i \in I\}$ be a set of L-fuzzy ideals of A. Then the intersection $\bigcap_{i \in I} \mu_i$ of A is also an L-fuzzy ideal of A.

Proof. (1) $\mu \oplus \rho$ is an L-fuzzy subspace of A by Proposition 3.3 of [3].

Let $x, y \in A$. Then

$$\begin{aligned} (\mu \oplus \rho)(x \cdot y) &= \sup\{\mu(x_1 \cdot y) \wedge \rho(x_2 \cdot y) : x_1 + x_2 = x\} \\ &\geq \sup\{(\mu(x_1) \vee \mu(y)) \wedge (\rho(x_2) \vee \rho(y)) : x_1 + x_2 = x\} \\ &\geq \sup\{[(\mu(x_1) \vee \mu(y)) \wedge (\rho(x_2))] \vee [(\mu(x_1) \vee \mu(y)) \wedge \rho(y)] : x_1 + x_2 = x\} \\ &\geq \sup\{(\mu(x_1) \wedge \rho(x_2)) \vee (\mu(y) \wedge \rho(y)) : x_1 + x_2 = x\} \\ &= (\mu \oplus \rho)(x) \vee (\mu \wedge \rho)(y) \\ &\geq (\mu \oplus \rho)(x) \text{ for } x_1, x_2 \in A. \end{aligned}$$

Similarly, we can prove that $(\mu \oplus \rho)(x \cdot y) \geq (\mu \oplus \rho)(y)$. Thus $(\mu \oplus \rho)(x \cdot y) \geq (\mu \oplus \rho)(x) \vee (\mu \oplus \rho)(y)$.

(2) $\bigcap_{i \in I} \mu_i$ is an L-fuzzy subspace of A by Proposition 3.4 of [3]. Let $x, y \in A$. Then

$$\begin{aligned} \left(\bigcap_{i \in I} \mu_i\right)(x \cdot y) &= \inf_{i \in I} \{\mu_i(x \cdot y)\} \geq \inf_{i \in I} \{\mu_i(x) \vee \mu_i(y)\} \\ &\geq \vee \left\{ \inf_{i \in I} \{\mu_i(x)\}, \inf_{i \in I} \{\mu_i(y)\} \right\} \\ &= \left(\bigcap_{i \in I} \mu_i\right)(x) \vee \left(\bigcap_{i \in I} \mu_i\right)(y). \end{aligned} \quad \square$$

Proposition 3.3. Let μ, ρ, σ be L-fuzzy ideals of A. Then

- (1) $(\mu \otimes \rho) \otimes \sigma = (\mu \otimes \sigma) \otimes \rho$;
- (2) $\mu \otimes (\rho \otimes \sigma) \subseteq ((\mu \otimes \rho) \otimes \sigma) \oplus (\rho \otimes (\mu \otimes \sigma)) \oplus ((\rho \otimes \mu) \otimes \sigma)$.

Proof. (1) Let $x \in A$. Then

$$\begin{aligned} ((\mu \otimes \rho) \otimes \sigma)(x) &= \sup\{(\mu \otimes \rho)(m) \wedge \sigma(n) : m \cdot n = x\} \\ &= \sup\{\sup\{\mu(a) \wedge \rho(b) : a \cdot b = m\} \wedge \sigma(n) : m \cdot n = x\} \\ &= \sup\{((\mu(a) \wedge \rho(b)) \wedge \sigma(n)) : (a \cdot b) \cdot n = x\} \\ &= \sup\{((\mu(a) \wedge \sigma(n)) \wedge \rho(b)) : (a \cdot n) \cdot b = x\} \\ &= \sup\{\sup\{\mu(a) \wedge \sigma(n) : a \cdot n = c\} \wedge \rho(b) : c \cdot b = x\} \\ &= \sup\{(\mu \otimes \sigma)(c) \wedge \rho(b) : c \cdot b = x\} \\ &= ((\mu \otimes \sigma) \otimes \rho)(x) \text{ for } a, b, c, m, n \in A. \end{aligned}$$

(2) Let $x \in A$. Then

$$\begin{aligned} (\mu \otimes (\rho \otimes \sigma))(x) &= \sup\{\mu(m) \wedge (\rho \otimes \sigma)(n) : m \cdot n = x\} \\ &= \sup\{\mu(m) \wedge \sup\{\rho(b) \wedge \sigma(c) : b \cdot c = n\} : m \cdot n = x\} \\ &= \sup\{\sup\{\mu(m) \wedge (\rho(b) \wedge \sigma(c)) : m \cdot (b \cdot c) = x\}\} \\ &= \sup\{\mu(m) \wedge (\rho(b) \wedge \sigma(c)) : (m \cdot b) \cdot c + b \cdot (m \cdot c) - (b \cdot m) \cdot c = x\} \\ &= \sup\{((\mu(m) \wedge \rho(b)) \wedge \sigma(c)) \wedge (\rho(b) \wedge (\mu(m) \wedge \sigma(c))) \wedge ((\rho(b) \wedge \mu(m)) \wedge \sigma(c)) : \\ &\quad (m \cdot b) \cdot c + b \cdot (m \cdot c) - (b \cdot m) \cdot c = x\} \end{aligned}$$

$$\begin{aligned}
&\leq \sup\{(\sup(\mu(m) \wedge \rho(b)) \wedge \sigma(c)) \wedge (\rho(b) \wedge \sup(\mu(m) \wedge \sigma(c))) \wedge (\sup(\rho(b) \wedge \mu(m)) \wedge \sigma(c)) : \\
&\quad (m \cdot b) \cdot c + b \cdot (m \cdot c) - (b \cdot m) \cdot c = x\} \\
&= \sup\{((\mu \otimes \rho)(r) \wedge \sigma(c)) \wedge (\rho(b) \wedge (\mu \otimes \sigma)(s)) \wedge ((\rho \otimes \mu)(t) \wedge \sigma(c)) : \\
&\quad r \cdot c + b \cdot s + t \cdot c = x, r = m \cdot b, s = m \cdot c, t = -b \cdot m\} \\
&\leq \sup\{\sup((\mu \otimes \rho)(r) \wedge \sigma(c)) \wedge \sup(\rho(b) \wedge (\mu \otimes \sigma)(s)) \wedge \sup((\rho \otimes \mu)(t) \wedge \sigma(c)) : \\
&\quad r \cdot c + b \cdot s + t \cdot c = x, r = m \cdot b, s = m \cdot c, t = -b \cdot m\} \\
&= \sup\{((\mu \otimes \rho) \otimes \sigma)(u) \wedge (\rho \otimes (\mu \otimes \sigma))(v) \wedge ((\rho \otimes \mu) \otimes \sigma)(w) : \\
&\quad u + v + w = x, u = r \cdot c, v = b \cdot s, w = t \cdot c\} \\
&= (((\mu \otimes \rho) \otimes \sigma) \oplus (\rho \otimes (\mu \otimes \sigma)) \oplus ((\rho \otimes \mu) \otimes \sigma))(x) \text{ for } m, n, b, c, r, s, t, u, v, w \in A.
\end{aligned}$$

□

Theorem 3.4. Let μ be an L -fuzzy subspace of A . Then μ is an L -fuzzy ideal of A if and only if $\chi_A \otimes \mu \subseteq \mu$ and $\mu \otimes \chi_A \subseteq \mu$, where $\chi_A(x) = 1$ for all $x \in A$.

Proof. ($\Leftarrow$) Suppose that $\chi_A \otimes \mu \subseteq \mu$. Let $x, y \in A$. Then

$$\begin{aligned}
\mu(x \cdot y) &\geq (\chi_A \otimes \mu)(x \cdot y) = \sup\{\chi_A(a) \wedge \mu(b) : a \cdot b = x \cdot y\} \\
&\geq \chi_A(x) \wedge \mu(y) \geq \mu(y) \text{ for } a, b \in A.
\end{aligned}$$

Suppose that $\mu \otimes \chi_A \subseteq \mu$. Let $x, y \in A$. Then

$$\begin{aligned}
\mu(x \cdot y) &\geq (\mu \otimes \chi_A)(x \cdot y) = \sup\{\mu(c) \wedge \chi_A(d) : c \cdot d = x \cdot y\} \\
&\geq \mu(x) \wedge \chi_A(y) \geq \mu(x) \text{ for } c, d \in A.
\end{aligned}$$

Thus μ is an L -fuzzy ideal of A .

($\Rightarrow$) Suppose μ is an L -fuzzy ideal of A . Let $x \in A$. Then

$$\begin{aligned}
(\chi_A \otimes \mu)(x) &= \sup\{\chi_A(a) \wedge \mu(b) : a \cdot b = x\} \\
&= \sup\{\mu(b) : a \cdot b = x\} \leq \mu(x) \text{ for } a, b \in A.
\end{aligned}$$

Similarly, we can prove $(\mu \otimes \chi_A)(x) \leq \mu(x)$.

□

Theorem 3.5. Let μ, ρ be L -fuzzy ideals of A . Then $\mu \otimes \rho$ is also an L -fuzzy ideal of A .

Proof. By Proposition 3.3 (2), we have

$$\begin{aligned}
\chi_A \otimes (\mu \otimes \rho) &\subseteq ((\chi_A \otimes \mu) \otimes \rho) \oplus (\mu \otimes (\chi_A \otimes \rho)) \oplus ((\mu \otimes \chi_A) \otimes \rho) \\
&\subseteq (\mu \otimes \rho) \oplus (\mu \otimes \rho) \oplus (\mu \otimes \rho) \\
&\subseteq (\mu \otimes \rho).
\end{aligned}$$

By Proposition 3.3 (1), it is obvious that

$$(\mu \otimes \rho) \otimes \chi_A = (\mu \otimes \chi_A) \otimes \rho \subseteq \mu \otimes \rho.$$

By Theorem 3.4, $\mu \otimes \rho$ is an L -fuzzy ideal of A .

□

Remark 3.6. Let $\{\mu_i : i \in I\}$ be a set of L -fuzzy ideals [resp. L -fuzzy subalgebras] in A . Then the union $\bigcup_{i \in I} \mu_i$ may not be an L -fuzzy ideal [resp. L -fuzzy subalgebra]. It can be proved by the same method as the proof of Proposition 5.7 in [1].

4 Coset of an L-fuzzy ideals

Definition 4.1. [17] Let μ be an L-fuzzy ideal of A . For each $x \in A$, the L-fuzzy subset $x + \mu : A \rightarrow L$ defined by $(x + \mu)(y) = \mu(y - x)$ is called a coset of the L-fuzzy ideal μ .

Theorem 4.2. If μ is an L-fuzzy ideal of A , then $x + \mu = y + \mu$ if and only if $\mu(x - y) = \mu(0) = 1$. In that case $\mu(x) = \mu(y)$.

Proof. If $x + \mu = y + \mu$, then evaluating both side of this equation at x we get $\mu(x - y) = \mu(x - x) = \mu(0)$ for $x, y \in A$. Conversely, if $\mu(x - y) = \mu(0) = 1$, then

$$(x + \mu)(z) = \mu(z - x) = \mu(z - y + y - x) \geq \mu(z - y) \wedge \mu(y - x) = \mu(z - y) = (y + \mu)(z)$$

for all $z \in A$. Thus $x + \mu \geq y + \mu$. On the other hand, we have

$$(y + \mu)(z) = \mu(z - y) = \mu(z - x + x - y) \geq \mu(z - x) \wedge \mu(x - y) = \mu(z - x) = (x + \mu)(z)$$

for all $z \in A$.

Thus $y + \mu \geq x + \mu$. As is clear from the above descriptions, we get the equation $x + \mu = y + \mu$. $\square$

Remark 4.3. Let $A_\mu = \{x \in A \mid \mu(x) = 1\}$. It is easy to see that A_μ is an ideal of A .

Remark 4.4. If μ is an L-fuzzy ideal in A , then $(x + \mu)(z) = \mu(y - x)$ for all $z \in y + A_\mu$. In particular, $(x + \mu)(z) = \mu(x)$ for all $z \in A_\mu$.

Proposition 4.5. Let μ be an L-fuzzy ideal and x_1, x_2, y_1, y_2 be any elements in A . If $x_1 + \mu = y_1 + \mu$ and $x_2 + \mu = y_2 + \mu$, then

- (1) $(x_1 + x_2) + \mu = (y_1 + y_2) + \mu$,
- (2) $(x_1 \cdot x_2) + \mu = (y_1 \cdot y_2) + \mu$,
- (3) $kx_1 + \mu = ky_1 + \mu$ for all $k \in F$.

Proof. The proof of (1), (2) by Proposition 3.4 in [17]. It is sufficient to prove (3).

Since $\mu(x_1 - y_1) = \mu(k(x_1 - y_1)) = \mu(kx_1 - ky_1) = 0$, we get that $kx_1 + \mu = ky_1 + \mu$ by Theorem 4.2. $\square$

Definition 4.6. The algebra A/μ of the L-fuzzy ideal μ is called the quotient algebra of Novikov algebra A . We define an addition, a scalar multiplication and a multiplication operations of the cosets as follows:

- (1) $(x + \mu) \oplus (y + \mu) = (x + y) + \mu$,
- (2) $k \odot (x + \mu) = kx + \mu$,
- (3) $(x + \mu) \otimes (y + \mu) = (x \cdot y) + \mu$ for all $k \in F, x, y \in A$.

The addition, the scalar multiplication and the multiplication operation of the cosets in Definition 4.6 are well defined by Proposition 4.5.

Theorem 4.7. The Novikov quotient algebra A/μ is isomorphic to the algebra A/A_μ .

Proof. Consider the surjective algebra homomorphism $\pi : A \rightarrow A/\mu$ defines by $\pi(x) = x + \mu$. By Theorem 4.2, $\text{Ker}(\pi) = A_\mu$. By the fundamental theorem of homomorphisms, there exists an isomorphism from A/A_μ to A/μ . The isomorphic correspondence is given by $x + \mu = x + A_\mu$ for $x \in A$.

5 L -fuzzy ideals on homomorphism

Definition 5.1. [18] Let X_1 and X_2 be sets. A map $f : X_1 \rightarrow X_2$ has a natural extension $\tilde{f} : L^{X_1} \rightarrow L^{X_2}$ defined by

$$\tilde{f}(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\} & f^{-1}(y) \neq \emptyset, \\ 0 & f^{-1}(y) = \emptyset \end{cases}$$

for all $\mu \in L^{X_1}$, $y \in X_2$. $\tilde{f}(\mu)$ is called the homomorphic image of the L -fuzzy set μ .

Let $\rho \in L^{X_2}$, we define the L -fuzzy set $\mu \in L^{X_1}$ by $\mu(x) = \rho(f(x))$ for all $x \in X_1$, where μ is called the preimage of ρ and denoted by $f^{-1}(\rho)$.

Definition 5.2. [1] Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be an algebra homomorphism. An L -fuzzy subset μ in A_1 is called f -invariant if for any $x, y \in A_1$, $f(x) = f(y)$ implies $\mu(x) = \mu(y)$.

Theorem 5.3. Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be an algebra homomorphism. If μ is an L -fuzzy subalgebra of A_2 , then $\tilde{f}^{-1}(\mu)$ is also an L -fuzzy subalgebra of A_1 .

Proof. By Definition 5.1, we have (1) $\tilde{f}^{-1}(\mu)(x + y) = \mu(f(x + y)) = \mu(f(x) + f(y)) \geq \mu(f(x)) \wedge \mu(f(y)) \geq \tilde{f}^{-1}(\mu)(x) \wedge \tilde{f}^{-1}(\mu)(y)$.

$$(2) \tilde{f}^{-1}(\mu)(kx) = \mu(f(kx)) = \mu(kf(x)) \geq \mu(f(x)) \tilde{f}^{-1}(\mu)(x).$$

$$(3) \tilde{f}^{-1}(\mu)(x \cdot y) = \mu(f(x \cdot y)) = \mu(f(x) \cdot f(y)) \geq \mu(f(x)) \wedge \mu(f(y)) \geq \tilde{f}^{-1}(\mu)(x) \wedge \tilde{f}^{-1}(\mu)(y) \text{ for all } k \in F, x, y \in A_1.$$

□

Theorem 5.4. Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be an algebra homomorphism. If μ is an L -fuzzy ideal of A_2 , then $\tilde{f}^{-1}(\mu)$ is also an L -fuzzy ideal of A_1 .

Proof. Similar with the proof of Theorem 5.3. □

Theorem 5.5. Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be an algebra homomorphism. If μ is an L -fuzzy subalgebra of A_1 , then $\tilde{f}(\mu)$ is also an L -fuzzy subalgebra of A_2 .

Proof. Since $f(0) = 0$ and $\mu(0) = 1$, it is clear that $\tilde{f}(\mu)(0) = 1$. By Proposition 3.2 of [3], $\tilde{f}(\mu)$ is an L -fuzzy subspace of A_2 .

Let $x, y \in A_2$. It is enough to show that $\tilde{f}(\mu)(x \cdot y) \geq \tilde{f}(\mu)(x) \wedge \tilde{f}(\mu)(y)$.

If $x \cdot y \in f(A_1)$, assume that $\tilde{f}(\mu)(x \cdot y) < \tilde{f}(\mu)(x) \wedge \tilde{f}(\mu)(y)$. Then $\tilde{f}(\mu)(x \cdot y) < \tilde{f}(\mu)(x)$ and $\tilde{f}(\mu)(x \cdot y) < \tilde{f}(\mu)(y)$. We can choose a number $t \in [0, 1]$ such that $\tilde{f}(\mu)(x \cdot y) < t < \tilde{f}(\mu)(x)$ and $\tilde{f}(\mu)(x \cdot y) < t < \tilde{f}(\mu)(y)$. There exist $a \in f^{-1}(x) \subseteq A_1$, $b \in f^{-1}(y) \subseteq A_1$ such that $\mu(a) > t$, $\mu(b) > t$.

Since $f(a \cdot b) = f(a) \cdot f(b) = x \cdot y$, we have $f^{-1}(x \cdot y) \neq \emptyset$, and

$$\tilde{f}(\mu)(x \cdot y) = \sup\{\mu(z) : z \in f^{-1}(x \cdot y)\} \geq \mu(a \cdot b) \geq \mu(a) \wedge \mu(b) > t > \tilde{f}(\mu)(x \cdot y).$$

This is a contradiction. Similarly, we can prove the other case.

If $x \cdot y \notin f(A_1)$, we have $x \notin f(A_1)$ or $y \notin f(A_1)$. By Definition 5.1, $\tilde{f}(\mu)(x) = 0$ or $\tilde{f}(\mu)(y) = 0$, it is obvious that $\tilde{f}(\mu)(x \cdot y) \geq \tilde{f}(\mu)(x) \wedge \tilde{f}(\mu)(y)$.

Hence, $\tilde{f}(\mu)$ is an L -fuzzy subalgebra of A_2 . □

Theorem 5.6. Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be a surjective algebra homomorphism. If μ is an L -fuzzy ideal of A_1 , then $\tilde{f}(\mu)$ is also an L -fuzzy ideal of A_2 .

Proof. Since $f(0) = 0$ and $\mu(0) = 1$, it is clear that $\tilde{f}(\mu)(0) = 1$. By Proposition 3.2 of [3], $\tilde{f}(\mu)$ is an L -fuzzy subspace of A_2 .

Let $x, y \in A_2$. It is enough to show that $\tilde{f}(\mu)(x \cdot y) \geq \tilde{f}(\mu)(x) \vee \tilde{f}(\mu)(y)$.

Assume that $\tilde{f}(\mu)(x \cdot y) < \tilde{f}(\mu)(x) \vee \tilde{f}(\mu)(y)$. Then $\tilde{f}(\mu)(x \cdot y) < \tilde{f}(\mu)(x)$ or $\tilde{f}(\mu)(x \cdot y) < \tilde{f}(\mu)(y)$. Without loss of generality, we can choose a number $t \in [0, 1]$ such that $\tilde{f}(\mu)(x \cdot y) < t < \tilde{f}(\mu)(x)$. There exists an $a \in f^{-1}(x) \subseteq A_1$ such that $\mu(a) > t$.

Since f is surjective, there exists $b \in A_1$ such that $\mu(b) = y$.

Since $f(a \cdot b) = f(a) \cdot f(b) = x \cdot y$, we have $f^{-1}(x \cdot y) \neq \emptyset$, and

$$\tilde{f}(\mu)(x \cdot y) = \sup\{\mu(z) : z \in f^{-1}(x \cdot y)\} \geq \mu(a \cdot b) \geq \mu(a) > t > \tilde{f}(\mu)(x \cdot y).$$

This is a contradiction.

Similarly, we can prove the other case. Hence, $\tilde{f}(\mu)$ is an L-fuzzy ideal in A_2 . $\square$

Theorem 5.7. Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be an algebra homomorphism. Then

(1) if μ, ρ are L-fuzzy subalgebras of A_1 , then $\tilde{f}(\mu \oplus \rho) = \tilde{f}(\mu) \oplus \tilde{f}(\rho)$,

(2) if $\{\mu_i : i \in I\}$ is a set of L-fuzzy subalgebras of A_1 , then $\tilde{f}(\bigcap_{i \in I} \mu_i) = \bigcap_{i \in I} \tilde{f}(\mu_i)$,

(3) if μ, ρ are L-fuzzy subalgebras of A_1 , then $\tilde{f}(\mu \otimes \rho) = \tilde{f}(\mu) \otimes \tilde{f}(\rho)$.

Proof. (1) and (2) can be proved by the same method as the proof of Theorem 5.1 in [10]. It is sufficient to prove (3).

Let $x \in A_2$. We prove that $\tilde{f}(\mu \otimes \rho)(x) = (\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x)$. If $x = y \cdot z \notin f(A_1)$, we have $y \notin f(A_1)$ or $z \notin f(A_1)$. By the proof of Theorem 5.5, we get $\tilde{f}(\mu \otimes \rho)(x) = 0$ and $(\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x) = \tilde{f}(\mu)(x) \otimes \tilde{f}(\rho)(x) = \sup\{\tilde{f}(\mu)(y) \wedge \tilde{f}(\rho)(z) : x = y \cdot z\} = 0$.

Let $x = y \cdot z \in f(A_1)$. Assume that $\tilde{f}(\mu \otimes \rho)(x) < (\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x)$. We can choose an element $t \in L$ such that $\tilde{f}(\mu \otimes \rho)(x) < t < \tilde{f}(\mu)(x) \otimes \tilde{f}(\rho)(x)$.

Since $\tilde{f}(\mu)(x) \otimes \tilde{f}(\rho)(x) = \sup\{\tilde{f}(\mu)(y) \wedge \tilde{f}(\rho)(z) : x = y \cdot z\}$, there exist $y, z \in A_2$ such that $x = y \cdot z$ with $\tilde{f}(\mu)(y) > t$ and $\tilde{f}(\rho)(z) > t$. Since $x \in f(A_1)$, there exists an $x_1 \in A_1$ such that $f(x_1) = x$ and $x_1 = y_1 \cdot z_1$ for $y_1 \in f^{-1}(y)$, $z_1 \in f^{-1}(z)$ with $\mu(y_1) > t$ and $\rho(z_1) > t$.

Since $f(y_1 \cdot z_1) = f(y_1) \cdot f(z_1) = y \cdot z = x$, we have

$$\begin{aligned} \tilde{f}(\mu \otimes \rho)(x) &= \sup\{(\mu \otimes \rho)(x_1) : f(x_1) = x\} \\ &= \sup\{\mu(a) \wedge \rho(b) : f(x_1) = f(a \cdot b) = x\} \\ &\geq \mu(y_1) \wedge \rho(z_1) > t. \end{aligned}$$

This is a contradiction.

Similarly, for the case $\tilde{f}(\mu \otimes \rho)(x) > (\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x)$, we get a contradiction. Hence, $\tilde{f}(\mu \otimes \rho) = \tilde{f}(\mu) \otimes \tilde{f}(\rho)$. $\square$

Theorem 5.8. Let A_1 and A_2 be Novikov algebras, and $f : A_1 \rightarrow A_2$ be a surjective algebra homomorphism. Then

(1) if μ, ρ are L-fuzzy ideals of A_1 , then $\tilde{f}(\mu \oplus \rho) = \tilde{f}(\mu) \oplus \tilde{f}(\rho)$,

(2) if $\{\mu_i : i \in I\}$ is a set of L-fuzzy ideals of A_1 , then $\tilde{f}(\bigcap_{i \in I} \mu_i) = \bigcap_{i \in I} \tilde{f}(\mu_i)$,

(3) if μ, ρ are L-fuzzy ideals of A_1 , then $\tilde{f}(\mu \otimes \rho) = \tilde{f}(\mu) \otimes \tilde{f}(\rho)$.

Proof. (1) and (2) can be proved by the same method as the proof of Theorem 5.1 in [10]. It is sufficient to prove (3).

Let $x \in A_2$. We prove that $\tilde{f}(\mu \otimes \rho)(x) = (\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x)$. Assume that $\tilde{f}(\mu \otimes \rho)(x) < (\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x)$. We can choose an element $t \in L$ such that $\tilde{f}(\mu \otimes \rho)(x) < t < \tilde{f}(\mu)(x) \otimes \tilde{f}(\rho)(x)$.

Since $\tilde{f}(\mu)(x) \otimes \tilde{f}(\rho)(x) = \sup\{\tilde{f}(\mu)(y) \wedge \tilde{f}(\rho)(z) : x = y \cdot z\}$, there exist $y, z \in A_2$ such that $x = y \cdot z$ with $\tilde{f}(\mu)(y) > t$ and $\tilde{f}(\rho)(z) > t$. Since f is surjective, there exists an $x_1 \in A_1$ such that $f(x_1) = x$ and $x_1 = y_1 \cdot z_1$ for $y_1 \in f^{-1}(y)$, $z_1 \in f^{-1}(z)$ with $\mu(y_1) > t$ and $\rho(z_1) > t$.

Since $f(y_1 \cdot z_1) = f(y_1) \cdot f(z_1) = y \cdot z = x$, we have

$$\begin{aligned}\tilde{f}(\mu \otimes \rho)(x) &= \sup\{(\mu \otimes \rho)(x_1) : f(x_1) = x\} \\ &= \sup\{\sup\{\mu(a) \wedge \rho(b) : f(a \cdot b) = x\}\} \\ &= \sup\{\mu(a) \wedge \rho(b) : f(a \cdot b) = x\} \\ &\geq \mu(y_1) \wedge \rho(z_1) > t.\end{aligned}$$

This is a contradiction.

Similarly, for the case $\tilde{f}(\mu \otimes \rho)(x) > (\tilde{f}(\mu) \otimes \tilde{f}(\rho))(x)$, we get a contradiction. Hence, $\tilde{f}(\mu \otimes \rho) = \tilde{f}(\mu) \otimes \tilde{f}(\rho)$. $\square$

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