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Research Article

Qin Yue*

Stability property of the prey free equilibrium point

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Abstract: We revisit a prey-predator model with stage structure for predator, which was proposed by Tapan Kumar Kar. By using the differential inequality theory and the comparison theorem of the differential equation, we show that the prey free equilibrium is globally asymptotically stable under some suitable assumption. Our study shows that although the predator species has other food resource, if the amount of the predator species is too large, it could also do irreversible harm to the prey species, and this could finally lead to the extinction of the prey species. Our result supplement and complement some known results.

Keywords: Stage structure, Predator-prey, Global attractivity

MSC: 34D23; 92B05; 34D40

1 Introduction

During the last decades, many scholars investigated the dynamic behaviors of the stage structured ecosystem, see [1-18] and the references cited therein. In their series papers, Chen et al[1-3] studied the stability, persistence and extinction property of a stage-structured predator-prey system, and they found that despite the extinction of the prey species, the predator species could still be permanent, they argued that the reason maybe relies on the predator species has other food resources. In [4], Chen et al showed that stage structure plays important roles on the persistent and extinction property of the cooperation system. The results of [3] is generalized by Pu et al[10] to the infinite delay case. Also, many scholars[19-32] investigated the extinction property of the ecosystem. For example, Zhao et al[27] proposed a cooperative system with strong and weak partner, and they showed that the strong species maybe driven to extinction under some suitable assumption. Chen et al[29] studied the extinction property of a two species nonlinear competition system. Yang et al[28] showed that single feedback control variable could lead to the extinction of the species.

Kar[11] proposed the following prey-predator model with stage structure for predator,

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{k}\right) - \alpha N_1 N_3, \\ \frac{dN_2}{dt} &= \beta N_3 - r_2 N_2, \\ \frac{dN_3}{dt} &= -r_3 N_3 + m \alpha N_1 N_3 + \gamma N_2 - \delta N_3^2,\end{aligned}\tag{1.1}$$

*Corresponding Author: **Qin Yue:** College of Finance and Mathematics, West Anhui University, Liuan 237000, Anhui, People's Republic of China, E-mail: zzj901010@163.com

where $N_1(t), N_2(t)$ and $N_3(t)$ are the population densities of prey, juvenile predator and adult predator, respectively. $r_1, k, \alpha, \beta, r_2, r_3, m$ and γ are all positive constants. Define

$$N_1 = \frac{Kr_2}{r_1}x_1, \quad N_2 = \frac{\beta x_2}{m\alpha}, \quad N_3 = \frac{r_2 x_3}{m\alpha}, \quad t = \frac{\tau}{r_2}.$$

Then system (1.1) can be rewrite as

$$\begin{aligned} \frac{dx_1}{d\tau} &= ax_1 - x_1^2 - bx_1x_3, \\ \frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &= -cx_3 + dx_1x_3 + ex_2 - fx_3^2. \end{aligned} \quad (1.2)$$

The possible non-negative equilibria of system (1.2) are $P_0(0, 0, 0)$, $P_1(a, 0, 0)$ and $P_2(x_1^*, x_2^*, x_3^*)$. Under the assumption $c \geq e$, the author investigated the stability property of the above three equilibria. The author also pointed out "We remark that if $e > c$, then there exists another equilibrium in the absence of prey. But it is not feasible since prey is the only source of food for the predator." Indeed, this equilibrium could be expressed as $P_3\left(0, \frac{e-c}{f}, \frac{e-c}{f}\right)$. We mention here that generally speaking, predator may have many resources as its food, and seldom did predator species take only one kind of prey species as its food resource. For example, the Chinese Alligator can be regarded as a stage-structured predator species since the mature is more than ten years old, and the Chinese Alligator almost eat all aquatic animals. Certainly, if one kind of prey species is scare, it will take other prey species as its food resource. Hence, we argue that it is necessary to reconsider the declaration of the T. K. Kar, and we should investigate the stability property of the equilibrium $P_3\left(0, \frac{e-c}{f}, \frac{e-c}{f}\right)$.

The aim of this paper is to give sufficient condition to ensure the global asymptotically stable of the equilibrium $P_3\left(0, \frac{e-c}{f}, \frac{e-c}{f}\right)$ of system (1.2), more precisely, we have the following result.

Theorem 1.1. Assume that

$$af + bc < be \quad (1.3)$$

holds, then $P_3\left(0, \frac{e-c}{f}, \frac{e-c}{f}\right)$ is globally asymptotically stable.

We mention here that the method we used here is quite different with that of the method used in [11]. Indeed, we only use the differential inequality theory and the comparison theorem of the differential equation. We will prove Theorem 1.1 in the next section, and a numeric example is presented in Section 3 to show the feasibility of the main results. We end this paper by a briefly discussion.

2 Proof of Theorem 1.1

Now let's consider the system

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\ \frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2, \end{aligned} \quad (2.1)$$

where $\alpha, \beta, \delta_1, \delta_2$ and γ are all positive constants, $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature species at time t . From Theorem 4.1 in Xiao and Lei[17], we have

Lemma 2.1 Assume that

$$\alpha > \delta_2 \left(1 + \frac{\delta_1}{\beta}\right) \quad (2.2)$$

holds, then the positive equilibrium $B(x_1^{**}, x_2^{**})$ of system (2.1) is globally stable, where

$$\begin{aligned} x_1^{**} &= \frac{\alpha x_2^{**}}{\beta + \delta_1}, \\ x_2^{**} &= \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\gamma(\beta + \delta_1)}. \end{aligned} \quad (2.3)$$

Now let's consider the system

$$\begin{aligned} \frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &= -cx_3 + ex_2 - fx_3^2, \end{aligned} \quad (2.4)$$

As a direct corollary of Lemma 2.1, we have

Lemma 2.2 Assume that

$$e > c \quad (2.5)$$

holds, then the positive equilibrium $E(\frac{e-c}{f}, \frac{e-c}{f})$ of system (2.4) is globally stable.

Proof of Theorem 1.1. Let $(x_1(t), x_2(t), x_3(t))$ be any positive solution of the system (1.2). From the second and the third equation of system (1.1), we have

$$\begin{aligned} \frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &\geq -cx_3 + ex_2 - fx_3^2, \end{aligned} \quad (2.6)$$

Now let's consider the system

$$\begin{aligned} \frac{du_2}{d\tau} &= u_3 - u_2, \\ \frac{du_3}{d\tau} &= -cu_3 + eu_2 - fu_3^2, \end{aligned} \quad (2.7)$$

Noting that condition (1.3) implies that $e > c$, and so, from Lemma 2.2, (2.7) admits a unique globally asymptotically stable positive equilibrium $E(\frac{e-c}{f}, \frac{e-c}{f})$. That is, let $(u_1(t), u_2(t))$ be any positive solution of the system (2.7), one has

$$\lim_{t \rightarrow +\infty} u_2(t) = \lim_{t \rightarrow +\infty} u_3(t) = \frac{e-c}{f}. \quad (2.8)$$

Let $(x_1(t), x_2(t), x_3(t))$ be any positive solution of system (1.2) with initial condition $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30})$, and let $(u_1(t), u_2(t))$ be the positive solution of system (2.7) with the initial condition $(u_2(0), u_3(0)) = (x_{20}, x_{30})$, it then follows from the differential inequality theory that

$$x_i(t) \geq u_i(t) \text{ for all } t \geq 0, i = 1, 2. \quad (2.9)$$

The positivity of the solution of system (1.2), (2.8) and (2.9) lead to

$$\liminf_{t \rightarrow +\infty} x_i(t) \geq \lim_{t \rightarrow +\infty} u_i(t) = \frac{e-c}{f}, \quad i = 1, 2. \quad (2.10)$$

Condition (1.3) implies that for enough small positive constant $\varepsilon > 0$, the following inequality holds.

$$af + bc - be + bfe < 0, \quad (2.11)$$

which is equivalent to

$$a - b\left(\frac{e-c}{f} - \varepsilon\right) < 0. \quad (2.12)$$

For $\varepsilon > 0$ enough small, which satisfies (2.12), it then follows from (2.10) that there exists an enough large $T_1 > 0$ such that

$$x_i(t) > \frac{e-c}{f} - \varepsilon, \quad i = 1, 2, \text{ for all } t > T_1. \quad (2.13)$$

Hence, for $t > T_1$, from the first equation of system (1.2) and (2.13), we have

$$\begin{aligned}\frac{dx_1}{d\tau} &= ax_1 - x_1^2 - bx_1x_3, \\ &\leq ax_1 - x_1^2 - bx_1\left(\frac{e-c}{f} - \varepsilon\right) \\ &\leq \left(a - b\left(\frac{e-c}{f} - \varepsilon\right)\right)x_1.\end{aligned}$$

Consequently,

$$x_1(t) \leq x_1(T_1) \exp\left\{\left(a - b\left(\frac{e-c}{f} - \varepsilon\right)\right)(t - T_1)\right\} \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (2.14)$$

That is,

$$\lim_{t \rightarrow +\infty} x_1(t) = 0. \quad (2.15)$$

For $\varepsilon > 0$ enough small, it follows from (2.15) that there exists a $T_2 > T_1$ such that

$$x_1(t) < \varepsilon \text{ for all } t \geq T_2. \quad (2.16)$$

For $t > T_2$, from the second and third equation of system (1.2) and (2.16), we have

$$\begin{aligned}\frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &\leq -cx_3 + d\varepsilon x_3 + ex_2 - fx_3^2,\end{aligned} \quad (2.17)$$

Now let's consider the system

$$\begin{aligned}\frac{dv_2}{d\tau} &= v_3 - v_2, \\ \frac{dv_3}{d\tau} &= -cv_3 + d\varepsilon v_3 + ev_2 - fv_3^2,\end{aligned} \quad (2.18)$$

it follows from $e > c$ and Lemma 2.1 that (2.18) admits a unique globally asymptotically stable positive equilibrium $E_1\left(\frac{e-c+d\varepsilon}{f}, \frac{e-c+d\varepsilon}{f}\right)$. That is, let $(v_1(t), v_2(t))$ be any positive solution of the system (2.18), one has

$$\lim_{t \rightarrow +\infty} v_1(t) = \frac{e-c+d\varepsilon}{f}, \quad \lim_{t \rightarrow +\infty} v_2(t) = \frac{e-c+d\varepsilon}{f}. \quad (2.19)$$

Let $(x_1(t), x_2(t), x_3(t))$ be any positive solution of system (1.2) with initial condition $(x_1(T_2), x_2(T_2), x_3(T_2)) = (x_{10}, x_{20}, x_{30})$, and let $(v_2(t), v_3(t))$ be the positive solution of system (2.18) with the initial condition $(v_2(T_2), v_3(T_2)) = (x_{20}, x_{30})$, it then follows from the differential inequality theory that

$$x_i(t) \leq v_i(t) \text{ for all } t \geq T_2. \quad (2.20)$$

The positivity of the solution of system (1.2), (2.19) and (2.20) lead to

$$\limsup_{t \rightarrow +\infty} x_i(t) \leq \lim_{t \rightarrow +\infty} v_i(t) = \frac{e-c+d\varepsilon}{f} \quad i = 1, 2. \quad (2.21)$$

(2.10) and (2.18) show that

$$\frac{e-c}{f} \leq \liminf_{t \rightarrow +\infty} x_i(t) \leq \limsup_{t \rightarrow +\infty} x_i(t) \leq \frac{e-c+d\varepsilon}{f}, \quad i = 1, 2. \quad (2.22)$$

Since ε could be any enough small positive constant, now, letting $\varepsilon \rightarrow 0$ in (2.22) leads to

$$\lim_{t \rightarrow +\infty} x_i(t) = \frac{e-c}{f}, \quad i = 1, 2. \quad (2.23)$$

(2.23) together with (2.15) shows that $P_3\left(0, \frac{e-c}{f}, \frac{e-c}{f}\right)$ is globally asymptotically stable. This completes the proof of Theorem 2.1.

3 Numeric simulation

Example 3.1. Consider the following stage structure predator prey system

$$\begin{aligned}\frac{dx_1}{d\tau} &= x_1 - x_1^2 - x_1x_3, \\ \frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &= -x_3 + x_1x_3 + 3x_2 - x_3^2.\end{aligned}\tag{3.1}$$

Here, corresponding to system (1.2), we take $a = b = c = d = f = 1$, $e = 3$. Since $af + bc = 2 < 3 = be$, it follows from Theorem 1.1 that $P_3(0, 2, 2)$ is globally asymptotically stable. Numeric simulation (Fig. 1) also supports this assertion.

Figure 1: Dynamics behaviors of of system (3.1), the initial conditions $(x_1(0), x_2(0), x_3(0)) = (3, 0.5, 2), (3, 1, 2), (1, 0.5, 2)$ and $(3, 2, 2)$, respectively.

4 Discussion

Kar[11] proposed a stage structured predator prey system (i.e., system (1.1)), he investigated the global stability property of the equilibria, however, he did not investigated the stability property of the prey free equilibrium P_3 . In this paper, by using the differential inequality theory and the comparison theorem of the differential equation, we could show that under some suitable condition, the boundary equilibrium P_3 is globally asymptotically stable. Our result has significant biological meaning. In system (1.2), without consider the relationship of the predator and prey species, then prey species is governed by the equation

$$\frac{dx_1}{d\tau} = ax_1 - x_1^2,\tag{4.1}$$

which is a Logistic equation, and the positive equilibrium $x_1^* = a$ is globally asymptotically stable. That is, without the influence of the predator species, the prey species could exist in long run.

Also, without consider the influence of the prey species, the predator species satisfies the system

$$\begin{aligned}\frac{dx_2}{d\tau} &= x_3 - x_2, \\ \frac{dx_3}{d\tau} &= -cx_3 + ex_2 - fx_3^2,\end{aligned}\tag{4.2}$$

From Lemma 2.2 we know that the system admits a unique positive equilibrium $E(\frac{e-c}{f}, \frac{e-c}{f})$, which is globally asymptotically stable, that is, the predator species has other food resource and it could survive without the prey species x_1 .

Theorem 1.1 shows that although the predator species has other food resources and the prey species x_1 is only one of the food resources of the predator species, however, if the amount of the predator species is too large, it could also do irreversible harm to the prey species, and this could finally lead to the extinction of the prey species.

5 Declarations

Competing interests

The authors declare that there is no conflict of interests.

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Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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