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Results on existence for generalized nD Navier-Stokes equations

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Abstract: In this paper we consider a class of nD Navier-Stokes equations of Kirchhoff type and prove the global existence of solutions by using a new approach introduced in [Jday R., Zennir Kh., Georgiev S.G., Existence and smoothness for new class of n -dimensional Navier-Stokes equations, Rocky Mountain J. Math., 2019, 49(5), 1595–1615].

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1 Introduction

In this article we investigate a Navier-Stokes equations of Kirchhoff type

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} - \left(m + aM \left(\int_{\mathbb{R}^n} |A^{1/2} u_i|^2 dx \right) \right) A u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \end{cases} \quad (1.1)$$

for $i \in \{1, \dots, n\}$, subject to the initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

where $a > 0$, $n \geq 2$ and A is a self-adjoint non-positive operator. The function M satisfies

$$(\text{Hyp0}) \quad \begin{cases} \exists m_0 > 0 : M \in C^1(\mathbb{R}^+) \text{ and } M(\lambda) > m_0, \forall \lambda \geq 0, \\ \exists \gamma, \delta > 0 : M(\lambda) \leq \gamma \int_{\mathbb{R}^n} |u_0(x)|^{2\delta} dx, \forall \lambda > 0. \end{cases}$$

Here $u_0(x) = (u_{10}(x), \dots, u_{n0}(x))$ is a given C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(t, x)$, $i = 1, \dots, n$, are the components of a given externally applied force, m is a positive constant, $w = (w_1, \dots, w_n)$ and p are independent unknown. In the case when $m = 0$, the equations (1.1), (1.2) are well known as the Euler equations. The first equation of (1.1) is inspired from the Newton law for a fluid element subject to the external force $f = (f_1, \dots, f_n)$ and to the forces arising from pressure and friction. The second equation of (1.1) says that the fluid is incompressible. (See [1, 2])

By the last equation of the system (1.1), we get

$$u_j \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0 \quad \text{for any } j \in \{1, \dots, n\}.$$

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Then

$$\begin{aligned}\sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} &= \sum_{j=1}^n \frac{\partial}{\partial x_j} (u_j u_i) - u_i \sum_{j=1}^n \frac{\partial u_j}{\partial x_j} \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} (u_j u_i).\end{aligned}$$

In the case $a = 0$, system (1.1) has been treated in [3]. Therefore the system (1.1), for the case $a = 1$ can be rewritten for $i \in \{1, \dots, n\}$ and $A = -\Delta$, as

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n \frac{\partial}{\partial x_j} (u_i u_j) - \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n. \end{cases} \quad (1.3)$$

There is a large literature regarding this type of problem. It is stated an open problem for (1.1), (1.2): if $n = 3$ and $u_0(x)$ be any smooth, divergence-free vector field satisfying the condition

$$|\partial_x^\alpha u_0(x)| \leq C_{\alpha, K} (1 + |x|)^{-K} \quad \text{on } \mathbb{R}^n, \quad (1.4)$$

for any α and K , and f to be identically equal to zero, then there exist smooth functions $p, u_i, i = 1, \dots, n$, on $\mathbb{R}^n \times [0, \infty)$ that satisfy (1.1), (1.2) and $p, u \in \mathcal{C}^\infty(\mathbb{R}^n \times [0, \infty))$, and the energy is bounded for all $t \geq 0$. (See [4–7].)

In this paper we extend the previous works to find for any $n \geq 3$ a new class of smooth initial data u_0 satisfying (1.4) and a new class of functions f_i , including $f_i = 0, i = 1, \dots, n$, such that the problem (1.1), (1.2) has a solution $p, u \in \mathcal{C}^\infty(\mathbb{R}^n \times [0, \infty))$.

This kind of systems appears in the models of nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [8] in the case $n = 1$ this type of problem describes a small amplitude vibration of an elastic string. The original equation is:

$$\rho h u_{tt} + \tau u_t = \left(P_0 + \frac{Eh}{2L} \int_0^L |u_x(x, t)|^2 ds \right) u_{xx} + f, \quad (1.5)$$

where $0 \leq x \leq L, t > 0$ and

$$\begin{cases} u \text{ is the lateral deflection} \\ x \text{ is the space coordinate variable while } t \text{ denotes the time variable} \\ E \text{ is the Young modulus} \\ \rho \text{ is the mass density} \\ L \text{ is the lenght} \\ h \text{ is the cross section area} \\ P_0 \text{ is the initial axial tension} \\ \tau \text{ is the resistance modulus} \\ f \text{ is the external force} \end{cases} \quad (1.6)$$

(For more see [9]). Here we assume that the initial data u_0 and the force term f are as follows

$$\text{(Hyp1)} \quad \begin{cases} u_{i0} \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \text{supp } u_{i0} \subset B, \\ \int_{\mathbb{R}^n} |u_0(x)|^\kappa dx \leq n Q^\kappa \mu(B), \quad \kappa \geq 2, \\ |\partial_x^\alpha u_{i0}(x)| \leq C_{\alpha, K} (1 + |x|)^{-K} \quad \text{on } \mathbb{R}^n, \quad i \in \{1, \dots, n\}, \end{cases}$$

for any multi-index α and for any positive constant K , where $C_{\alpha, K}$ is a positive constant depending on α and K , Q is a positive constant such that

$$|u_0(x)| \leq Q, \quad |u_{0x_i}(x)| \leq Q, \quad |u_{0x_i x_j}(x)| \leq Q, \quad x \in B, \quad i, j \in \{1, \dots, n\},$$

$\mu(B)$ is the measure of B ,

$$(\text{Hyp2}) \quad \begin{cases} f_i \in \mathcal{C}^\infty([0, \infty), \mathcal{C}_0^\infty(\mathbb{R}^n)), & \text{supp}_x f_i \subset B, \\ |\partial_t^m \partial_x^\alpha f_i(t, x)| \leq C_{\alpha, m, K} (1 + |x| + t)^{-K} & \text{on } [0, \infty) \times \mathbb{R}^n, \\ i \in \{1, \dots, n\}, \end{cases}$$

for any α, m, K , where B is a compact subset of $\mathbb{R}^n$, respectively.

Several numerical methods for solving of (1.1) are used. In [10] Lagrangian and semi-Lagrangian velocity and displacement methods are introduced for the numerical solution of (1.1). In the scalar case, methods of characteristics for time discretization of convection diffusion problems are extensively used (see [11] and references therein). These methods are based on time discretization of the material time derivative combined with finite differences or finite elements for space discretization. When the characteristic methods are formulated in a fixed reference domain they are called pure Lagrangian methods. The classical methods of characteristics are semi-Lagrangian and first-order in time. There exists an extensive literature for these methods (see [12, 13] and references therein). The error estimates of the norm $O(h^k) + O(\Delta t) + O(h^{k+1}/\delta t)$ in $l^\infty(L^2(\Omega))$ -norm are obtained under the assumption that the normal velocity vanishes on the boundary Ω , where h is the space step and δt is the time step (See [14–16]).

In order to increase the order of time and space approximations, higher order schemes for the discretization of the material derivative and higher order finite element spaces are used (see [17–19] and references therein). Second order characteristic method for solving of (1.1) is used in [20, 21].

In this paper we propose a new method for investigation of the Cauchy problem (1.1), (1.2) which is different than the well-known methods. In Section 2 we give some auxiliary results. In Section 3 we prove the main result introduced in Theorem (3.1)

2 Preliminaries

We will start with the following useful Lemma.

Lemma 2.1. Let $x^0 = (x_1^0, \dots, x_n^0) \in \partial B$ and f_i satisfies (Hyp2), $i \in \{1, \dots, n\}$. If p and $u_i \in \mathcal{C}^1([a, b], \mathcal{C}_0^2(\mathbb{R}^n))$, $\text{supp}_x u_i \subset B$, $\text{supp}_x p \subset B$, $i \in \{1, \dots, n\}$, satisfy the system

$$\begin{cases} \int_{x^0}^x \int_{x^0}^s (u_i(t, \sigma) - g_i(\sigma)) d\sigma ds \\ + \sum_{j=1}^n \int_a^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{s_j}) u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\ - \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_a^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{x_j}) d\bar{\sigma}_j d\bar{s}_j d\tau \\ + \int_a^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau \\ = \int_a^t \int_{x^0}^x \int_{x^0}^s f_i(\tau, \sigma) d\sigma ds d\tau, \quad i \in \{1, \dots, n\}, \\ \sum_{j=1}^n \int_a^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau = 0, \quad t \in [a, b], \quad x \in \mathbb{R}^n, \end{cases} \quad (2.1)$$

then p and u_i , $i \in \{1, \dots, n\}$, satisfy the problem

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n \frac{\partial}{\partial x_j} (u_i u_j) - \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0, \quad (t, x) \in [a, b] \times \mathbb{R}^n, \\ u(a, x) = g(x), \quad g(x) = (g_1(x), \dots, g_n(x)), \quad x \in \mathbb{R}^n \end{cases} \quad (2.2)$$

where $g_i \in \mathcal{C}_0^2(\mathbb{R}^n)$, $\text{supp } g_i \subset B$, $i \in \{1, \dots, n\}$.

Here

$$\int_{x^0}^x = \int_{x_1^0}^{x_1} \dots \int_{x_n^0}^{x_n},$$

$$d\sigma = d\sigma_n \dots d\sigma_1,$$

$$\int_{\bar{x}_j^0}^{\bar{s}_j} = \int_{x_1^0}^{s_1} \dots \int_{x_{j-1}^0}^{s_{j-1}} \int_{x_{j+1}^0}^{s_{j+1}} \dots \int_{x_n^0}^{s_n},$$

$$\int_{\bar{x}_j^0}^{\bar{x}_j} = \int_{x_1^0}^{x_1} \dots \int_{x_{j-1}^0}^{x_{j-1}} \int_{x_{j+1}^0}^{x_{j+1}} \dots \int_{x_n^0}^{x_n},$$

$$\tilde{\sigma}_{s_j} = (\sigma_1, \dots, \sigma_{j-1}, s_j, \sigma_{j+1}, \dots, \sigma_n),$$

$$d\bar{\sigma}_j = d\sigma_n \dots d\sigma_{j+1} d\sigma_{j-1} \dots d\sigma_1,$$

$$\tilde{\sigma}_{x_j} = (\sigma_1, \dots, \sigma_{j-1}, x_j, \sigma_{j+1}, \dots, \sigma_n),$$

$$d\bar{s}_j = ds_n \dots ds_{j+1} ds_{j-1} \dots ds_1,$$

$$\tilde{\sigma}_{s_i} = (\sigma_1, \dots, \sigma_{i-1}, s_i, \sigma_{i+1}, \dots, \sigma_n), \quad i, j \in \{1, \dots, n\}.$$

Proof. We differentiate once in t and twice in $x_1, \dots, x_n$, all equations of the system (2.1) and we see that the function u satisfies the system (1.3). Now we put $t = a$ in the first n equations of the system (2.1) and we get

$$\int_{x^0}^x \int_{x^0}^s (u_i(a, \sigma) - g_i(\sigma)) d\sigma ds = 0, \quad i \in \{1, \dots, n\}.$$

We differentiate the last system twice in $x_1, \dots, x_n$, and we obtain

$$u_i(a, x) = g_i(x), \quad x \in \mathbb{R}^n, \quad i \in \{1, \dots, n\},$$

i.e., the function u_i , $i \in \{1, \dots, n\}$, satisfies the initial condition (2.2). This completes the proof. $\square$

The proof of the existence result is based on a fixed point theorem for sum of two operators one of which is expansive.

Definition 2.2. Let (X, d) be a metric space and Y be a subset of X . The mapping $T : Y \rightarrow X$ is said to be expansive if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y)$$

for any $x, y \in M$.

Next result we will use to prove our fixed point theorem.

Theorem 2.3. [22] Let X be a nonempty closed convex subset of a Banach space Y . Suppose that T and S map X into Y such that

1. S is continuous and $S(X)$ resides in a compact subset of Y .
2. $T : X \rightarrow Y$ is expansive and onto.

Then there exists a point $x^* \in X$ such that

$$Sx^* + Tx^* = x^*.$$

Theorem 2.4. Let X be a nonempty closed convex subset of a Banach space E and Y is a nonempty compact subset of E such that $X \subset Y$, $Y \neq X$. Suppose that T and S map X into E such that

1. S is continuous and $S(X)$ resides in Y .
2. $T : X \rightarrow E$ is linear, continuous and expansive, and $T : X \rightarrow Y$ is onto.

Then there exists an $x^* \in X$ such that

$$Tx^* + Sx^* = x^*.$$

Proof. Since Y is compact and $S(X)$ resides in Y , we have that the first condition of Theorem 2.3 holds. Because $T : X \rightarrow E$ is expansive, we have that the second condition of Theorem 2.3 holds. Note that $T^{-1} : Y \rightarrow E$ exists, it is linear and contractive with a constant $l \in (0, 1)$. Let $z \in S(X)$ be arbitrarily chosen and fixed. Set

$$A = \{y - z : y \in Y\}.$$

Take $y_0 \in Y$ arbitrarily. Define the sequence $\{y_n\}_{n \in \mathbb{N}}$ as follows.

$$y_{n+1} = T^{-1}y_n - z, \quad n \in \mathbb{N} \cup \{0\}.$$

Then

$$\|y_2 - y_1\| = \|T^{-1}y_1 - T^{-1}y_0\| \leq l\|y_1 - y_0\|,$$

$$\|y_3 - y_2\| = \|T^{-1}y_2 - T^{-1}y_1\| \leq l\|y_2 - y_1\| \leq l^2\|y_1 - y_0\|.$$

Using the principle of the mathematical induction, we get

$$\|y_{n+1} - y_n\| \leq l^n \|y_1 - y_0\|, \quad n \in \mathbb{N}.$$

Now, for $m > n$, $m, n \in \mathbb{N}$, we find

$$\begin{aligned} \|y_m - y_n\| &\leq \|y_m - y_{m-1}\| + \cdots + \|y_{n+1} - y_n\| \leq (l^{m-1} + \cdots + l^n) \|y_1 - y_0\| \\ &\leq l^n \sum_{j=0}^{\infty} l^j \|y_1 - y_0\| = \frac{l^n}{1-l} \|y_1 - y_0\|. \end{aligned}$$

Therefore $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence of elements of $Y \subset E$. Since E is a Banach space, it follows that the sequence $\{y_n\}_{n \in \mathbb{N}}$ is convergent to an element $y^* \in E$. Because $\{y_n\}_{n \in \mathbb{N}} \subset Y$ and $Y \subset E$ is compact, we have that $y^* \in Y$. Thus

$$y^* = T^{-1}y^* - z$$

or

$$z^* = Tz^* + z, \quad z^* = T^{-1}y^* \in X.$$

Because $z \in S(X)$ was arbitrarily chosen, we conclude that $S(X) \subset (I - T)(X)$, i.e., the third condition of Theorem 2.2 holds. Hence and Theorem 2.3, it follows that there exists an $x^* \in X$ such that

$$Tx^* + Sx^* = x^*.$$

This completes the proof. $\square$

Below we will suppose that $x^0 = (x_1^0, \dots, x_n^0) \in \partial B$ is arbitrarily chosen and fixed. Also, we will use the following notations.

$$\begin{aligned} \int_{\bar{x}_{j,m}^0}^{\bar{s}_{j,m}} &= \int_{x_1^0}^{s_1} \dots \int_{x_{j-1}^0}^{s_{j-1}} \int_{x_{j+1}^0}^{s_{j+1}} \dots \int_{x_{m-1}^0}^{s_{m-1}} \int_{x_{m+1}^0}^{s_{m+1}} \dots \int_{x_n^0}^{s_n}, \\ \int_{\bar{x}_{j,m,r}^0}^{\bar{s}_{j,m,r}} &= \int_{x_1^0}^{s_1} \dots \int_{x_{j-1}^0}^{s_{j-1}} \int_{x_{j+1}^0}^{s_{j+1}} \dots \int_{x_{m-1}^0}^{s_{m-1}} \int_{x_{m+1}^0}^{s_{m+1}} \dots \int_{x_{r-1}^0}^{s_{r-1}} \int_{x_{r+1}^0}^{s_{r+1}} \dots \int_{x_n^0}^{s_n}, \\ \tilde{\sigma}_{s_j, s_m} &= (\sigma_1, \dots, \sigma_{j-1}, s_j, \sigma_{j+1}, \dots, \sigma_{m-1}, s_m, \sigma_{m+1}, \dots, \sigma_n), \\ \tilde{\sigma}_{s_i, s_m, s_r} &= (\sigma_1, \dots, \sigma_{i-1}, s_i, \sigma_{i+1}, \dots, \sigma_{m-1}, s_m, \sigma_{m+1}, \dots, \sigma_{r-1}, s_r, \sigma_{r+1}, \dots, \sigma_n), \\ i, j, m, r &\in \{1, \dots, n\}. \end{aligned}$$

With $\mu(A)$ we will denote the measure of a set $A \subset \mathbb{R}^k$, $k \geq 1$.

Let

$$\begin{aligned} B^* &= \max \left\{ 1, \mu(B), \mu \left(B \cap (\mathbb{R}_{x_1} \times \dots \times \mathbb{R}_{x_{k_1-1}} \times \mathbb{R}_{x_{k_1+1}} \times \dots \times \mathbb{R}_{x_n}) \right), \right. \\ &\mu \left(B \cap (\mathbb{R}_{x_1} \times \dots \times \mathbb{R}_{x_{k_1-1}} \times \mathbb{R}_{x_{k_1+1}} \times \dots \times \mathbb{R}_{x_{k_2-1}} \times \mathbb{R}_{x_{k_2+1}} \times \dots \times \mathbb{R}_{x_n}) \right), \\ &\mu \left(B \cap (\mathbb{R}_{x_1} \times \dots \times \mathbb{R}_{x_{k_1-1}} \times \mathbb{R}_{x_{k_1+1}} \times \dots \times \mathbb{R}_{x_{k_2-1}} \times \mathbb{R}_{x_{k_2+1}} \times \dots \times \mathbb{R}_{x_{k_3-1}} \times \mathbb{R}_{x_{k_3+1}} \times \dots \times \mathbb{R}_{x_n}) \right), \\ &\left. k_1, k_2, k_3 \in \{1, \dots, n\} \right\}, \mathbb{R}^n = \mathbb{R}_{x_1} \times \dots \times \mathbb{R}_{x_n}, \end{aligned}$$

as we set

$$\begin{aligned} \mu \left(B \cap (\mathbb{R}_{x_1} \times \dots \times \mathbb{R}_{x_{k_1-1}} \times \mathbb{R}_{x_{k_1+1}} \times \dots \times \mathbb{R}_{x_{k_2-1}} \times \mathbb{R}_{x_{k_2+1}} \times \dots \times \mathbb{R}_{x_{k_3-1}} \times \mathbb{R}_{x_{k_3+1}} \times \dots \times \mathbb{R}_{x_n}) \right) &:= 1, \\ k_1, k_2, k_3 \in \{1, 2, 3\} \quad \text{for } n = 3. \end{aligned}$$

3 Main result – proof

Our main result is as follows.

Theorem 3.1. Suppose that w_{i0} satisfies (Hyp1) and f_i satisfies (Hyp2), $i \in \{1, \dots, n\}$. Then the Cauchy problem (1.1), (1.2) has a solution p , $u_i \in \mathcal{C}^\infty([0, \infty), \mathcal{C}_0^\infty(\mathbb{R}^n))$, $i \in \{1, \dots, n\}$, such that

$$\int_{\mathbb{R}^n} |u(t, x)|^2 dx \leq C \quad \text{for all } t \geq 0,$$

where $|u(t, x)| = \sqrt{(u_1(t, x))^2 + \dots + (u_n(t, x))^2}$.

Firstly, we will prove that the Cauchy problem for $i \in \{1, \dots, n\}$

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} - \left(m + M \left(\int_{\mathbb{R}^n} |\nabla_x u_i|^2 dx \right) \right) \Delta u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}(t, x) = 0 \quad \text{in } [0, 1] \times \mathbb{R}^n, \end{cases} \quad (3.1)$$

$$w(0, x) = w_0(x) \quad \text{in } \mathbb{R}^n, \quad (3.2)$$

has a $C^1([0, 1], C_0^2(\mathbb{R}^n))$ -solution (u^1, p^1) such that

$$\text{supp}_x u_i^1, \quad \text{supp}_x p^1 \subset B, \quad i \in \{1, \dots, n\}.$$

Let

$$N^1 = \max_{t \in [0, 1], x \in B} |f(t, x)|.$$

We choose the constants $\gamma, \delta, l > 0$ as follows

$$\ln \left(Q + 3n(1 + (m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n^2 Q^2 (B^*)^2 + (B^*)^2 N^1 \right) \leq Q. \quad (3.3)$$

Let

$$E^1 = \left\{ v = (v_1, \dots, v_{n+1}) : v_i \in C^1([0, 1], C_0^2(\mathbb{R}^n)), \text{supp}_x v_i \subset B, \right. \\ \left. i \in \{1, \dots, n+1\} \right\}$$

be endowed with the norm

$$\|v\| = \max_{q \in \{1, \dots, n+1\}} \left\{ \max_{t \in [0, 1], x \in B} |v_q(t, x)|, \max_{t \in [0, 1], x \in B} |v_{qt}(t, x)|, \right. \\ \max_{t \in [0, 1], x \in B} |v_{qx_i}(t, x)|, \max_{t \in [0, 1], x \in B} |v_{qx_i x_j}(t, x)|, \\ \left. i, j \in \{1, \dots, n\} \right\}.$$

With $\tilde{K}^1$ we denote the set of all equi-continuous families in E^1 , i.e., if $\mathcal{F} \subset \tilde{K}^1$ is a family of elements of E^1 , then for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$|v_q(t_1, x^1) - v_q(t_2, x^2)| < \epsilon, \quad |v_{qt}(t_1, x^1) - v_{qt}(t_2, x^2)| < \epsilon, \\ |v_{qx_i}(t_1, x^1) - v_{qx_i}(t_2, x^2)| < \epsilon, \quad |v_{qx_i x_j}(t_1, x^1) - v_{qx_i x_j}(t_2, x^2)| < \epsilon$$

for any $i, j \in \{1, \dots, n\}$, for any $q \in \{1, \dots, n+1\}$, and for any $v \in \mathcal{F}$, whenever $|t_1 - t_2| < \delta$, $|x^1 - x^2| < \delta$. Let also,

$$\bar{K}^1 = \overline{\tilde{K}^1}, \quad K^1 = \{v \in \bar{K}^1 : \|v\| \leq Q\},$$

$$Q^1 = \{v \in \bar{K}^1 : \|v\| \leq (1 + l)Q\}.$$

Note that K^1 is a compact subset of Q^1 . For $(u, p) \in Q^1$ we define the operators.

$$L_i^1(u, p)(t, x) = -lu_i(t, x) + l \int_{x^0}^x \int_{x^0}^s (u_i(t, \sigma) - u_{i0}(\sigma)) d\sigma ds$$

$$\begin{aligned}
& + l \sum_{j=1}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{s_j}) u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\
& - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{x_j}) d\bar{\sigma}_j d\bar{s}_j d\tau \\
& + l \int_0^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau - l \int_0^t \int_{x^0}^x \int_{x^0}^s f_i(\tau, \sigma) d\sigma ds d\tau, \\
L_{n+1}^1(u, p)(t, x) & = -lp(t, x) + l \sum_{j=1}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau, \\
N_i^1(u, p)(t, x) & = (1 + l)u_i(t, x), \quad i \in \{1, \dots, n\}, \\
N_{n+1}^1(u, p)(t, x) & = (1 + l)p(t, x), \\
L^1(u, p)(t, x) & = \left(L_1^1(u, p)(t, x), \dots, L_{n+1}^1(u, p)(t, x) \right), \\
\bar{N}^1(u, p)(t, x) & = \left(N_1^1(u, p)(t, x), \dots, N_{n+1}^1(u, p)(t, x) \right), \\
(t, x) & \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

Proposition 3.2. $L^1 : K^1 \mapsto K^1$ is continuous and $L^1(K^1)$ resides in a compact subset of Q^1 .

Proof.

1. For $(u, p) \in K^1$ we have that $L_i^1(u, p) \in \mathcal{C}^1([0, 1], \mathcal{C}^2(\mathbb{R}^n))$, $i \in \{1, \dots, n+1\}$ and for $x \in \mathbb{R}^n \setminus B$, using the definition of L^1 , we have that $L^1(u, p)(t, x) = 0$ for all $t \in [0, 1]$. Hence, $L_i^1(u, p) \in \mathcal{C}^1([0, 1], \mathcal{C}_0^2(\mathbb{R}^n))$ and $\text{supp}_x L_i^1(u, p) \subset B$, $i \in \{1, \dots, n+1\}$.
2. Let $(u, p) \in K^1$. Then

$$\begin{aligned}
\frac{\partial}{\partial t} L_i^1(u, p)(t, x) & = -l \frac{\partial}{\partial t} u_i(t, x) + l \int_{x^0}^x \int_{x^0}^s \frac{\partial}{\partial t} u_i(t, \sigma) d\sigma ds \\
& + l \sum_{j=1}^n \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(t, \tilde{\sigma}_{s_j}) u_j(t, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds \\
& - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(t, \tilde{\sigma}_{x_j}) d\bar{\sigma}_j d\bar{s}_j \\
& + l \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p(t, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds
\end{aligned}$$

$$-l \int_{x^0}^x \int_{x^0}^s f_i(t, \sigma) d\sigma ds, \quad i \in \{1, \dots, n\}, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.$$

Hence, using (3.3) and (Hyp0) for some constantes $\gamma, \delta > 0$,

$$\begin{aligned} \left| \frac{\partial}{\partial t} L_i^1(u, p)(t, x) \right| &\leq l \left| \frac{\partial}{\partial t} u_i(t, x) \right| + l \left| \int_{x^0}^x \int_{x^0}^s \left| \frac{\partial}{\partial t} u_i(t, \sigma) \right| d\sigma ds \right| \\ &\quad + l \sum_{j=1}^n \left| \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} |u_i(t, \tilde{\sigma}_{s_j}) u_j(t, \tilde{\sigma}_{s_j})| d\bar{\sigma}_j ds \right| \\ &\quad + l \left(m + \gamma \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right)^\delta \right) \sum_{j=1}^n \left| \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} |u_i(t, \tilde{\sigma}_{x_j})| d\bar{\sigma}_j d\bar{s}_j \right| + l \left| \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} |p(t, \tilde{\sigma}_{s_i})| d\bar{\sigma}_i ds \right| \\ &\quad + l \left| \int_{x^0}^x \int_{x^0}^s |f(t, \sigma)| d\sigma ds \right| \\ &\leq lQ + l(B^*)^2 Q + nl(B^*)^2 Q^2 + ln \left(m + \gamma Q^{2\delta-1} \right) (B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1 \\ &\leq l \left(Q + n(B^*)^2 Q^2 + (n(m + \gamma Q^{2\delta-1}) + 2)(B^*)^2 Q + (B^*)^2 N^1 \right) \\ &\leq Q, \quad i \in \{1, \dots, n\}, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \end{aligned}$$

3. Let $(u, p) \in K^1$. Then

$$\frac{\partial}{\partial t} L_{n+1}^1(u, p)(t, x) = -l \frac{\partial}{\partial t} p(t, x) + l \sum_{j=1}^n \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j(t, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds, \quad (t, x) \in [0, 1] \times \mathbb{R}^n,$$

and using (3.3), we get

$$\begin{aligned} \left| \frac{\partial}{\partial t} L_{n+1}^1(u, p)(t, x) \right| &\leq l \frac{\partial}{\partial t} |p(t, x)| + l \sum_{j=1}^n \left| \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} |u_j(t, \tilde{\sigma}_{s_j})| d\bar{\sigma}_j ds \right| \\ &\leq lQ + l(B^*)^2 Q \\ &= l \left(Q + (B^*)^2 Q \right) \\ &\leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \end{aligned}$$

4. Let $(u, p) \in K^1$ and $k \neq i, k, i \in \{1, \dots, n\}$. Then

$$\begin{aligned} \frac{\partial}{\partial x_k} L_i^1(u, p)(t, x) &= -l \frac{\partial}{\partial x_k} u_i(t, x) + l \int_{x^0}^x \int_{\bar{x}_k^0}^{\bar{s}_k} (u(t, \tilde{\sigma}_{s_k}) - u_{i0}(\tilde{\sigma}_{s_k})) d\bar{\sigma}_k ds \\ &\quad + l \sum_{j=1, j \neq k}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} u_i(\tau, \tilde{\sigma}_{s_k, s_j}) u_j(\tau, \tilde{\sigma}_{s_k, s_j}) d\bar{\sigma}_{k,j} ds d\tau \end{aligned}$$

$$\begin{aligned}
& + l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{s}_k^0}^{\bar{s}_k} u_i(\tau, \tilde{\sigma}_{x_k}) u_k(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau \\
& - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq k}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{s}_{k,j}^0}^{\bar{s}_{k,j}} u_i(\tau, \tilde{\sigma}_{s_k, x_j}) d\bar{\sigma}_{j,k} d\bar{s}_j d\tau \\
& - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{s}_k^0}^{\bar{s}_k} \frac{\partial}{\partial x_k} u_i(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau \\
& + l \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} p(\tau, \tilde{\sigma}_{s_k, s_i}) d\bar{\sigma}_{i,k} ds d\tau \\
& - l \int_0^t \int_{x^0}^x \int_{\bar{x}_k^0}^{\bar{s}_k} f_i(\tau, \tilde{\sigma}_{s_k}) d\bar{\sigma}_k ds d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

Again using (3.3) and (Hyp0) for some constants $\gamma, \delta > 0$, we obtain

$$\begin{aligned}
& \left| \frac{\partial}{\partial x_k} L_i^1(u, p)(t, x) \right| \leq l \frac{\partial}{\partial k} |u_i(t, x)| \\
& + l \left| \int_{x^0}^x \int_{\bar{x}_k^0}^{\bar{s}_k} |u_i(t, \tilde{\sigma}_{s_k}) - u_{i0}(\tilde{\sigma}_{s_k})| d\bar{\sigma}_k ds \right| \\
& + l \sum_{j=1, j \neq k}^n \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} |u_i(\tau, \tilde{\sigma}_{s_k, s_j}) u_j(\tau, \tilde{\sigma}_{s_k, s_j})| d\bar{\sigma}_{k,j} ds d\tau \right| \\
& + l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{s}_k^0}^{\bar{s}_m} |u_i(\tau, \tilde{\sigma}_{x_k}) u_k(\tau, \tilde{\sigma}_{x_k})| d\bar{\sigma}_k d\bar{s}_k d\tau \right| \\
& + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq k}^n \left| \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{s}_{k,j}^0}^{\bar{s}_{k,j}} |u_i(\tau, \tilde{\sigma}_{s_k, x_j})| d\bar{\sigma}_{j,k} d\bar{s}_j d\tau \right| \\
& + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{s}_k^0}^{\bar{s}_k} \left| \frac{\partial}{\partial x_k} u_i(\tau, \tilde{\sigma}_{x_k}) \right| d\bar{\sigma}_k d\bar{s}_k d\tau \right| \\
& + l \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} |p(\tau, \tilde{\sigma}_{s_k, s_i})| d\bar{\sigma}_{i,k} ds d\tau \right| + l \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_k^0}^{\bar{s}_k} |f(\tau, \tilde{\sigma}_{s_k})| d\bar{\sigma}_k ds d\tau \right| \\
& \leq lQ + 2l(B^*)^2 Q + (n-1)l(B^*)^2 Q^2 + l(B^*)^2 Q^2 \\
& + l(m + \gamma Q^{2\delta-1})(n-1)(B^*)^2 Q + (m + \gamma Q^{2\delta-1})l(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1
\end{aligned}$$

$$\leq l \left(Q + (3 + n(m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n(B^*)^2 Q^2 + (B^*)^2 N^1 \right)$$

$$\leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.$$

5. Let $(u, p) \in K^1$. Then

$$\begin{aligned} \frac{\partial}{\partial x_i} L_i^1(u, p)(t, x) &= -l \frac{\partial}{\partial x_i} u_i(t, x) l \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} (u_i(t, \tilde{\sigma}_{s_i}) - u_{i0}(\tilde{\sigma}_{s_i})) d\bar{\sigma}_i ds \\ &\quad + l \sum_{j=1, j \neq i}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} u_i(\tau, \tilde{\sigma}_{s_i, s_j}) u_j(\tau, \tilde{\sigma}_{s_i, s_j}) d\bar{\sigma}_{i,j} ds d\tau \\ &\quad + l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} (u_i(\tau, \tilde{\sigma}_{x_i}))^2 d\bar{\sigma}_i d\bar{s}_i d\tau \\ &\quad - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq i}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} u_i(\tau, \tilde{\sigma}_{s_i, x_j}) d\bar{\sigma}_{i,j} d\bar{s}_j d\tau \\ &\quad - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \frac{\partial}{\partial x_i} u_i(\tau, \tilde{\sigma}_{x_i}) d\bar{\sigma}_i d\bar{s}_i d\tau \\ &\quad + l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} p(\tau, \tilde{\sigma}_{x_i}) d\bar{\sigma}_i d\bar{s}_i d\tau \\ &\quad - l \int_0^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} f_i(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \end{aligned}$$

From here, by (Hyp0) for some constantes $\gamma, \delta > 0$

$$\begin{aligned} \left| \frac{\partial}{\partial x_i} L_i^1(u, p)(t, x) \right| &\leq l \left| \frac{\partial}{\partial x_i} u_i(t, x) \right| + l \left| \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} |u_i(t, \tilde{\sigma}_{s_i}) - u_{i0}(\tilde{\sigma}_{s_i})| d\bar{\sigma}_i ds \right| \\ &\quad + l \sum_{j=1, j \neq i}^n \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} |u_i(\tau, \tilde{\sigma}_{s_i, s_j}) u_j(\tau, \tilde{\sigma}_{s_i, s_j}) d\bar{\sigma}_{i,j} ds d\tau \right| \\ &\quad + l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} |u_i(\tau, \tilde{\sigma}_{x_i})|^2 d\bar{\sigma}_i d\bar{s}_i d\tau \right| \\ &\quad + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq i}^n \left| \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} |u_i(\tau, \tilde{\sigma}_{s_i, x_j})| d\bar{\sigma}_{i,j} d\bar{s}_j d\tau \right| \\ &\quad + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \left| \frac{\partial}{\partial x_i} u_i(\tau, \tilde{\sigma}_{x_i}) \right| d\bar{\sigma}_i d\bar{s}_i d\tau \right| \end{aligned}$$

$$\begin{aligned}
& + l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} |p(\tau, \tilde{\sigma}_{x_i})| d\bar{\sigma}_i d\bar{s}_i d\tau \right| \\
& + l \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} |f_i(\tau, \tilde{\sigma}_{s_i})| d\bar{\sigma}_i ds d\tau \right| \\
& \leq lQ + 2l(B^*)^2 Q + (n-1)l(B^*)^2 Q^2 + l(B^*)^2 Q^2 \\
& + (m + \gamma Q^{2\delta-1})l(n-1)(B^*)^2 Q + l(m + \gamma Q^{2\delta-1})(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1 \\
& \leq l \left(Q + (3 + n(m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n(B^*)^2 Q^2 + (B^*)^2 N^1 \right) \\
& \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

6. Let $(u, p) \in K^1$. Then

$$\begin{aligned}
\frac{\partial}{\partial x_k} L_{n+1}^1(u, p)(t, x) &= -l \frac{\partial}{\partial x_k} p(t, x) + l \sum_{j=1, j \neq k}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} u_j(\tau, \tilde{\sigma}_{s_k, s_j}) d\bar{\sigma}_{j,k} ds d\tau \\
&+ l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} u_k(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{\partial}{\partial x_k} L_{n+1}^1(u, p)(t, x) \right| &\leq l \left| \frac{\partial}{\partial x_k} p(t, x) \right| + l \sum_{j=1, j \neq k}^n \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} |u_j(\tau, \tilde{\sigma}_{s_k, s_j})| d\bar{\sigma}_{j,k} ds d\tau \right| \\
&+ l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} |u_k(\tau, \tilde{\sigma}_{x_k})| d\bar{\sigma}_k d\bar{s}_k d\tau \right| \\
&\leq lQ + l(n-1)(B^*)^2 Q + l(B^*)^2 Q \\
&\leq l \left(Q + n(B^*)^2 Q \right) \\
&\leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

7. Let $(u, p) \in K^1$ and $k \neq i, k, i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_k^2} L_i^1(u, p)(t, x) &= -l \frac{\partial^2}{\partial x_k^2} u_i(t, x) + l \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} (u_i(t, \tilde{\sigma}_{x_k}) - u_{i0}(\tilde{\sigma}_{x_k})) d\bar{\sigma}_k d\bar{s}_k \\
&+ l \sum_{j=1, j \neq k}^n \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} u_i(\tau, \tilde{\sigma}_{x_k, s_j}) u_j(\tau, \tilde{\sigma}_{x_k, s_j}) d\bar{\sigma}_{j,k} d\bar{s}_k d\tau
\end{aligned}$$

$$\begin{aligned}
& +l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} \frac{\partial}{\partial x_k} (u_i(\tau, \tilde{\sigma}_k) u_j(\tau, \tilde{\sigma}_{x_k})) d\bar{\sigma}_k d\bar{s}_k d\tau \\
& -l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \sum_{j=1, j \neq k}^n \int_0^t \int_{\bar{x}_{k,j}^0}^{\bar{x}_{k,j}} \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} u_i(\tau, \tilde{\sigma}_{x_k, x_j}) d\bar{\sigma}_{j,k} d\bar{s}_{j,k} d\tau \right. \\
& -l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} \frac{\partial^2}{\partial x_k^2} u_i(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau \right. \\
& +l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} p(\tau, \tilde{\sigma}_{x_k, s_i}) d\bar{\sigma}_{k,i} d\bar{s}_k d\tau \\
& \left. -l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} f_i(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \right.
\end{aligned}$$

Now, using (3.3) and (Hyp0) for some constants $\gamma, \delta > 0$,

$$\begin{aligned}
\left| \frac{\partial^2}{\partial x_k^2} L_i^1(u, p)(t, x) \right| & \leq l \left| \frac{\partial^2}{\partial x_k^2} u_i(t, x) \right| + l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} |u_i(t, \tilde{\sigma}_{x_k}) - u_{i0}(\tilde{\sigma}_{x_k})| d\bar{\sigma}_k d\bar{s}_k \right| \\
& +l \sum_{j=1, j \neq k}^n \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} |u_i(\tau, \tilde{\sigma}_{x_k, s_j}) u_j(\tau, \tilde{\sigma}_{x_k, s_j})| d\bar{\sigma}_{j,k} d\bar{s}_{j,k} d\tau \right| \\
& +l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} \frac{\partial}{\partial x_k} (u_i(\tau, \tilde{\sigma}_{x_k}) u_j(\tau, \tilde{\sigma}_{x_k})) d\bar{\sigma}_k d\bar{s}_k d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \sum_{j=1, j \neq k}^n \left| \int_0^t \int_{\bar{x}_{k,j}^0}^{\bar{x}_{k,j}} \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} |u_i(\tau, \tilde{\sigma}_{x_k, x_j})| d\bar{\sigma}_{j,k} d\bar{s}_{j,k} d\tau \right| \right. \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} \frac{\partial^2}{\partial x_k^2} u_i(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau \right| \right. \\
& +l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} |p(\tau, \tilde{\sigma}_{x_k, s_i})| d\bar{\sigma}_{k,i} d\bar{s}_k d\tau \right| \\
& \left. +l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} |f_i(\tau, \tilde{\sigma}_{x_k})| d\bar{\sigma}_k d\bar{s}_k d\tau \right| \right) \\
& \leq lQ + 2l(B^*)^2 Q + (n-1)l(B^*)^2 Q^2 + 2l(B^*)^2 Q^2 \\
& +ln(m + \gamma Q^{2\delta-1})(B^*)^2 Q + (m + \gamma Q^{2\delta-1})l(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1
\end{aligned}$$

$$\leq l \left(Q + (3 + (m + \gamma Q^{2\delta-1})(n+1))(B^*)^2 Q + (n+1)l(B^*)^2 Q^2 + (B^*)^2 N^1 \right) \\ \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.$$

8. Let $(u, p) \in K^1$. Then

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} L_i^1(u, p)(t, x) &= -l \frac{\partial^2}{\partial x_i^2} u_i(t, x) + l \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{s}_i^0}^{\bar{s}_i} (u_i(t, \tilde{\sigma}_{x_i}) - u_{i0}(\tilde{\sigma}_{x_i})) d\bar{\sigma}_i d\bar{s}_i \\ &+ \sum_{j=1, j \neq i}^n l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} u_i(\tau, \tilde{\sigma}_{x_i, s_j}) u_j(\tau, \tilde{\sigma}_{x_i, s_j}) d\bar{\sigma}_{i,j} d\bar{s}_{i,j} d\tau \\ &+ 2l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} u_i(\tau, \tilde{\sigma}_{x_i}) \frac{\partial}{\partial x_i} u_i(\tau, \tilde{\sigma}_{x_i}) d\bar{\sigma}_i d\bar{s}_i d\tau \\ &- l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq i}^n \int_0^t \int_{\bar{x}_{i,j}^0}^{\bar{x}_{i,j}} \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} u_i(\tau, \tilde{\sigma}_{x_i, x_j}) d\bar{\sigma}_{i,j} d\bar{s}_{i,j} d\tau \\ &- l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \frac{\partial^2}{\partial x_i^2} u_i(\tau, \tilde{\sigma}_{x_i}) d\bar{\sigma}_i d\bar{s}_i d\tau \\ &+ l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \frac{\partial}{\partial x_i} p(\tau, \tilde{\sigma}_{x_i}) d\bar{\sigma}_i d\bar{s}_i d\tau \\ &- l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} f_i(\tau, \tilde{\sigma}_{x_i}) d\bar{\sigma}_i d\bar{s}_i d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \end{aligned}$$

Using (3.3), we arrive to the following estimate

$$\begin{aligned} \left| \frac{\partial^2}{\partial x_i^2} L_i^1(u, p)(t, x) \right| &\leq l \left| \frac{\partial^2}{\partial x_i^2} p_i(t, x) \right| + l \left| \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{s}_i^0}^{\bar{s}_i} |u_i(t, \tilde{\sigma}_{x_i}) - u_{i0}(\tilde{\sigma}_{x_i})| d\bar{\sigma}_i d\bar{s}_i \right| \\ &+ \sum_{j=1, j \neq i}^n l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} |u_i(\tau, \tilde{\sigma}_{x_i, s_j}) u_j(\tau, \tilde{\sigma}_{x_i, s_j})| d\bar{\sigma}_{i,j} d\bar{s}_{i,j} d\tau \right| \\ &+ 2l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \left| u_i(\tau, \tilde{\sigma}_{x_i}) \frac{\partial}{\partial x_i} u_i(\tau, \tilde{\sigma}_{x_i}) \right| d\bar{\sigma}_i d\bar{s}_i d\tau \right| \\ &+ l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq i}^n \left| \int_0^t \int_{\bar{x}_{i,j}^0}^{\bar{x}_{i,j}} \int_{\bar{x}_{i,j}^0}^{\bar{s}_{i,j}} |u_i(\tau, \tilde{\sigma}_{x_i, x_j})| d\bar{\sigma}_{i,j} d\bar{s}_{i,j} d\tau \right| \\ &+ l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \left| \frac{\partial^2}{\partial x_i^2} u_i(\tau, \tilde{\sigma}_{x_i}) \right| d\bar{\sigma}_i d\bar{s}_i d\tau \right| \end{aligned}$$

$$\begin{aligned}
& +l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} \left| \frac{\partial}{\partial x_i} p(\tau, \tilde{\sigma}_{x_i}) \right| d\bar{\sigma}_i d\bar{s}_i d\tau \right| \\
& +l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_i^0}^{\bar{s}_i} |f_i(\tau, \tilde{\sigma}_{x_i})| d\bar{\sigma}_i d\bar{s}_i d\tau \right| \\
& \leq lQ + 2l(B^*)^2 Q + (n-1)l(B^*)^2 Q^2 + 2l(B^*)^2 Q^2 \\
& +l(m + \gamma Q^{2\delta-1})(n-1)(B^*)^2 Q + l(m + \gamma Q^{2\delta-1})(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1 \\
& \leq l \left(Q + (3 + n(m + \gamma Q^{2\delta-1}))(B^*)^2 Q + (n+1)(B^*)^2 Q^2 + (B^*)^2 N^1 \right) \\
& \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

9. Let $(u, p) \in K^1$ and $r \neq k, k \neq i, r \neq i, r, k, i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_r \partial x_k} L_i^1(u, p)(t, x) &= -l \frac{\partial^2}{\partial x_k \partial x_r} u_i(t, x) + l \int_{x^0}^x \int_{\bar{x}_{r,k}^0}^{\bar{s}_{r,k}} (u_i(t, \tilde{\sigma}_{s_r, s_k}) - u_{i0}(\tilde{\sigma}_{s_r, s_k})) d\bar{\sigma}_{r,k} ds \\
&+ l \sum_{j=1, j \neq k, j \neq r}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{r,k,j}^0}^{\bar{s}_{r,k,j}} u_i(\tau, \tilde{\sigma}_{s_r, s_k, s_j}) u_j(\tau, \tilde{\sigma}_{s_r, s_k, s_j}) d\bar{\sigma}_{r,k,j} ds d\tau \\
&+ l \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{r,k}^0}^{\bar{s}_{r,k}} u_i(\tau, \tilde{\sigma}_{x_r, s_k}) u_r(\tau, \tilde{\sigma}_{x_r, s_k}) d\bar{\sigma}_{r,k} d\bar{s}_r d\tau \\
&+ l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{r,k}^0}^{\bar{s}_{r,k}} u_i(\tau, \tilde{\sigma}_{s_r, x_k}) u_k(\tau, \tilde{\sigma}_{s_r, x_k}) d\bar{\sigma}_{r,k} d\bar{s}_k d\tau \\
&- l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq r, j \neq k}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{r,k,j}^0}^{\bar{s}_{r,k,j}} u_i(\tau, \tilde{\sigma}_{s_r, s_k, x_j}) d\bar{\sigma}_{j,k,r} d\bar{s}_j d\tau \\
&- l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{r,k}^0}^{\bar{s}_{r,k}} \frac{\partial}{\partial x_r} u_i(\tau, \tilde{\sigma}_{s_k, x_r}) d\bar{\sigma}_{r,k} d\bar{s}_r d\tau \\
&- l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{r,k}^0}^{\bar{s}_{r,k}} \frac{\partial^2}{\partial x_r \partial x_k} u_i(\tau, \tilde{\sigma}_{s_r, x_k}) d\bar{\sigma}_{k,r} d\bar{s}_k d\tau \\
&+ l \int_0^t \int_{x^0}^x \int_{\bar{x}_{r,k,i}^0}^{\bar{s}_{r,k,i}} p(\tau, \tilde{\sigma}_{s_r, s_k, s_i}) d\bar{\sigma}_{r,k,i} ds d\tau
\end{aligned}$$

$$-l \int_0^t \int_{x_0}^x \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} f_i(\tau, \tilde{\sigma}_{s_r, s_k}) d\bar{\sigma}_{k,r} ds d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n,$$

and

$$\begin{aligned} & \left| \frac{\partial^2}{\partial x_r \partial x_k} L_i^1(u, p)(t, x) \right| \leq l \left| \frac{\partial^2}{\partial x_k \partial x_r} u_i(t, x) \right| + l \left| \int_0^t \int_{x_0}^x \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} |u_i(\tau, \tilde{\sigma}_{s_r, s_k}) - u_{i0}(\tilde{\sigma}_{s_r, s_k})| d\bar{\sigma}_{r,k} ds \right| \\ & + l \sum_{j=1, j \neq k, j \neq r}^n \left| \int_0^t \int_{x_0}^x \int_{\bar{x}_{r,k,j}^0}^{\bar{x}_{r,k,j}} |u_i(\tau, \tilde{\sigma}_{s_r, s_k, s_j}) u_j(\tau, \tilde{\sigma}_{s_r, s_k, s_j})| d\bar{\sigma}_{r,k,j} ds d\tau \right| \\ & + l \left| \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} |u_i(\tau, \tilde{\sigma}_{x_r, s_k}) u_r(\tau, \tilde{\sigma}_{x_r, s_k})| d\bar{\sigma}_{r,k} d\bar{s}_r d\tau \right| \\ & + l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} |u_i(\tau, \tilde{\sigma}_{s_r, x_k}) u_k(\tau, \tilde{\sigma}_{s_r, x_k})| d\bar{\sigma}_{r,k} d\bar{s}_k d\tau \right| \\ & + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq r, j \neq k}^n \left| \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{r,k,j}^0}^{\bar{x}_{r,k,j}} |u_i(\tau, \tilde{\sigma}_{s_r, s_k, x_j})| d\bar{\sigma}_{j,k,r} d\bar{s}_j d\tau \right| \\ & + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} \left| \frac{\partial}{\partial x_r} u_i(\tau, \tilde{\sigma}_{s_k, x_r}) \right| d\bar{\sigma}_{r,k} d\bar{s}_r d\tau \right| \\ & + l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} \left| \frac{\partial^2}{\partial x_r \partial x_k} u_i(\tau, \tilde{\sigma}_{s_r, x_k}) \right| d\bar{\sigma}_{k,r} d\bar{s}_k d\tau \right| \\ & + l \left| \int_0^t \int_{x_0}^x \int_{\bar{x}_{r,k,i}^0}^{\bar{x}_{r,k,i}} |p(\tau, \tilde{\sigma}_{s_r, s_k, s_i})| d\bar{\sigma}_{r,k,i} ds d\tau \right| \\ & + l \left| \int_0^t \int_{x_0}^x \int_{\bar{x}_{r,k}^0}^{\bar{x}_{r,k}} |f_i(\tau, \tilde{\sigma}_{s_r, s_k})| d\bar{\sigma}_{k,r} ds d\tau \right| \\ & \leq lQ + 2l(B^*)^2 Q + (n-2)l(B^*)^2 Q^2 + l(B^*)^2 Q^2 \\ & + l(B^*)^2 Q^2 + l(m + \gamma Q^{2\delta-1})(n-2)(B^*)^2 Q + l(m + \gamma Q^{2\delta-1})(B^*)^2 Q \\ & + l(m + \gamma Q^{2\delta-1})(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1 \\ & \leq l \left(Q + (3 + n(m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n(B^*)^2 Q^2 + (B^*)^2 N^1 \right) \\ & \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \end{aligned}$$

10. Let $(u, p) \in K^1$ and $k \neq i, i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
 \frac{\partial^2 L_i^1}{\partial x_i \partial x_k}(u, p)(t, x) = & -l \frac{\partial^2}{\partial x_i \partial x_k} u_i(t, x) + l \int_{x^0}^x \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} (u_i(t, \tilde{\sigma}_{s_i, s_k}) - u_{i0}(\tilde{\sigma}_{s_i, s_k})) d\bar{\sigma}_{k,i} ds \\
 & + l \sum_{j=1, j \neq k, j \neq i}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,k,j}^0}^{\bar{s}_{i,k,j}} u_i(\tau, \tilde{\sigma}_{s_i, s_k, s_j}) u_j(\tau, \tilde{\sigma}_{s_i, s_k, s_j}) d\bar{\sigma}_{j,k,i} ds d\tau \\
 & + l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} (u_i(\tau, \tilde{\sigma}_{s_k, x_i}))^2 d\bar{\sigma}_{i,k} d\bar{s}_i d\tau \\
 & + l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} u_i(\tau, \tilde{\sigma}_{s_i, x_k}) u_j(\tau, \tilde{\sigma}_{s_i, x_k}) d\bar{\sigma}_{k,i} d\bar{s}_k d\tau \\
 & - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq k, j \neq i}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{i,k,j}^0}^{\bar{s}_{i,k,j}} u_i(\tau, \tilde{\sigma}_{s_i, s_k, x_j}) d\bar{\sigma}_{j,k,i} d\bar{s}_j d\tau \\
 & - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} \frac{\partial u_i}{\partial x_i}(\tau, \tilde{\sigma}_{s_k, x_i}) d\bar{\sigma}_{i,k} d\bar{s}_i d\tau \\
 & - l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} \frac{\partial}{\partial x_k} u_i(\tau, \tilde{\sigma}_{s_i, x_k}) d\bar{\sigma}_{k,i} d\bar{s}_k d\tau \\
 & + l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} p(\tau, \tilde{\sigma}_{s_k, x_i}) d\bar{\sigma}_{i,k} d\bar{s}_i d\tau \\
 & - l \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,mk}^0}^{\bar{s}_{i,k}} f_i(\tau, \tilde{\sigma}_{s_i, s_k}) d\bar{\sigma}_{i,k} ds d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
 \end{aligned}$$

From here and from (3.3) and (Hyp0) for some constants $\gamma, \delta > 0$, we obtain

$$\begin{aligned}
 \left| \frac{\partial^2 L_i^1}{\partial x_i \partial x_k}(u, p)(t, x) \right| \leq & l \left| \frac{\partial^2}{\partial x_i \partial x_k} u_i(t, x) \right| + l \left| \int_{x^0}^x \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} |u_i(t, \tilde{\sigma}_{s_i, s_k}) - u_{i0}(\tilde{\sigma}_{s_i, s_k})| d\bar{\sigma}_{k,i} ds \right| \\
 & + l \sum_{j=1, j \neq k, j \neq i}^n \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,k,j}^0}^{\bar{s}_{i,k,j}} |u_i(\tau, \tilde{\sigma}_{s_i, s_k, s_j}) u_j(\tau, \tilde{\sigma}_{s_i, s_k, s_j})| d\bar{\sigma}_{j,k,i} ds d\tau \right| \\
 & + l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} (u_i(\tau, \tilde{\sigma}_{s_k, x_i}))^2 d\bar{\sigma}_{i,k} d\bar{s}_i d\tau \right|
 \end{aligned}$$

$$\begin{aligned}
& +l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} |u_i(\tau, \tilde{\sigma}_{s_i, x_k}) u_j(\tau, \tilde{\sigma}_{s_i, x_k})| d\bar{\sigma}_{k,i} d\bar{s}_k d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq k, j \neq i}^n \left| \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{i,k,j}^0}^{\bar{s}_{i,k,j}} |u_i(\tau, \tilde{\sigma}_{s_i, s_k, x_j})| d\bar{\sigma}_{j,k,i} d\bar{s}_j d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} \left| \frac{\partial u_i}{\partial x_i}(\tau, \tilde{\sigma}_{s_k, x_i}) \right| d\bar{\sigma}_{i,k} d\bar{s}_i d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} \left| \frac{\partial}{\partial x_k} u_i(\tau, \tilde{\sigma}_{s_i, x_k}) \right| d\bar{\sigma}_{k,i} d\bar{s}_k d\tau \right| \\
& +l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{k,i}^0}^{\bar{s}_{k,i}} |p(\tau, \tilde{\sigma}_{s_k, x_i})| d\bar{\sigma}_{i,k} d\bar{s}_i d\tau \right| \\
& +l \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,k}^0}^{\bar{s}_{i,k}} |f_i(\tau, \tilde{\sigma}_{s_i, s_k})| d\bar{\sigma}_{i,k} ds d\tau \right| \\
& \leq lQ + 2l(B^*)^2 Q + (n-2)l(B^*)^2 Q^2 + l(B^*)^2 Q^2 \\
& \quad + l(B^*)^2 Q^2 + l(m + \gamma Q^{2\delta-1})(B^*)^2 Q \\
& \quad + l(m + \gamma Q^{2\delta-1})(n-2)(B^*)^2 Q + ml(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1 \\
& \leq l \left(Q + (2 + n(m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n(B^*)^2 Q^2 + (B^*)^2 N^1 \right) \\
& \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

11. Let $(w, p) \in K^1$ and $r \neq i, k = i, k, i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_r} L_i^1(u, p)(t, x) &= -l \frac{\partial^2}{\partial x_i \partial x_r} u_i(t, x) + l \int_{x^0}^x \int_{\bar{x}_{r,i}^0}^{\bar{s}_{r,i}} (u_i(t, \tilde{\sigma}_{s_r, s_i}) - u_{i0}(\tilde{\sigma}_{s_r, s_i})) d\bar{\sigma}_{i,r} ds \\
& +l \sum_{j=1, j \neq i, j \neq r}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{r,i,j}^0}^{\bar{s}_{r,i,j}} u_i(\tau, \tilde{\sigma}_{s_r, s_i, s_j}) u_j(\tau, \tilde{\sigma}_{s_r, s_i, s_j}) d\bar{\sigma}_{j,i,r} ds d\tau \\
& +l \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} u_i(\tau, \tilde{\sigma}_{s_i, x_r}) u_j(\tau, \tilde{\sigma}_{s_i, x_r}) d\bar{\sigma}_{r,i} d\bar{s}_r d\tau \\
& +l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{r,i}^0}^{\bar{s}_{r,i}} (u_i(\tau, \tilde{\sigma}_{s_r, x_i}))^2 d\bar{\sigma}_{i,r} d\bar{s}_i d\tau
\end{aligned}$$

$$\begin{aligned}
& -l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq i, j \neq r}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{r,i,j}^0}^{\bar{s}_{r,i,j}} u_i(\tau, \tilde{\sigma}_{s_r, s_i, x_j}) d\bar{\sigma}_{r,i,j} d\bar{s}_j d\tau \\
& -l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} \frac{\partial}{\partial x_r} u_i(\tau, \tilde{\sigma}_{s_i, x_r}) d\bar{\sigma}_{r,i} d\bar{s}_r d\tau \\
& -l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} \frac{\partial}{\partial x_i} u_i(\tau, \tilde{\sigma}_{x_i, s_r}) d\bar{\sigma}_{r,i} d\bar{s}_i d\tau \\
& +l \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} p(\tau, \tilde{\sigma}_{x_i, s_r}) d\bar{\sigma}_{i,r} d\bar{s}_i d\tau \\
& -l \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} f_i(\tau, \tilde{\sigma}_{s_i, s_r}) d\bar{\sigma}_{i,r} ds d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
& \left| \frac{\partial^2}{\partial x_i \partial x_r} L_i^1(u, p)(t, x) \right| \leq l \left| \frac{\partial^2}{\partial x_i \partial x_r} u_i(t, x) \right| + l \left| \int_{x^0}^x \int_{\bar{x}_{r,i}^0}^{\bar{s}_{r,i}} |u_i(t, \tilde{\sigma}_{s_r, s_i}) - u_{i0}(\tilde{\sigma}_{s_r, s_i})| d\bar{\sigma}_{i,r} ds \right| \\
& +l \sum_{j=1, j \neq i, j \neq r}^n \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{r,i,j}^0}^{\bar{s}_{r,i,j}} |u_i(\tau, \tilde{\sigma}_{s_r, s_i, s_j}) u_j(\tau, \tilde{\sigma}_{s_r, s_i, s_j})| d\bar{\sigma}_{j,i,r} ds d\tau \right| \\
& +l \left| \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} |u_i(\tau, \tilde{\sigma}_{s_i, x_r}) u_j(\tau, \tilde{\sigma}_{s_i, x_r})| d\bar{\sigma}_{r,i} d\bar{s}_r d\tau \right| \\
& +l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} (u_i(\tau, \tilde{\sigma}_{s_r, x_i}))^2 d\bar{\sigma}_{i,r} d\bar{s}_i d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1, j \neq i, j \neq r}^n \left| \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_{r,i,j}^0}^{\bar{s}_{r,i,j}} |u_i(\tau, \tilde{\sigma}_{s_r, s_i, x_j})| d\bar{\sigma}_{r,i,j} d\bar{s}_j d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} \left| \frac{\partial}{\partial x_r} u_i(\tau, \tilde{\sigma}_{s_i, x_r}) \right| d\bar{\sigma}_{r,i} d\bar{s}_r d\tau \right| \\
& +l \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} \left| \frac{\partial}{\partial x_i} u_i(\tau, \tilde{\sigma}_{x_i, s_r}) \right| d\bar{\sigma}_{r,i} d\bar{s}_i d\tau \right| \\
& +l \left| \int_0^t \int_{\bar{x}_i^0}^{\bar{x}_i} \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} |p(\tau, \tilde{\sigma}_{x_i, s_r})| d\bar{\sigma}_{i,r} d\bar{s}_i d\tau \right|
\end{aligned}$$

$$\begin{aligned}
& + l \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{i,r}^0}^{\bar{s}_{i,r}} |f_i(\tau, \tilde{\sigma}_{s_i, s_r})| d\bar{\sigma}_{i,r} ds d\tau \right| \\
& \leq lQ + 2l(B^*)^2 Q + (n-2)l(B^*)^2 Q^2 \\
& \quad + l(B^*)^2 Q^2 + l(B^*)^2 Q^2 + l(m + \gamma Q^{2\delta-1})(n-2)(B^*)^2 Q \\
& \quad + l(m + \gamma Q^{2\delta-1})(B^*)^2 Q \\
& \quad + (m + \gamma Q^{2\delta-1})l(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 N^1 \\
& \leq l \left(Q + (3 + (m + \gamma Q^{2\delta-1})n)(B^*)^2 Q^2 + n(B^*)^2 Q^2 + (B^*)^2 N^1 \right) \\
& \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

12. Let $(u, p) \in K^1$ and $k \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_k^2} L_{n+1}^1(u, p)(t, x) &= -l \frac{\partial^2}{\partial x_k^2} p(t, x) + l \sum_{j=1, j \neq k}^n \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} u_j(\tau, \tilde{\sigma}_{x_k, s_j}) d\bar{\sigma}_{j,k} d\bar{s}_k d\tau \\
& \quad + l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} \frac{\partial}{\partial x_k} u_k(\tau, \tilde{\sigma}_{x_k}) d\bar{\sigma}_k d\bar{s}_k d\tau, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

Using the last equality and (3.3), we go to

$$\begin{aligned}
\left| \frac{\partial^2}{\partial x_k^2} L_{n+1}^1(u, p)(t, x) \right| &\leq l \left| \frac{\partial^2}{\partial x_k^2} p(t, x) \right| + l \sum_{j=1, j \neq k}^n \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,j}^0}^{\bar{s}_{k,j}} |u_j(\tau, \tilde{\sigma}_{x_k, s_j})| d\bar{\sigma}_{j,k} d\bar{s}_k d\tau \right| \\
& \quad + l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_k^0}^{\bar{s}_k} \left| \frac{\partial}{\partial x_k} u_k(\tau, \tilde{\sigma}_{x_k}) \right| d\bar{\sigma}_k d\bar{s}_k d\tau \right| \\
& \leq lQ + l(n-1)(B^*)^2 Q + l(B^*)^2 Q \\
& = l \left(Q + n(B^*)^2 Q \right) \\
& \leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n.
\end{aligned}$$

13. Let $(u, p) \in K^1$ and $k \neq r$, $k, r \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_k \partial x_r} L_{n+1}^1(u, p)(t, x) &= -l \frac{\partial^2}{\partial x_k \partial x_r} p(t, x) + l \sum_{j=1, j \neq k, j \neq r}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,j,r}^0}^{\bar{s}_{k,j,r}} u_j(\tau, \tilde{\sigma}_{s_k, s_j, s_r}) d\bar{\sigma}_{j,k,r} ds d\tau \\
& \quad + l \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{k,r}^0}^{\bar{s}_{k,r}} u_j(\tau, \tilde{\sigma}_{s_k, x_r}) d\bar{\sigma}_{k,r} d\bar{s}_r d\tau
\end{aligned}$$

$$+l \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,r}^0}^{\bar{s}_{k,r}} u_k(\tau, \tilde{\sigma}_{x_k, s_r}) d\bar{\sigma}_{k,r} d\bar{s}_k d\tau. \quad (t, x) \in [0, 1] \times \mathbb{R}^n.$$

From here,

$$\begin{aligned} \left| \frac{\partial^2}{\partial x_k \partial x_r} L_{n+1}^1(u, p)(t, x) \right| &\leq l \left| \frac{\partial^2}{\partial x_k \partial x_r} p(t, x) \right| + l \sum_{j=1, j \neq k, j \neq r}^n \left| \int_0^t \int_{x^0}^x \int_{\bar{x}_{k,j,r}^0}^{\bar{s}_{k,j,r}} |u_j(\tau, \tilde{\sigma}_{s_k, s_j, s_r})| d\bar{\sigma}_{j,k,r} ds d\tau \right| \\ &\quad + l \left| \int_0^t \int_{\bar{x}_r^0}^{\bar{x}_r} \int_{\bar{x}_{k,r}^0}^{\bar{s}_{k,r}} |u_j(\tau, \tilde{\sigma}_{s_k, x_r})| d\bar{\sigma}_{k,r} d\bar{s}_r d\tau \right| \\ &\quad + l \left| \int_0^t \int_{\bar{x}_k^0}^{\bar{x}_k} \int_{\bar{x}_{k,r}^0}^{\bar{s}_{k,r}} |u_k(\tau, \tilde{\sigma}_{x_k, s_r})| d\bar{\sigma}_{k,r} d\bar{s}_k d\tau \right| \\ &\leq lQ + l(n-2)(B^*)^2 Q + l(B^*)^2 Q + l(B^*)^2 Q \\ &= l(Q + n(B^*)^2 Q) \\ &\leq Q, \quad (t, x) \in [0, 1] \times \mathbb{R}^n. \end{aligned}$$

From (1)-(13) it follows that $L^1 : K^1 \rightarrow K^1$ and it is continuous. Since K^1 is a compact subset of Q^1 , we have that $L^1(K^1)$ resides in a compact subset of Q^1 . This completes the proof. $\square$

Proposition 3.3. $\bar{N}^1 : K^1 \rightarrow Q^1$ is an expansive operator and onto.

Proof. For $U = (u, p)$, $\tilde{U} = (\tilde{u}, \tilde{p}) \in K^1$ we have

$$\|\bar{N}^1(U) - \bar{N}^1(\tilde{U})\| = (1 + l)\|U - \tilde{U}\|,$$

i.e., $\bar{N}^1 : K^1 \rightarrow Q^1$ is an expansive operator with $h = 1 + l$. Let $(\tilde{u}, \tilde{p}) \in Q^1$. Then $(\frac{\tilde{u}}{1+l}, \frac{\tilde{p}}{1+l}) \in K^1$ and

$$\bar{N}^1 \left(\frac{\tilde{u}}{1+l}, \frac{\tilde{p}}{1+l} \right) = (\tilde{u}, \tilde{p}),$$

i.e., the operator $\bar{N}^1 : K^1 \rightarrow Q^1$ is onto. This completes the proof. $\square$

By Proposition 3.2, Proposition 3.3 and Theorem 2.4, it follows that the operator $L^1 + \bar{N}^1$ has a fixed point $(u^1, p^1) \in K^1$. For it we have

$$\begin{aligned} &\int_{x^0}^x \int_{\sigma^0}^s (u_i^1(t, \sigma) - u_{i0}(\sigma)) d\sigma ds + \sum_{j=1}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i^1(\tau, \tilde{\sigma}_{s_j}) u_j^1(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\ &\quad - \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_0^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} u_i^1(\tau, \tilde{\sigma}_{x_j}) d\bar{\sigma}_j d\bar{s}_j d\tau \\ &\quad + \int_0^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p^1(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau \end{aligned}$$

$$= \int_0^t \int_{x^0}^x \int_{x^0}^s f_i(\tau, \sigma) d\sigma ds d\tau, \quad i \in \{1, \dots, n\},$$

$$\sum_{j=1}^n \int_0^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j^1(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau = 0, \quad t \in [0, 1], \quad x \in \mathbb{R}^n.$$

Hence and Lemma 2.1, it follows that (u^1, p^1) is a $\mathcal{C}^1([0, 1], \mathcal{C}_0^2(\mathbb{R}^n))$ solution of the problem (3.1), (3.2) for which $\text{supp}_x p^1, \text{supp}_x u_i^1 \subset B, i \in \{1, \dots, n\}$.

Now we consider the Cauchy problem

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} - \left(m + M \left(\int_{\mathbb{R}^n} |\nabla_x u_i|^2 dx \right) \right) \Delta u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), & i \in \{1, \dots, n\}, \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}(t, x) = 0 & \text{in } [1, 2] \times \mathbb{R}^n, \end{cases} \quad (3.4)$$

$$u(1, x) = u^1(1, x) \quad \text{in } \mathbb{R}^n. \quad (3.5)$$

Let

$$M^2 = \max_{t \in [1, 2], x \in B} |f(t, x)|.$$

We choose the constant $l_1 > 0$ such that

$$l_1 n \left(Q + 3n(1 + (m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n^2 Q^2 (B^*)^2 + (B^*)^2 N^2 \right) \leq Q.$$

Let

$$E^2 = \left\{ v = (v_1, \dots, v_{n+1}) : v_i \in \mathcal{C}^1([1, 2], \mathcal{C}_0^2(\mathbb{R}^n)), \text{supp}_x v_i \subset B, i \in \{1, \dots, n+1\} \right\}$$

be endowed with the norm

$$\begin{aligned} \|v\| = & \max_{q \in \{1, \dots, n+1\}} \left\{ \max_{t \in [1, 2], x \in B} |v_q(t, x)|, \max_{t \in [1, 2], x \in B} |v_{qt}(t, x)|, \max_{t \in [1, 2], x \in B} |v_{qx_i}(t, x)|, \right. \\ & \left. \max_{t \in [1, 2], x \in B} |v_{qx_i x_j}(t, x)|, i, j \in \{1, \dots, n\} \right\}. \end{aligned}$$

With $\tilde{K}^2$ we denote the set of all equi-continuous families in E^2 . Let also,

$$\overline{K}^2 = \overline{\tilde{K}^2}, \quad K^2 = \{v \in \overline{K}^2 : \|v\| \leq Q\},$$

$$Q^2 = \{v \in \overline{K}^2 : \|v\| \leq (1 + l_1)Q\}.$$

Note that K^2 is a compact subset of Q^2 . For $(u, p) \in Q^2$ we define the operators.

$$\begin{aligned} L_i^2(u, p)(t, x) = & -l_1 u_i(t, x) + l_1 \int_{x^0}^x \int_{x^0}^s \left(u_i(t, \sigma) - u_i^1(1, \sigma) \right) d\sigma ds \\ & + l_1 \sum_{j=1}^n \int_1^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{s_j}) u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\ & - l_1 \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_1^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{x_j}) d\bar{\sigma}_j d\bar{s}_j d\tau \end{aligned}$$

$$\begin{aligned}
& +l_1 \int_1^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau \\
& -l_1 \int_1^t \int_{x^0}^x \int_{x^0}^s f_i(\tau, \sigma) d\sigma ds d\tau, \\
L_{n+1}^2(u, p)(t, x) &= -l_1 p(t, x) + l_1 \sum_{j=1}^n \int_1^t \int_1^y \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau dy \\
& +l_1(t-1) \int_{x^0}^x \int_{x^0}^s (p_t(1, \sigma) - p_t^1(1, \sigma)) d\sigma ds \\
& +l_1 \int_{x^0}^x \int_{x^0}^s (p(1, \sigma) - p^1(1, \sigma)) d\sigma ds,
\end{aligned}$$

$$M_i^2(u, p)(t, x) = (1 + l_1)u_i(t, s), \quad i \in \{1, \dots, n\},$$

$$M_{n+1}^2(u, p)(t, x) = (1 + l_1)p(t, x),$$

$$L^2(u, p)(t, x) = \left(L_1^2(u, p)(t, x), \dots, L_{n+1}^2(u, p)(t, x) \right),$$

$$\overline{M}^2(u, p)(t, x) = \left(M_1^2(u, p)(t, x), \dots, M_{n+1}^2(u, p)(t, x) \right),$$

$$(t, x) \in [1, 2] \times \mathbb{R}^n.$$

As in above, we obtain that the operator $L^2 + \overline{M}^2$ has a fixed point $(u^2, p^2) \in K^2$. For it we have

$$\begin{aligned}
0 &= \int_{x^0}^x \int_{x^0}^s (u_i^2(t, \sigma) - u_i^1(1, \sigma)) d\sigma ds \\
&+ \sum_{j=1}^n \int_1^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i^2(\tau, \tilde{\sigma}_{s_j}) u_j^2(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\
&- \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_1^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i^2(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\
&+ \int_1^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p^2(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau \\
&- \int_1^t \int_{x^0}^x \int_{x^0}^s f_i(\tau, \sigma) d\sigma ds d\tau, \quad i \in \{1, \dots, n\}, \\
&= \sum_{j=1}^n \int_1^t \int_1^y \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j^2(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau dy \\
&+ (t-1) \int_{x^0}^x \int_{x^0}^s (p_t^2(1, \sigma) - p_t^1(1, \sigma)) d\sigma ds \\
&+ \int_{x^0}^x \int_{x^0}^s (p^2(1, \sigma) - p^1(1, \sigma)) d\sigma ds, \quad (t, x) \in [1, 2] \times \mathbb{R}^n.
\end{aligned} \tag{3.6}$$

We differentiate the first n equations of the system (3.6) once in t , twice in x_1 , and so on, twice in x_n , and we arrive to

$$\frac{\partial}{\partial t} u_i^2 + \sum_{j=1}^n \frac{\partial}{\partial x_j} (u_i^2 u_j^2) - \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u_i^2 + \frac{\partial}{\partial x_i} p^2 = f_i(t, x), \quad (3.7)$$

$i \in \{1, \dots, n\}, (t, x) \in [1, 2] \times \mathbb{R}^n$.

Now we differentiate the last equation of (3.6) twice in t , twice in x_1 , and so on, twice in x_n , and we get

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} u_j^2(t, x) = 0, \quad (t, x) \in [1, 2] \times \mathbb{R}^n.$$

For $i \in \{1, \dots, n\}$ we put in (3.6) $t = 1$ and we find

$$\int_{x^0}^x \int_{x^0}^s (u_i^2(1, \sigma) - u_i^1(1, \sigma)) d\sigma ds = 0,$$

which we differentiate twice in x_1 , and so on, twice in x_n , and we find

$$u_i^2(1, x) = u_i^1(1, x), \quad x \in \mathbb{R}^n, \quad (3.8)$$

whereupon

$$\frac{\partial}{\partial x_j} u_i^2(1, x) = \frac{\partial}{\partial x_j} u_i^1(1, x), \quad x \in \mathbb{R}^n, \quad (3.9)$$

and

$$\frac{\partial^2}{\partial x_m \partial x_r} u_i^2(1, x) = \frac{\partial^2}{\partial x_m \partial x_r} u_i^1(1, x), \quad i, m, r \in \{1, \dots, n\}, \quad x \in \mathbb{R}^n. \quad (3.10)$$

In the $(n+1)$ -th equation of the system (3.6) we put $t = 1$ and we find

$$\int_{x^0}^x \int_{x^0}^s (p^2(1, \sigma) - p^1(1, \sigma)) d\sigma ds = 0,$$

which we differentiate twice in x_1 , and so on, twice in x_n , and we find

$$p^2(1, x) = p^1(1, x), \quad x \in \mathbb{R}^n.$$

Hence,

$$\frac{\partial p^2}{\partial x_i}(1, x) = \frac{\partial p^1}{\partial x_i}(1, x), \quad x \in \mathbb{R}^n, \quad i \in \{1, \dots, n\}, \quad (3.11)$$

and

$$\frac{\partial^2 p^2}{\partial x_m \partial x_r}(1, x) = \frac{\partial^2 p^1}{\partial x_m \partial x_r}(1, x), \quad x \in \mathbb{R}^n, \quad r, m \in \{1, \dots, n\}. \quad (3.12)$$

Now we differentiate in t the $n+1$ -th equation of the system (3.6) and we get

$$\sum_{j=1}^n \int_1^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j^2(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau + \int_{x^0}^x \int_{x^0}^s (p_t^2(1, \sigma) - p_t^1(1, \sigma)) d\sigma ds = 0,$$

in which we put $t = 1$ and we get

$$\int_{x^0}^x \int_{x^0}^s (p_t^2(1, \sigma) - p_t^1(1, \sigma)) d\sigma ds = 0.$$

The last equation we differentiate twice in x_1 , and so on, twice in x_n , and we get

$$p_t^2(1, x) = p_t^1(1, x), \quad x \in \mathbb{R}^n. \quad (3.13)$$

By (3.8)-(3.13), using (3.7), we obtain

$$\begin{aligned}
 \frac{\partial}{\partial t} u_i^2(1, x) &\leq - \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(u_i^2(1, x) u_j^2(1, x) \right) \\
 &\quad + \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i(1, x)}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u_i^2(1, x) - \frac{\partial p^2}{\partial x_i}(1, x) + f_i(1, x) \\
 &\leq - \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(u_i^1(1, x) u_j^1(1, x) \right) \\
 &\quad + \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i(1, x)}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u_i^1(1, x) - \frac{\partial p^2}{\partial x_i}(1, x) + f_i(1, x) \\
 &\leq \frac{\partial}{\partial t} u_i^1(1, x), \quad x \in \mathbb{R}^n, \quad i \in \{1, \dots, n\}.
 \end{aligned}$$

Consequently the function

$$(\bar{u}, \bar{p}) = \begin{cases} (u^1, p^1) & \text{for } t \in [0, 1], \quad x \in \mathbb{R}^n, \\ (u^2, p^2) & \text{for } t \in [1, 2], \quad x \in \mathbb{R}^n, \end{cases}$$

is a solution to the problem

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} - \left(m + M \left(\int_{\mathbb{R}^n} |\nabla_x u_i|^2 dx \right) \right) \Delta u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), & i \in \{1, \dots, n\}, \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}(t, x) = 0 & \text{in } [0, 2] \times \mathbb{R}^n, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

such that $\bar{u}_i, \bar{p} \in C^1([0, 2], C_0^2(\mathbb{R}^n))$, $\text{supp}_x \bar{u}_i, \text{supp}_x \bar{p} \subset B$, $i \in \{1, \dots, n\}$. Note that

$$|\bar{u}_i|, |\bar{p}| \leq Q \quad \text{on } [0, 2] \times \mathbb{R}^n, \quad i \in \{1, \dots, n\}.$$

Now we consider the Cauchy problem

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} - \left(m + M \left(\int_{\mathbb{R}^n} |\nabla_x u_i|^2 dx \right) \right) \Delta u_i + \frac{\partial p}{\partial x_i} = f_i(t, x), & i \in \{1, \dots, n\}, \\ \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}(t, x) = 0 & \text{in } [2, 3] \times \mathbb{R}^n, \\ u(2, x) = u^2(2, x) & \text{in } \mathbb{R}^n. \end{cases}$$

Let

$$M^3 = \max_{t \in [2, 3], x \in B} |f(t, x)|.$$

We choose the constant $l_2 > 0$ such that

$$l_2 n \left(Q + 3n(1 + (m + \gamma Q^{2\delta-1}))(B^*)^2 Q + n^2 Q^2 (B^*)^2 + (B^*)^2 M^3 \right) \leq Q.$$

Let

$$E^3 = \left\{ v = (v_1, \dots, v_{n+1}) : v_i \in C^1([2, 3], C_0^2(\mathbb{R}^n)), \text{supp}_x v_i \subset B, i \in \{1, \dots, n+1\} \right\}$$

be endowed with the norm

$$||v|| = \max_{q \in \{1, \dots, n+1\}} \left\{ \max_{t \in [2, 3], x \in B} |v_q(t, x)|, \max_{t \in [2, 3], x \in B} |v_{qt}(t, x)|, \max_{t \in [2, 3], x \in B} |v_{qx_i}(t, x)| \right\},$$

$$\max_{t \in [2,3], x \in B} |v_{qx_i x_j}(t, x)|, i, j \in \{1, \dots, n\} \}.$$

With $\tilde{K}^3$ we denote the set of all equi-continuous families in E^3 . Let also,

$$\overline{K}^3 = \overline{\tilde{K}^3}, \quad K^3 = \{v \in \overline{K}^3 : \|v\| \leq Q\},$$

$$Q^3 = \{v \in \overline{K}^3 : \|v\| \leq (1 + l_2)Q\}.$$

Note that K^3 is a compact subset of Q^3 . For $(u, p) \in Q^3$ we define the operators.

$$\begin{aligned} L_i^3(u, p)(t, x) &= -l_2 u_i(t, x) + l_2 \int_{x^0}^x \int_{x^0}^s \left(u_i(t, \sigma) - u_i^2(2, \sigma) \right) d\sigma ds \\ &\quad + l_2 \sum_{j=1}^n \int_2^t \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{s_j}) u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau \\ &\quad - l_2 \left(m + M \left(\sum_{j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^2 \right) \right) \sum_{j=1}^n \int_2^t \int_{\bar{x}_j^0}^{\bar{x}_j} \int_{\bar{x}_j^0}^{\bar{s}_j} u_i(\tau, \tilde{\sigma}_{x_j}) d\bar{\sigma}_j d\bar{s}_j d\tau \\ &\quad + l_2 \int_2^t \int_{x^0}^x \int_{\bar{x}_i^0}^{\bar{s}_i} p(\tau, \tilde{\sigma}_{s_i}) d\bar{\sigma}_i ds d\tau \\ &\quad - l_2 \int_2^t \int_{x^0}^x \int_{x^0}^s f_i(\tau, \sigma) d\sigma ds d\tau, \\ L_{n+1}^3(u, p)(t, x) &= -l_2 p(t, x) + l_2 \sum_{j=1}^n \int_2^t \int_2^y \int_{x^0}^x \int_{\bar{x}_j^0}^{\bar{s}_j} u_j(\tau, \tilde{\sigma}_{s_j}) d\bar{\sigma}_j ds d\tau dy \\ &\quad + l_2(t-2) \int_{x^0}^x \int_{x^0}^s \left(p_t(2, \sigma) - p_t^2(2, \sigma) \right) d\sigma ds \\ &\quad + l_2 \int_{x^0}^x \int_{x^0}^s \left(p(2, \sigma) - p^2(2, \sigma) \right) d\sigma ds, \end{aligned}$$

$$M_i^3(u, p)(t, x) = (1 + l_2)u_i(t, s), \quad i \in \{1, \dots, n\},$$

$$M_{n+1}^3(u, p)(t, x) = (1 + l_2)p(t, x),$$

$$L^3(u, p)(t, x) = \left(L_1^3(u, p)(t, x), \dots, L_{n+1}^3(u, p)(t, x) \right),$$

$$\overline{M}^3(u, p)(t, x) = \left(M_1^3(u, p)(t, x), \dots, M_{n+1}^3(u, p)(t, x) \right),$$

$$(t, x) \in [2, 3] \times \mathbb{R}^n.$$

Consequently the function

$$(\tilde{u}, \tilde{p}) = \begin{cases} (u^1, p^1) & \text{for } t \in [0, 1], \quad x \in \mathbb{R}^n, \\ (u^2, p^2) & \text{for } t \in [1, 2], \quad x \in \mathbb{R}^n, \\ (u^3, p^3) & \text{for } t \in [2, 3], \quad x \in \mathbb{R}^n, \\ \dots & \end{cases}$$

is a solution to the problem (1.1), (1.2) such that $\tilde{u}_i, \tilde{p} \in \mathcal{C}^1([0, \infty), \mathcal{C}_0^2(\mathbb{R}^n))$, $\text{supp}_x \tilde{u}_i, \text{supp}_x \tilde{p} \subset B$, $i \in \{1, \dots, n\}$. Since $f_i \in \mathcal{C}^2([0, \infty), \mathcal{C}_0^\infty(\mathbb{R}^n))$, using (1.1), we conclude that $\tilde{p}, \tilde{u}_i \in \mathcal{C}^\infty([0, \infty) \times \mathbb{R}^n)$, $i \in \{1, \dots, n\}$. Note that

$$|\tilde{u}_i|, |\tilde{p}| \leq Q \quad \text{on } [0, \infty) \times \mathbb{R}^n, \quad i \in \{1, \dots, n\}.$$

Therefore

$$\int_{\mathbb{R}^n} |\tilde{u}(t, x)|^2 dx = \int_B |\tilde{u}(t, x)|^2 dx \leq nQ^2 \mu(B) < \infty.$$

References

- [1] Jawadi A., Boutyour H., Cadou J.M., Asymptotic numerical method for steady flow of power-law fluids, *J. Non-Newton. Fluid Mech.*, 2013, 202, 22–31.
- [2] Van De Vosse F.N., Segal A., Van Steenhoven A. A., Janssen J.D., A finite element approximation of the unsteady two-dimensional Navier-Stokes equations, *Internat. J. Numer. Methods Fluids*, 1986, 6(7), 427–443.
- [3] Jday R., Zennir Kh., Georgiev S.G., Existence and smoothness for new class of n-dimentional Navier-Stokes equations, *Rocky Mountain J. Math.*, 2019, 49(5), 1595–1615.
- [4] Higueras I., Happenhofer N., Koch O., Kupka F., Optimized strong stability preserving IMEX Runge-Kutta methods, *J. Comput. Appl. Math.*, 2014, 272, 116–140.
- [5] Hokpunna A., Manhart M., Compact fourth-order finite-volume method for numerical solutions of Navier-Stokes equations on staggered grids, *J. Comput. Phys.*, 2010, 229(20), 7545–7570.
- [6] Jamal M., Braikat B., Boutmir S., Damil N., Potier-Ferry M., A high order implicit algorithm for solving instationary non-linear problems, *Comput. Mech.*, 2002, 28(5), 375–380.
- [7] Grote M.J., Mitkova T., High-order explicit local time-stepping methods for damped wave equations, *J. Comput. Appl. Math.*, 2013, 239, 270–289.
- [8] Kirchhoff G., *Vorlesungen uber Mechanik*, 3rd ed., Teubner, Leipzig, 1983.
- [9] Zennir Kh., General decay of solutions for damped wave equation of Kirchhoff type with density in $\mathbb{R}^n$, *Ann. Univ. Ferrara*, 2015, 61, 381–394.
- [10] Aggoune W., Zahrouni H., Potier-Ferry M., High-order prediction-correction algorithms for unilateral contact problems, *J. Comput. Appl. Math.*, 2004, 168(1-2), 1–9.
- [11] Baguet S., Cochelin B., On the behaviour of the ANM continuation in the presence of bifurcations, *Comm. Numer. Methods Engrg.*, 2003, 19(6), 459–471.
- [12] Benítez M., Bermúdez A., Pure Lagrangian and non-Lagrangian finite element methods for the numerical solution of Navier-Stokes equations, *Appl. Numer. Math.*, 2015, 95, 62–81.
- [13] Fafard M., Henchi K., Gendron G., Ammar S., Application of an asymptotic method to transient dynamic problems, *J. Sound Vib.*, 1997, 208(1), 73–99.
- [14] Cadou J.M., Potier-Ferry M., Cochelin B., Damil N., ANM for stationary Navier-Stokes equations and with Petrov-Galerkin formulation, *Internat. J. Numer. Methods Engrg.*, 2001, 50(4), 825–845.
- [15] Castillo E., Codina R., Stabilized stress-velocity-pressure finite element formulations of the Navier-Stokes problem for fluids with non-linear viscosity, *Comput. Methods Appl. Mech. Engrg.*, 2014, 279, 554–578.
- [16] Bogaers A.E.J., Kok S., Reddy B.D., Franz T., Quasi-Newton methods for implicit black-box FSI coupling, *Comput. Methods Appl. Mech. Engrg.*, 2014, 279, 113–132.
- [17] Girault G., Guevel Y., Cadou J.M., An algorithm for the computation of multiple Hopf bifurcation points based on Padé approximants, *Internat. J. Numer. Methods Fluids*, 2012, 68(9), 1189–1206.
- [18] Guevel Y., Boutyour H., Cadou J.M., Automatic detection and branch switching methods for steady bifurcation in fluid mechanics, *J. Comput. Phys.*, 2011, 230(9), 3614–3629.

- [19] Kazemi-Kamyab V., van Zuijlen A.H., Bijl H., Analysis and application of high order implicit Runge-Kutta schemes for unsteady conjugate heat transfer: A strongly-coupled approach, *J. Comput. Phys.*, 2014, 272, 471–486.
- [20] Najah A., Cochelin B., Damil N., Potier-Ferry M., A critical review of asymptotic numerical methods, *Arch. Comput. Methods Eng.*, 1998, 5(1), 31–50.
- [21] Verstappen R., Wissink J.G., Cazemier W., Veldman A.E.P., Direct numerical simulations of turbulent flow in a driven cavity, *Future Gener. Comput. Syst.*, 1994, 10(2-3), 345–350.
- [22] Xiang T., Yuan R., A class of expansive type Krasnoselskii fixed point theorem, *Nonlinear Analysis*, 2009, 71(7-8), 3229–3239.