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Time-dependent attractor of wave equations with nonlinear damping and linear memory

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Abstract: In this article, we consider the long-time behavior of solutions for the wave equation with nonlinear damping and linear memory. Within the theory of process on time-dependent spaces, we verify the process is asymptotically compact by using the contractive functions method, and then obtain the existence of the time-dependent attractor in $H_0^1(\Omega) \times L^2(\Omega) \times L_\mu^2(\mathbb{R}^+; H_0^1(\Omega))$.

Keywords: wave equation; linear memory; contractive functions; time-dependent attractor

MSC: 35B25; 37L30; 45K05

1 Introduction

Let Ω be an open bounded set of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For any $\tau \in \mathbb{R}$, we consider the following equations

$$\begin{cases} \varepsilon(t)u_{tt} + a(x)g(u_t) - \Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = h(x), & \text{in } \Omega \times (\tau, \infty), \\ u(x, t) = 0, & x \in \partial\Omega, t \in \mathbb{R}, \\ u(x, t) = u_0(x, t), u_t(x, t) = \partial_t u_0(x, t), & x \in \Omega, t \leq \tau, \end{cases} \quad (1.1)$$

where $u = u(x, t) : \Omega \times [\tau, \infty) \rightarrow \mathbb{R}$ is an unknown function, and $u_0 : \Omega \times (-\infty, \tau] \rightarrow \mathbb{R}$ is a given past history of u , $h(\cdot) \in L^2(\Omega)$ is independent of time, μ is a summable positive function. $\eta = \eta^t(x, s) := u(x, t) - u(x, t-s)$, $s \in \mathbb{R}^+$. $\varepsilon \in C^1(\mathbb{R})$ is a decreasing bounded function and satisfies

$$\lim_{t \rightarrow +\infty} \varepsilon(t) = 0. \quad (1.2)$$

In particular, there exists $L > 0$, such that

$$\sup_{t \in \mathbb{R}} [|\varepsilon(t)| + |\varepsilon'(t)|] \leq L. \quad (1.3)$$

The function $a(x)$ satisfies:

$$a(x) \in L^\infty(\Omega), \quad a(x) \geq \alpha_0 > 0 \quad \text{in } \Omega, \quad (1.4)$$

where α_0 is a constant.

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Like in [1,2], the nonlinear damping $g \in C^1(\mathbb{R})$, $g(0) = 0$, g is strictly increasing, and satisfies

$$\liminf_{|s| \rightarrow \infty} g'(s) > 0, \quad (1.5)$$

$$|g(s)| \leq C(1 + |s|^p), \quad 1 \leq p < 5. \quad (1.6)$$

The nonlinear term $f \in C^1(\mathbb{R})$, $f(0) = 0$, and for some $C_0 \geq 0$ satisfies

$$|f'(s)| \leq C_0(1 + |s|^2), \quad \forall s \in \mathbb{R}, \quad (1.7)$$

along with the dissipation condition

$$\liminf_{|s| \rightarrow \infty} \frac{f(s)}{s} > -\lambda_1, \quad \forall s \in \mathbb{R}, \quad (1.8)$$

where λ_1 is the first eigenvalue of the strictly positive operator $A = -\Delta$.

With respect to the memory component, as in [3 – 6], we assume that

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \int_0^\infty \mu(s) ds = m_0 < \infty, \quad (1.9)$$

$$\mu'(s) \leq -\rho\mu(s) \leq 0, \quad \forall s \geq 0, \quad (1.10)$$

where ρ is a positive constant.

The problem (1.1) can be viewed as a description of viscoelastic solids with fading memory and dissipation due to the viscous resistance of the surrounding medium, as well as of composite materials, phase-fields, and wave phenomena [7-9].

When μ is a Dirac measure at some fixed time instant or when it vanishes, the equation (1.1) reduces to the nonlinear damped wave equation, which has been investigated extensively by many authors. For instance, in the case that ε is a positive constant independent of time, the long-time behavior of the solution can be well characterized by using the concept of global attractors in the framework of semigroup. The existence and regular properties of the global attractor have been studied in [2,10-12]. When ε is a positive constant independent of time and the forcing term h depends on time, the system is a non-autonomous wave equation, the long-time behavior of the solution can be understood in the framework of process. We refer the reader to [11,13-16] for some specific results involving the uniform attractor (or pullback attractors) about non-autonomous case.

When ε is still a positive constant and the nonlinear damping $a(x)g(t)$ is either linear damping au_t or strong damping Δu_t , Conti and Pata ([17]), Borni and Pata ([18]), Pata and Zucchi ([19]) investigated the existence of global attractors for (1.1). Sun, Cao and Duan ([20]) obtained the existence and asymptotic regularity of the uniform attractor about the non-autonomous system with strong damping, while the robust exponential attractors was scrutinized by Kloden, Real and Sun ([21]).

However, provided that ε depends explicitly on time in (1.1), such as a positive decreasing function of time $\varepsilon(t)$ vanishing at infinity, leading to time-dependent terms at functional level, these problems become more complex and interesting, because the corresponding dynamical system is still understood within the non-autonomous framework even the forcing term is independent of time, and the classical theory generally fails to capture the dissipation mechanism of the system, as mentioned in [22,23].

To circumvent these issue, in [22], Conti, Pata and Temam presented a notion of time-dependent attractor by exploiting the minimality with respect to the pullback attraction property, and constructed a sufficient condition proving the existence of time-dependent attractor based on the theory established by Plinio, Duane and Temam([23]). Meanwhile, within the new framework, the authors studied the following weak damped wave equations with time-dependent speed of propagation

$$\varepsilon(t)u_{tt} + au_t - \Delta u + f(u) = g(x). \quad (1.11)$$

Besides, they proved that the time-dependent global attractor of (1.11) converged in a suitable sense to the attractor of the parabolic equation $au_t - \Delta u + f(u) = g(x)$ when $\varepsilon(t) \rightarrow 0$ as $t \rightarrow +\infty$ ([24]). Successively,

in [25], they continued to show the existence of an invariant time-dependent global attractor to the following specific one-dimensional wave equation $\varepsilon(t)u_{tt} - u_{xx} + [1 + \varepsilon f'(u)]u_t + f(u) = h$, which converges in suitable sense to the classical Fourier equation.

Recently, Meng et al. investigated the long-time behavior of the solution for the wave equation with nonlinear damping $g(u_t)$ on the time-dependent space, in which they found a new technical method verifying compactness of the process via defining the contractive functions, see [1]. In [26], Meng and Liu also showed the necessary and sufficient conditions of the existence of time-dependent global attractor borrowed from the ideas in [10]. Liu and Ma ([27]) studied the existence of the pullback attractors for the plate equation with time-dependent forcing term on the strong time-dependent Hilbert space. Successively, exploiting the methods and framework of [22,24], Liu and Ma obtained the existence and regularity of the time-dependent attractor for the plate equation with critical growth nonlinearity, as well as the asymptotic structure in [28].

As we know, in the study of the long-time behavior, especially for attractors, obtaining certain asymptotic compactness of the solution operator is a key step. However, if the equation contains the history memory, for instance, just for our problem (1.1), it makes impossible to utilize $(I - P_m)u$ as the test function to capture the asymptotic compactness of the solution process, that is to say, the methods introduced in [10,26] is out of action to our problem. On the other hand, because of the critical nonlinear damping, the technique of operator decomposition in [22] is not suitable to deal with (1.1) anymore. Thus, for our problem, we need to make a priori estimates to solution on a new triple solution space, and then verify the compactness of the solution process by exploiting the method of contractive function.

It is worth mentioning that we use the more weaker dissipative condition (1.8) than [1,22]; indeed, for simplicity, in which the authors made use of the dissipative condition like $\liminf_{|s| \rightarrow \infty} f'(s) > -\lambda_1$.

For convenience, hereafter, C (or c) denotes an arbitrary positive constant which may be different from line to line even in the same line.

The rest of this article consists of two Sections. In the next Section, we define some functions sets and iterate some useful lemmas. In Section 3, the existence of the time-dependent global attractor is obtained.

2 Preliminaries

As in Borini, Pata [18], Pata, Zucchi [19] and Dafermos [4], we introduce the past history of u , i.e. $\eta = \eta^t(x, s)$, as a new variable of the system, which will be ruled by a supplementary equation: denoting

$$\eta_t = \frac{\partial}{\partial t} \eta, \quad \eta_s = \frac{\partial}{\partial s} \eta, \quad (2.1)$$

then we can rewrite (1.1) as

$$\begin{cases} \varepsilon(t)u_{tt} + a(x)g(u_t) - \Delta u - \int_0^\infty \mu(s)\triangle \eta^t(s)ds + f(u) = h(x), \\ \eta_t^t = -\eta_s^t + u_t, \end{cases} \quad (2.2)$$

with initial boundary conditions

$$\begin{cases} u(x, t) = 0, & x \in \partial\Omega, \quad t \geq \tau, \\ \eta^t(x, s) = 0, & (x, s) \in \partial\Omega \times \mathbb{R}^+, \quad t \geq \tau, \\ u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x), & x \in \Omega, \\ \eta^\tau(x, s) = \eta^0(x, s), & (x, s) \in \Omega \times \mathbb{R}^+, \end{cases} \quad (2.3)$$

where

$$\begin{cases} u_0(x) = u_0(x, \tau), \quad u_1(x) = \partial_t u_0(x, t)|_{t=\tau}, \\ \eta^0 = \eta^0(x, s) = u_0(x, \tau) - u_0(x, \tau - s). \end{cases}$$

Without loss of generality, set $H = L^2(\Omega)$ with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $s \in \mathbb{R}^+$, we define the hierarchy of (compactly) nested Hilbert spaces

$$H^s = D(A^{\frac{s}{2}}), \quad \langle w, v \rangle_s = \langle A^{\frac{s}{2}} w, A^{\frac{s}{2}} v \rangle, \quad \|w\|_s = \|A^{\frac{s}{2}} w\|;$$

especially, we have the embeddings $H^{s+1} \hookrightarrow H^s$.

For $s \in \mathbb{R}^+$, let $L_\mu^2(\mathbb{R}^+; H^s)$ be the family of Hilbert spaces of functions $\varphi : \mathbb{R}^+ \rightarrow H^s$, equipped with the inner product and norm, respectively,

$$\langle \varphi_1, \varphi_2 \rangle_{\mu, s} = \langle \varphi_1, \varphi_2 \rangle_{\mu, H^s} = \int_0^\infty \mu(s) \langle \varphi_1(s), \varphi_2(s) \rangle_{H^s} ds,$$

$$\|\varphi\|_{\mu, s}^2 = \|\varphi\|_{\mu, H^s}^2 = \int_0^\infty \mu(s) \|\varphi(s)\|_{H^s}^2 ds.$$

Now, for $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$, we have the following time-dependent spaces

$$\mathcal{H}_t^s = H^{s+1} \times H^s \times L_\mu^2(\mathbb{R}^+; H^{s+1}),$$

with the norm

$$\|z\|_{\mathcal{H}_t^s}^2 = \|(u, u_t, \eta^t)\|_{\mathcal{H}_t^s}^2 = \|u\|_{s+1}^2 + \varepsilon(t) \|u_t\|_s^2 + \|\eta^t\|_{\mu, s+1}^2.$$

Especially, denote $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\mathcal{H}_t^1 = H^2 \times H^1 \times (L_\mu^2(\mathbb{R}^+; H^2) \cap H_\mu^1(\mathbb{R}^+; H^1)) = D(A) \times H_0^1(\Omega) \times (L_\mu^2(\mathbb{R}^+; D(A)) \cap H_\mu^1(\mathbb{R}^+; H_0^1(\Omega))),$$

$$\mathcal{H}_t = \mathcal{H}_t^0 = H^1 \times H \times L_\mu^2(\mathbb{R}^+; H^1) = H_0^1(\Omega) \times L^2(\Omega) \times L_\mu^2(\mathbb{R}^+; H_0^1(\Omega)).$$

Here

$$H_\mu^1(\mathbb{R}^+; H^s) = \{\varphi : \varphi(t), \partial_t \varphi(t) \in L_\mu^2(\mathbb{R}^+; H^s)\}.$$

For every $t \in \mathbb{R}$, let X_t be a family of normed spaces, we introduce the R -ball of X_t

$$\mathbb{B}_t(R) = \{z \in X_t : \|z\|_{X_t} \leq R\}.$$

We denote the Hausdorff semi-distance of two (nonempty) sets $B, C \subset X_t$ by:

$$\delta_t(B, C) = \sup_{x \in B} \text{dist}_{X_t}(x, C) = \sup_{x \in B} \inf_{y \in C} \|x - y\|_{X_t}.$$

For any given $\epsilon > 0$, the ϵ -neighbourhood of a set $B \subset X_t$ is defined as

$$\mathcal{O}_t^\epsilon(B) = \bigcup_{x \in B} \{y \in X_t : \|y - x\|_{X_t} < \epsilon\} = \bigcup_{x \in B} \{x + \mathbb{B}_t(\epsilon)\}.$$

Finally, given any set $B \subset X_t$, the symbol $\bar{B}$ stands for the closure of B in X_t .

Now we iterate some basic notations and abstract results, which are necessary for getting our main results.

Definition 2.1.^[22] Let X_t be a family of normed spaces. A process is a two-parameter family of mappings $\{U(t, \tau) : X_\tau \rightarrow X_t, t \geq \tau \in \mathbb{R}\}$ with properties

(i) $U(\tau, \tau) = \text{Id}$ is the identity on X_τ , $\tau \in \mathbb{R}$;

(ii) $U(t, s)U(s, \tau) = U(t, \tau)$, $\forall t \geq s \geq \tau$.

Definition 2.2.^[22] A family $\mathfrak{C} = \{C_t\}_{t \in \mathbb{R}}$ of bounded sets $C_t \subset X_t$ is called uniformly bounded if there exist a constant $R > 0$ such that $C_t \subset \mathbb{B}_t(R)$, $\forall t \in \mathbb{R}$.

Definition 2.3.^[22] A time-dependent absorbing set for the process $U(t, \tau)$ is a uniformly bounded family $\mathfrak{B} = \{B_t\}_{t \in \mathbb{R}}$ with the following property: for every $R > 0$ there exists a t_0 such that

$$\tau \leq t - t_0 \Rightarrow U(t, \tau)\mathbb{B}_\tau(R) \subset B_t.$$

Definition 2.4.^[22] A (uniformly bounded) family $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}}$ is called pullback attracting if for all $\epsilon > 0$, the family $\{\mathcal{O}_t^\epsilon(K_t)\}_{t \in \mathbb{R}}$ is pullback absorbing.

Definition 2.5.^[22] We call a time-dependent global attractor is the smallest element of $\mathbb{K}$, i.e. the family $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}} \in \mathbb{K}$ such that $A_t \subset K_t$, $\forall t \in \mathbb{R}$, for any element $\mathfrak{K} = \{K_t\}_{t \in \mathbb{R}} \in \mathbb{K}$.

Definition 2.6.^[22,24] We say $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$ is invariant if

$$U(t, \tau)A_\tau = A_t, \quad \forall t \geq \tau.$$

Theorem 2.7.^[22] (1) If $U(t, \tau)$ is asymptotically compact, then there exists a unique time-dependent attractor $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$;

(2) If $U(t, \tau)$ is a T -closed process for some $T > 0$, which possesses a time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$, then $\mathfrak{A}$ is invariant.

Lemma 2.8.^[1,29] Let $g(\cdot)$ satisfy condition (1.5). Then for any $\delta > 0$, there exists a positive constant C_δ , such that $|u - v|^2 \leq \delta + C_\delta(g(u) - g(v))(u - v)$ for all $u, v \in \mathbb{R}$.

Theorem 2.9.^[1] Let $U(\cdot, \cdot)$ be a process in a family of Banach space $\{X_t\}_{t \in \mathbb{R}}$. Then $U(\cdot, \cdot)$ has a time-dependent global attractor $\mathfrak{U}^* = \{A_t^*\}_{t \in \mathbb{R}}$ satisfying

$$A_t^* = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B_\tau},$$

if and only if

(i) $U(\cdot, \cdot)$ has a pullback absorbing family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$;

(ii) $U(\cdot, \cdot)$ is pullback asymptotically compact.

Theorem 2.10.^[1] Let $U(\cdot, \cdot)$ be a process on $\{X_t\}_{t \in \mathbb{R}}$ and has a pullback absorbing family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$.

Moreover, assume that for any $\varepsilon > 0$ there exist $T(\varepsilon) \leq t$, $\phi_T^t \in \mathcal{C}(B_T)$, such that

$$\|U(t, T)x - U(t, T)y\| \leq \varepsilon + \phi_T^t(x, y), \quad \forall x, y \in B_T,$$

for any fixed $t \in \mathbb{R}$. Then $U(\cdot, \cdot)$ is pullback asymptotically compact, where $\mathcal{C}(B_T)$ denotes the set of all contractive function on $B_T \times B_T$.

For nonlinear function g , by condition (1.6) we have

$$|g(s)|^{\frac{p+1}{p}} = |g(s)|^{\frac{1}{p}} \cdot |g(s)| \leq C(1 + |s|)|g(s)| \leq C' + C'g(s) \cdot s, \quad (2.4)$$

furthermore, there holds

$$|g(s)| \leq C + C(g(s)s)^{\frac{p}{p+1}}. \quad (2.5)$$

Lemma 2.11.^[20] Let $F(u) = \int_0^u f(y)dy$. From (1.8), we can get for $0 < \nu < 1$ and $c_i > 0$ ($i = 1, 2$), there hold

$$2\langle F(u), 1 \rangle \geq -(1 - \nu)\|u\|_1^2 - c_1, \quad (2.6)$$

$$\langle f(u), u \rangle \geq -(1 - \nu)\|u\|_1^2 - c_2, \quad \forall u \in H^1. \quad (2.7)$$

3 Existence of the time-dependent global attractor

3.1 Well-posedness and time-dependent absorbing set

Now we state the results about the well-posedness of system (2.2)-(2.3) which can be found in [19,20]. In fact, the existence of solution $(u(t), u_t(t), \eta^t(s))$ to (2.2)-(2.3) is obtained by using the standard Galerkin approximation method, which is based on Lemma 3.2 below.

Lemma 3.1. Under the assumptions (1.2)-(1.10), for every initial data $z_\tau = (u_0, u_1, \eta^0) \in \mathcal{H}_\tau$, there exists a unique solution $z(t) = (u(t), u_t(t), \eta^t(s))$ of problem (2.2)-(2.3) in space $\mathcal{H}_t$, and for any $\tau \in \mathbb{R}$, $t \geq \tau$, it satisfies

$$u \in C([\tau, t]; H^1), \quad u_t \in C([\tau, t]; H), \quad \eta^t \in C([\tau, t]; L_\mu^2(\mathbb{R}^+; H^1)).$$

Furthermore, let $z_i(\tau) \in \mathcal{H}_\tau$ be the initial data such that $\|z_i(\tau)\|_{\mathcal{H}_\tau} \leq R$ ($i = 1, 2$), and $z_i(t)$ be the solution of problem (2.2)-(2.3). Then there exists $C = C(R) > 0$, such that

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_t} \leq e^{C(t-\tau)} \|z_1(\tau) - z_2(\tau)\|_{\mathcal{H}_\tau}, \quad \forall t \geq \tau. \quad (3.1)$$

Thus, the system (2.2)-(2.3) generates a strongly continuous process $U(t, \tau)$, where

$$U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t \text{ acting as } U(t, \tau)z(\tau) = \{u(t), u_t(t), \eta^t(s)\},$$

with the initial data $z_\tau = z(\tau) = \{u_0, u_1, \eta^0\} \in \mathcal{H}_\tau$.

To prove Lemma 3.1, we first need the following estimate:

Lemma 3.2. Under the assumptions (1.2)-(1.10), for any initial data $z(\tau) \in \mathbb{B}_\tau(R) \subset \mathcal{H}_\tau$, there exists $R_0 > 0$, such that

$$\|U(t, \tau)z(\tau)\|_{\mathcal{H}_t} \leq R_0, \quad \forall \tau \leq t.$$

Proof. Denote

$$E_0(t) = \frac{1}{2} \|U(t, \tau)z\|_{\mathcal{H}_t}^2 + \int_{\Omega} F(u) dx - \int_{\Omega} h \cdot u dx.$$

Multiplying (2.2)₁ with u_t in L^2 and exploiting (2.2)₂, we achieve

$$\frac{d}{dt} E_0(t) + \int_{\Omega} a(x)g(u_t) \cdot u_t dx + \langle \eta^t(s), \eta_s^t(s) \rangle_{\mu,1} - \frac{\varepsilon'(t)}{2} \|u_t\|^2 = 0. \quad (3.2)$$

By Hölder, Young inequalities, and combining with (1.10) we obtain

$$\begin{aligned} \langle \eta^t, \eta_s^t \rangle_{\mu,1} &= \frac{1}{2} \int_0^\infty \mu(s) \frac{d}{ds} \|\eta^t(s)\|_1^2 ds = -\frac{1}{2} \int_0^\infty \mu'(s) \|\eta^t(s)\|_1^2 ds \\ &\geq \frac{\rho}{2} \int_0^\infty \mu(s) \|\eta^t(s)\|_1^2 ds = \frac{\rho}{2} \|\eta^t(s)\|_{\mu,1}^2; \end{aligned} \quad (3.3)$$

Using that g is strictly increasing, $\varepsilon(t)$ is decreasing, and (3.3), we have

$$\int_{\Omega} a(x)g(u_t) \cdot u_t dx - \frac{\varepsilon'(t)}{2} \|u_t\|^2 + \langle \eta^t(s), \eta_s^t(s) \rangle_{\mu,1} \geq \alpha_0 \int_{\Omega} g(u_t) \cdot u_t dx - \frac{\varepsilon'(t)}{2} \|u_t\|^2 + \frac{\rho}{2} \|\eta^t(s)\|_{\mu,1}^2 > 0.$$

Integrating (3.2) over $[\tau, t]$ we get

$$E_0(t) \leq E_0(\tau), \quad \forall t \geq \tau. \quad (3.4)$$

From (2.6) and Sobolev's embeddings we deduce

$$\begin{aligned} E_0(t) &\geq \frac{1}{2} \varepsilon(t) \|u_t\|^2 + \frac{\nu}{2} \|u\|_1^2 + \frac{1}{2} \|\eta^t\|_{\mu,1}^2 - \frac{\nu}{4} \|u\|_1^2 - \frac{1}{\lambda_1 \nu} \|h\|^2 - \frac{c_1}{2} \\ &\geq \frac{\nu}{4} [\varepsilon(t) \|u_t\|^2 + \|u\|_1^2 + \|\eta^t\|_{\mu,1}^2] - \left(\frac{1}{\lambda_1 \nu} \|h\|^2 + \frac{c_1}{2} \right) \\ &\geq -\left(\frac{1}{\lambda_1 \nu} \|h\|^2 + \frac{c_1}{2} \right). \end{aligned} \quad (3.5)$$

Consequently, there exist some proper positive constant C_1, C_2 and C_3 , such that

$$C_1 \|U(t, \tau)z\|_{\mathcal{H}_t}^2 - C_2 \leq E_0(t) \leq C_3 \|U(t, \tau)z\|_{\mathcal{H}_t}^2.$$

And from (3.4), (3.5) yields

$$\begin{aligned} &\int_{\tau}^t \int_{\Omega} a(x)g(u_t) \cdot u_t dx - \frac{1}{2} \int_{\tau}^t \varepsilon'(s) \|u_t(s)\|^2 ds + \frac{\rho}{2} \int_{\tau}^t \|\eta^t(s)\|_{\mu,1}^2 ds \\ &\leq E_0(\tau) - E_0(t) \leq E_0(\tau) + \frac{1}{\lambda_1 \nu} \|h\|^2 + \frac{c_1}{2}. \end{aligned} \quad (3.6)$$

On the other hand, (1.6), (2.5) along with the Hölder and Young inequality imply

$$\begin{aligned} \left| \int_{\Omega} a(x)g(u_t) \cdot u dx \right| &\leq C \int_{\Omega} a(x)|u| dx + C \int_{\Omega} a(x)(g(u_t) \cdot u_t)^{\frac{p}{p+1}} |u| dx \\ &\leq C \int_{\Omega} a(x)|u| dx + C \left(\int_{\Omega} a(x)g(u_t) \cdot u_t dx \right)^{\frac{p}{p+1}} \left(\int_{\Omega} a(x)|u|^{p+1} \right)^{\frac{1}{p+1}} \\ &\leq C \int_{\Omega} a(x)|u| dx + \eta \|u\|_1^2 + C_{\eta} \|u\|_1^{\frac{p-1}{p}} \int_{\Omega} a(x)g(u_t) \cdot u_t dx, \end{aligned} \quad (3.7)$$

where $\eta > 0$ is a small enough constant, which will be determined later.

Multiplying (2.2)₁ by $u_t + \delta u$ and integrating over Ω , we get

$$\frac{d}{dt}E(t) + I(t) = 0, \quad (3.8)$$

where

$$\begin{aligned} E(t) &= E_0(t) + \delta \varepsilon(t) \langle u_t, u \rangle, \\ I(t) &= \delta \|u\|_1^2 + \left(-\frac{\varepsilon'(t)}{2} - \delta \varepsilon(t)\right) \|u_t\|^2 + \langle \eta^t(s), \eta_s^t(s) \rangle_{\mu,1} - \delta \varepsilon'(t) \langle u_t, u \rangle + \delta \int_0^\infty \mu(s) \langle \nabla \eta^t(s), \nabla u(t) \rangle ds \\ &\quad + \delta \langle f(u), u \rangle - \delta \langle h, u \rangle + \int_{\Omega} a(x)g(u_t) \cdot (u_t + \delta u) dx, \end{aligned}$$

therefore, we get

$$E(t) = E(\tau) - \int_{\tau}^t I(s) ds. \quad (3.9)$$

Together with Hölder, Young, Poincaré inequalities, it follows that

$$\begin{aligned} E(t) &\geq \frac{1}{2} \varepsilon(t) \|u_t\|^2 + \frac{\nu}{4} \|u\|_2^2 + \frac{1}{2} \|\eta^t\|_{\mu,1}^2 - \delta \varepsilon(t) \|u_t\| \cdot \|u\| \\ &\geq \left(\frac{\nu}{4} - \frac{\delta^2 L}{\lambda_1}\right) \|u\|_1^2 + \frac{1}{4} \varepsilon(t) \|u_t\|^2 + \frac{1}{2} \|\eta^t\|_{\mu,1}^2 - \left(\frac{1}{\lambda_1 \nu} \|h\|^2 + \frac{c_1}{2}\right) \\ &\geq \frac{\nu}{8} (\|u\|_1^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t\|_{\mu,1}^2) - \left(\frac{1}{\lambda_1 \nu} \|h\|^2 + \frac{c_1}{2}\right), \end{aligned} \quad (3.10)$$

here we use $\frac{\nu}{4} - \frac{\delta^2 L}{\lambda_1} > \frac{\nu}{8}$ for $\delta < \nu$ small enough.

Thanks to (1.5)-(1.6), for any $\delta > 0$, there exist $C_{\delta} > 0$ such that

$$\int_{\Omega} a(x)g(u_t) \cdot u_t dx \geq 2\delta \|u_t\|^2 - C_{\delta} |\Omega|;$$

moreover,

$$\begin{aligned} \delta \left| \int_0^\infty \mu(s) \langle \nabla \eta^t(s), \nabla u(t) \rangle ds \right| &\leq \delta \int_0^\infty \mu(s) \|\nabla \eta^t(s)\| \cdot \|\nabla u(t)\| ds \\ &\leq \frac{\rho}{4} \|\eta^t(s)\|_{\mu,1}^2 + \frac{\delta^2 m_0}{\rho} \|u\|_1^2. \end{aligned} \quad (3.11)$$

Collecting all the above estimates and due to (2.7), Hölder, Young inequalities, it leads to

$$\begin{aligned}
 I(t) &\geq 2\delta \|u_t\|^2 - C_\delta |\Omega| - \delta [C \int_{\Omega} a(x)|u|dx + \eta \|u\|_1^2 + C_\eta \|u\|_1^{\frac{p-1}{p}} \int_{\Omega} a(x)g(u_t) \cdot u_t dx] + \frac{\rho}{4} \|\eta^t(s)\|_{\mu,1}^2 \\
 &\quad + (\frac{|\varepsilon'(t)|}{2} - \delta \varepsilon(t)) \|u_t\|^2 - \frac{1}{2} |\varepsilon'(t)| \|u_t\|^2 - \frac{\delta^2 L}{2\lambda_1} \|u\|_1^2 + \delta v \|u\|_1^2 - \delta c_2 - \delta \|h\| \cdot \|u\| - \frac{\delta^2 m_0}{\rho} \|u\|_1^2 \\
 &\geq (2\delta - \delta \varepsilon(t)) \|u_t\|^2 + (\frac{\delta v}{2} - \delta \eta - \frac{\delta^2 L}{2\lambda_1} - \frac{\delta^2 m_0}{\rho}) \|u\|_1^2 + \frac{\rho}{4} \|\eta^t(s)\|_{\mu,1}^2 - \delta C_\eta \|u\|_1^{\frac{p-1}{p}} \int_{\Omega} a(x)g(u_t) \cdot u_t dx \\
 &\quad - C_\delta |\Omega| - \frac{\delta}{2v} \|h\|^2 - (C\|a\|_{L^\infty})^2 \\
 &\geq \frac{\delta v}{4} (\varepsilon(t) \|u_t\|^2 + \|u\|_1^2 + \|\eta^t(s)\|_{\mu,1}^2) - C_\delta (E_0(\tau) \int_{\Omega} a(x)g(u_t) \cdot u_t dx - C_\delta (|\Omega| + \|h\|^2) - \delta c_2 \\
 &\quad - (C\|a\|_{L^\infty})^2,
 \end{aligned} \tag{3.12}$$

where for $\delta < \min\{v, \rho\}$ small enough. Then, from (3.9)-(3.12), (3.6) yields

$$\begin{aligned}
 &\frac{v}{8} (\|u\|_1^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t(s)\|_{\mu,1}^2) - m_1 \\
 &\leq -\delta \int_{\tau}^t [\frac{\delta v}{4} (\|u(r)\|_1^2 + \varepsilon(r) \|u_t(r)\|^2 + \|\eta^r(s)\|_{\mu,1}^2) - m_2] dr + E(\tau),
 \end{aligned}$$

where $m_1 = \frac{1}{\lambda_1 v} \|h\|^2 + \frac{c_1}{2} + C_\delta E_0(\tau) (E_0(\tau) + \frac{1}{\lambda_1 v} \|h\|^2 + \frac{c_1}{2})$, $m_2 = C_\delta (|\Omega| + \|h\|^2) + \delta c_2 + (C\|a\|_{L^\infty})^2$. So, for any $K_0 > \frac{4m_2}{\delta v}$, there exists $t_0 > \tau$ such that

$$\|u(t_0)\|_1^2 + \varepsilon(t_0) \|u_t(t_0)\|^2 + \|\eta^{t_0}(s)\|_{\mu,1}^2 \leq K_0.$$

As a result, let $B_t = \bigcup_{t \geq \tau} U(t, \tau) B_0$, where

$$B_0 = \{(u_0, u_1, \eta^0) \in \mathcal{H}_\tau : \|u_0\|_1^2 + \varepsilon(\tau) \|u_1\|^2 + \|\eta^0(s)\|_{\mu,1}^2 \leq K_0\},$$

then B_t is a bounded absorbing set for process $U(t, \tau)$.

In addition, from the above discussion, there exists a positive constant R_0 such that

$$\|u(t)\|_1^2 + \varepsilon(t) \|u_t(t)\|^2 + \|\eta^t(s)\|_{\mu,1}^2 \leq R_0, \quad \forall t \geq t_0 > \tau.$$

□

The proof for Lemma 3.1. Let $z_1(\tau), z_2(\tau) \in \mathcal{H}_\tau$ such that $\|z_i(\tau)\|_{\mathcal{H}_\tau} \leq R$, $i = 1, 2$, and denote by C a generic positive constant depending on R but independent of z_i . We first observe that the energy estimates in Lemma 3.2 above ensure:

$$\|U(t, \tau) z_i(\tau)\|_{\mathcal{H}_t} \leq C. \tag{3.13}$$

We set $\{u_i(t), \partial_t u_i(t), \eta_i^t(s)\} = U(t, \tau) z_i(\tau)$ and denote $\bar{z}(t) = \{\bar{u}(t), \bar{u}_t(t), \bar{\eta}^t(s)\} = U(t, \tau) z_1(\tau) - U(t, \tau) z_2(\tau)$. Then the difference between the two solutions with initial data $\bar{z}(\tau) = z_1(\tau) - z_2(\tau)$ fulfills

$$\varepsilon(t) \bar{u}_{tt} - \Delta \bar{u} + a(x)(g(u_{1t}) - g(u_{2t})) - \int_0^\infty \mu(s) \Delta \bar{\eta}^t(s) ds + f(u_1) - f(u_2) = 0.$$

Multiplying the above equation with $2\bar{u}_t$ and integrating over Ω , we obtain

$$\frac{d}{dt} \|\bar{z}\|_{\mathcal{H}_t}^2 - \varepsilon'(t) \|\bar{u}_t\|^2 + 2 \int_{\Omega} a(x)((g(u_{1t}) - g(u_{2t}))) \bar{u}_t dx + 2 \langle \bar{\eta}^t(s), \bar{\eta}_s^t(s) \rangle_{\mu,1} = -2 \langle f(u_1) - f(u_2), \bar{u}_t \rangle.$$

From (1.4) and the strict increase of g , we have

$$2 \int_{\Omega} a(x)(g(u_{1t}) - g(u_{2t})) \bar{u}_t dx \geq 0.$$

By exploiting (1.7), (3.13), and Hölder, Young inequality, and the embedding $H^1 \hookrightarrow L^6$, it yields

$$\begin{aligned} -2\langle f(u_1) - f(u_2), \bar{u}_t \rangle &\leq C \int_{\Omega} (1 + |u_1|^2 + |u_2|^2) \cdot |\bar{u}| \cdot |\bar{u}_t| dx \\ &\leq C[1 + \|u_1\|_{L^6}^2 + \|u_2\|_{L^6}^2] \cdot \|\bar{u}\|_1 \cdot \|\bar{u}_t\| \\ &\leq C[1 + \|u_1\|_1^2 + \|u_2\|_1^2] \cdot \|\bar{u}\|_1 \cdot \|\bar{u}_t\| \\ &\leq C\|\bar{u}\|_1 \|\bar{u}_t\| \\ &\leq 2\|\bar{u}_t\|^2 + C\|\bar{u}\|_1^2, \end{aligned}$$

from (3.3) yields

$$\langle \bar{\eta}^t, \bar{\eta}_s^t \rangle_{\mu,1} \geq \frac{\rho}{2} \|\bar{\eta}^t\|_{\mu,1}^2.$$

Consequently, we end up with the differential inequality

$$\frac{d}{dt} \|\bar{z}(t)\|_{\mathcal{H}_t}^2 \leq C(L+1) \frac{1}{\varepsilon(t)} \|\bar{z}(t)\|_{\mathcal{H}_t}^2,$$

then applying the Gronwall's Lemma on $[\tau, t]$, we obtain

$$\|\bar{z}(t)\|_{\mathcal{H}_t}^2 \leq e^{C(L+1) \int_{\tau}^t \frac{1}{\varepsilon(s)} ds} \|\bar{z}(\tau)\|_{\mathcal{H}_{\tau}}^2.$$

3.2 A priori estimates

The main purpose of this part is to establish (3.25)-(3.27), which will be used to obtain the asymptotic compactness of the process.

Let $(u_i(t), u_{i_t}(t), \eta_i^t(\cdot))$ ($i = 1, 2$) be the corresponding solution of (2.2)-(2.3) with initial datum $(u_0^i(\tau), v_0^i(\tau), \eta_i^T) \in \{B_{\tau}\}_{\tau \in \mathbb{R}}$. For convenience, we introduce notations

$$g_i(t) = g(u_{i_t}(t)), \quad f_i(t) = f(u_i(t)), \quad i = 1, 2,$$

and

$$w = u_1(t) - u_2(t), \quad \zeta^t = \eta_1^t - \eta_2^t,$$

then $w(t)$ satisfies

$$\begin{cases} \varepsilon(t)w_{tt} - \Delta w + a(x)(g_1(t) - g_2(t)) - \int_0^{\infty} \mu(s)\Delta \zeta^t(s)ds + f_1(t) - f_2(t) = 0, & t > T, \\ w(x, T) = u_0^1(T) - u_0^2(T), \quad w_t(x, T) = v_0^1(T) - v_0^2(T), \quad \zeta^T = \eta_1^T - \eta_2^T, \\ w|_{\partial\Omega} = \zeta^t|_{\partial\Omega} = 0. \end{cases} \quad (3.14)$$

Denote

$$\mathcal{E}_w(t) = \frac{1}{2}(\|w\|_1^2 + \varepsilon(t)\|w_t\|^2 + \|\zeta^t\|_{\mu,1}^2).$$

Taking the inner product (3.14)₁ with w_t in $L^2(\Omega)$, and by (3.14)₂, we find

$$\frac{d}{dt} \mathcal{E}_w(t) + \langle a(x)(g_1(t) - g_2(t)), w_t \rangle - \frac{\varepsilon'(t)}{2} \|w_t\|^2 + \langle \zeta^t, \zeta_s^t \rangle_{\mu,1} + \langle f_1 - f_2, w_t(\xi) \rangle = 0. \quad (3.15)$$

Integrating (3.15) over $[s, t]$, we have

$$\begin{aligned} \mathcal{E}_w(t) - \mathcal{E}_w(s) + \int_s^t \langle a(x)(g_1(\xi) - g_2(\xi)), w_t(\xi) \rangle d\xi - \frac{1}{2} \int_s^t \varepsilon'(\xi) \|w_t(\xi)\|^2 d\xi \\ + \frac{\rho}{2} \int_s^t \|\zeta^t(\xi)\|_{\mu,1}^2 d\xi + \int_s^t \langle f_1 - f_2, w_t(\xi) \rangle d\xi \leq 0, \end{aligned} \quad (3.16)$$

where $T \leq s \leq t$. Thanks to $\varepsilon'(t) < 0$, then from (3.16) yields

$$\int_s^t \langle a(x)(g_1(\xi) - g_2(\xi)), w_t(\xi) \rangle d\xi + \frac{\rho}{2} \int_s^t \|\zeta^t(\xi)\|_{\mu,1}^2 d\xi \leq \varepsilon_w(s) - \int_s^t \langle f_1 - f_2, w_t(\xi) \rangle d\xi,$$

thus, for $a(x) \geq \alpha_0 > 0$, $g' > 0$, there holds

$$\int_s^t \|\zeta^t(\xi)\|_{\mu,1}^2 d\xi \leq \frac{2}{\rho} \varepsilon_w(s) - \frac{2}{\rho} \int_s^t \langle f_1 - f_2, w_t(\xi) \rangle d\xi. \quad (3.17)$$

Combining with (1.3) and Lemma 2.8 we get that, for any $\delta > 0$, there exists $C_\delta > 0$, such that

$$\varepsilon(\xi)|w_t|^2 \leq L|w_t|^2 \leq \delta L + LC_\delta(g_1(\xi) - g_2(\xi))w_t(\xi),$$

namely

$$\int_T^t \varepsilon(\xi)\|w_t(\xi)\|^2 d\xi \leq \delta L|\Omega|(t - T) + \frac{C_\delta L}{\alpha_0} \varepsilon_w(T) - \frac{LC_\delta}{\alpha_0} \int_T^t \langle f_1 - f_2, w_t(\xi) \rangle ds. \quad (3.18)$$

Multiplying (3.14)₁ by w , and integrating over $\Omega \times [T, t]$, we obtain

$$\begin{aligned} & \int_T^t \|w(s)\|_1^2 ds + \varepsilon(t)\langle w_t(t), w(t) \rangle \\ &= \langle \varepsilon(T)w_t(T), w(T) \rangle + \int_T^t \varepsilon'(s)\langle w_t(s), w(s) \rangle ds + \int_T^t \varepsilon(s)\|w_t(s)\|^2 ds - \int_T^t \int_0^\infty \mu(s)\langle \nabla \zeta^t(s), \nabla w(t) \rangle ds \\ & \quad - \int_T^t \langle f_1 - f_2, w(s) \rangle ds - \int_T^t \langle a(x)(g_1 - g_2), w(s) \rangle ds. \end{aligned}$$

Then, using the following inequality

$$\int_0^\infty \mu(s)\langle \nabla \zeta^t(s), \nabla w(t) \rangle ds \leq \frac{1}{2\sigma} \|\zeta^t\|_{\mu,1}^2 + \frac{\sigma m_0}{2} \|w\|_1^2, \quad (3.19)$$

yields

$$\begin{aligned} & \left(1 - \frac{\sigma m_0}{2}\right) \int_T^t \|w(s)\|_1^2 ds + \varepsilon(t)\langle w_t(t), w(t) \rangle \\ & \leq \langle \varepsilon(T)w_t(T), w(T) \rangle + \int_T^t \varepsilon'(s)\langle w_t(s), w(s) \rangle ds + \int_T^t \varepsilon(s)\|w_t(s)\|^2 ds + \frac{1}{2\sigma} \int_T^t \|\zeta^\xi(s)\|_{\mu,1}^2 d\xi \\ & \quad - \int_T^t \langle f_1 - f_2, w(s) \rangle ds - \int_T^t \langle a(x)(g_1 - g_2), w(s) \rangle ds. \end{aligned} \quad (3.20)$$

Therefore, let $0 < \sigma < \frac{2}{m_0}$, from (3.17)-(3.18), (3.20) yields

$$\begin{aligned}
 & 2 \int_T^t \mathcal{E}_w(s) ds \\
 \leq & \left[\left(1 + \frac{2}{2 - \sigma m_0}\right) \frac{LC_\delta}{\alpha_0} + \frac{2}{\sigma(2 - \sigma m_0)\rho} + 1 \right] \mathcal{E}_w(T) + \left(1 + \frac{2}{2 - \sigma m_0}\right) \delta L |\Omega| (t - T) \\
 & - \left[\left(1 + \frac{2}{2 - \sigma m_0}\right) \frac{LC_\delta}{\alpha_0} + \frac{2}{\sigma(2 - \sigma m_0)\rho} + 1 \right] \int_T^t \langle f_1 - f_2, w_t(s) \rangle ds + \frac{2}{2 - \sigma m_0} \int_T^t \varepsilon'(s) \langle w_t(s), w(s) \rangle ds \\
 & - \frac{2}{2 - \sigma m_0} \varepsilon(t) \langle w_t(t), w(t) \rangle + \frac{2}{2 - \sigma m_0} \varepsilon(T) \langle w_t(T), w(T) \rangle - \frac{2}{2 - \sigma m_0} \int_T^t \langle a(x)(g_1 - g_2), w(s) \rangle ds \\
 & - \frac{2}{2 - \sigma m_0} \int_T^t \langle f_1 - f_2, w(s) \rangle ds.
 \end{aligned} \quad (3.21)$$

Integrating (3.16) over $[T, t]$, we have

$$\begin{aligned}
 & (t - T) \mathcal{E}_w(t) + \int_T^t \int_s^t a(x) \langle g_1(\xi) - g_2(\xi), w_t(\xi) \rangle d\xi ds - \frac{1}{2} \int_T^t \int_s^t \varepsilon'(\xi) \|w_t(\xi)\|^2 d\xi ds \\
 & + \frac{\rho}{2} \int_T^t \int_s^t \|\zeta^t\|_{\mu,1}^2 d\xi ds \leq - \int_T^t \int_s^t \langle f_1(\xi) - f_2(\xi), w_t(\xi) \rangle d\xi ds + \int_T^t \mathcal{E}_w(s) ds.
 \end{aligned} \quad (3.22)$$

Since $\langle g_1(\xi) - g_2(\xi), w_t(\xi) \rangle - \frac{1}{2} \varepsilon'(\xi) \|w_t(\xi)\|^2 + \|\zeta^t\|_{\mu,2}^2 > 0$, together with (3.21)-(3.22), there holds

$$\begin{aligned}
 & (t - T) \mathcal{E}_w(t) \\
 \leq & \left[\left(\frac{1}{2} + \frac{1}{2 - \sigma m_0}\right) \frac{LC_\delta}{\alpha_0} + \frac{1}{\sigma(2 - \sigma m_0)\rho} + \frac{1}{2} \right] \mathcal{E}_w(T) + \left(\frac{1}{2} + \frac{1}{2 - \sigma m_0}\right) \delta L |\Omega| (t - T) \\
 & + \frac{1}{2 - \sigma m_0} \langle \varepsilon(T) w_t(T), w(T) \rangle - \frac{1}{2 - \sigma m_0} \langle \varepsilon(t) w_t(t), w(t) \rangle + \frac{1}{2 - \sigma m_0} \int_T^t \varepsilon'(s) \langle w_t(s), w(s) \rangle ds \\
 & - \frac{1}{2 - \sigma m_0} \int_T^t \langle f_1(s) - f_2(s), w(s) \rangle ds - \frac{1}{2 - \sigma m_0} \int_T^t a(x) \langle (g_1(s) - g_2(s)), w(s) \rangle ds \\
 & - \left[\left(\frac{1}{2} + \frac{1}{2 - \sigma m_0}\right) \frac{LC_\delta}{\alpha_0} + \frac{1}{\sigma(2 - \sigma m_0)\rho} + \frac{1}{2} \right] \int_T^t \langle f_1 - f_2, w_t(s) \rangle ds - \int_T^t \int_s^t \langle f_1 - f_2, w_t(\xi) \rangle d\xi ds.
 \end{aligned} \quad (3.23)$$

Next, we will deal with $\int_T^t \int_\Omega a(x) (g_1(s) - g_2(s)) w(s) dx ds$.

Multiplying (2.2) by u_{i_t} and integrating over Ω , we achieve

$$\frac{1}{2} \frac{d}{dt} [\varepsilon(t) \|u_{i_t}\|^2 + \|u_i\|_1^2 + \|\eta^i\|_{\mu,1}^2 + 2 \int_\Omega F(u_i) dx - 2 \int_\Omega h(x) u_i dx] + \langle \eta^i, \eta_s^i \rangle_{\mu,1} + \langle a(x) g(u_{i_t}), u_{i_t} \rangle = 0,$$

which combines with (3.4)-(3.6) and the existence of time-dependent absorbing set, means that

$$\int_T^t \langle a(x) g(u_{i_t}), u_{i_t} \rangle ds \leq E_0(T) - E_0(t) \leq C_T,$$

where the constant C_T depends on T . Then, similar to the computation in [1,16] it follows that

$$\begin{aligned}
 & \left| \int_T^t \int_{\Omega} a(x) g(u_{i_t}) w(x, s) dx ds \right| \\
 & \leq \left(\int_T^t \int_{\Omega} a(x) |g(u_{i_t})|^{\frac{p+1}{p}} dx ds \right)^{\frac{p}{p+1}} \left(\int_T^t \int_{\Omega} a(x) |w(x, s)|^{q+1} dx ds \right)^{\frac{1}{q+1}} \\
 & \leq \left(\int_T^t \int_{\Omega} a(x) (C' + C' g(u_{i_t}) \cdot u_{i_t}) dx ds \right)^{\frac{p}{p+1}} \left(\int_T^t \int_{\Omega} a(x) |w(x, s)|^{p+1} dx ds \right)^{\frac{1}{p+1}} \\
 & \leq \left[C' |\Omega| \|a\|_{L^\infty} (t - T) + C' \int_T^t \int_{\Omega} a(x) g(u_{i_t}) \cdot u_{i_t} dx ds \right]^{\frac{p}{p+1}} \left(\int_T^t \int_{\Omega} a(x) |w(x, s)|^{p+1} dx ds \right)^{\frac{1}{p+1}} \\
 & \leq (C' |\Omega| \|a\|_{L^\infty} (t - T) + C' C_T)^{\frac{p}{p+1}} \left(\int_T^t \int_{\Omega} a(x) |w(x, s)|^{p+1} dx ds \right)^{\frac{1}{p+1}} \\
 & \leq \left[(C' |\Omega| \|a\|_{L^\infty} (t - T))^{\frac{p}{p+1}} + (C' C_T)^{\frac{p}{p+1}} \right] \left(\int_T^t \int_{\Omega} a(x) |w(x, s)|^{p+1} dx ds \right)^{\frac{1}{p+1}}.
 \end{aligned} \tag{3.24}$$

Now, combining with (3.23), (3.24) we have

$$\begin{aligned}
 & (t - T) \mathcal{E}_w(t) \\
 & \leq \left[\left(\frac{1}{2} + \frac{1}{2 - \sigma m_0} \right) \frac{LC_\delta}{\alpha_0} + \frac{1}{\sigma(2 - \sigma m_0)\rho} + \frac{1}{2} \right] \mathcal{E}_w(T) + \left(\frac{1}{2} + \frac{1}{2 - \sigma m_0} \right) \delta L |\Omega| (t - T) \\
 & \quad + \frac{1}{2 - \sigma m_0} \langle \mathcal{E}(T) w_t(T), w(T) \rangle - \frac{1}{2 - \sigma m_0} \langle \mathcal{E}(t) w_t(t), w(t) \rangle + \frac{1}{2 - \sigma m_0} \int_T^t \mathcal{E}'(s) \langle w_t(s), w(s) \rangle ds \\
 & \quad - \int_T^t \int_s^t \langle f_1 - f_2, w_t(\xi) \rangle d\xi ds - \left[\left(\frac{1}{2} + \frac{1}{2 - \sigma m_0} \right) \frac{LC_\delta}{\alpha_0} + \frac{1}{\sigma(2 - \sigma m_0)\rho} + \frac{1}{2} \right] \int_T^t \langle f_1 - f_2, w_t(s) \rangle ds \\
 & \quad + \frac{\left[(C' |\Omega| \|a\|_{L^\infty} (t - T))^{\frac{p}{p+1}} + (C' C_T)^{\frac{p}{p+1}} \right]}{2 - \sigma m_0} \left(\int_T^t \int_{\Omega} a(x) |w(x, s)|^{p+1} dx ds \right)^{\frac{1}{p+1}} \\
 & \quad - \frac{1}{2 - \sigma m_0} \int_T^t \langle f_1(s) - f_2(s), w(s) \rangle ds.
 \end{aligned}$$

Set

$$\begin{aligned}
& \phi_T^t((u_0^1(T), v_0^1(T)), (u_0^2(T), v_0^2(T))) \\
&= -\frac{1}{(t-T)(2-\sigma m_0)} \langle \varepsilon(t) w_t, w \rangle + \frac{1}{(t-T)(2-\sigma m_0)} \int_T^t \varepsilon'(s) \langle w_t(s), w(s) \rangle ds \\
&\quad - \frac{1}{(t-T)(2-\sigma m_0)} \int_T^t \langle f_1(s) - f_2(s), w(s) \rangle ds - \frac{1}{(t-T)} \int_T^t \int_s^t \langle f_1 - f_2, w_t(\xi) \rangle d\xi ds \\
&\quad + \frac{\left[(C'|\Omega| \|a\|_{L^\infty}(t-T))^{\frac{p}{p+1}} + (C' C_T)^{\frac{p}{p+1}} \right]}{(t-T)(2-\sigma m_0)} \left(\int_T^t \int_\Omega a(x) |w(x, s)|^{p+1} dx ds \right)^{\frac{1}{p+1}} \\
&\quad - \frac{\left[\left(\frac{1}{2} + \frac{1}{2-\sigma m_0} \right) \frac{LC_\delta}{\alpha_0} + \frac{1}{\sigma(2-\sigma m_0)\rho} + \frac{1}{2} \right]}{t-T} \int_T^t \langle f_1 - f_2, w_t(s) \rangle ds,
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
C_M = & \left(\frac{1}{2} + \frac{1}{2-\sigma m_0} \right) \frac{LC_\delta}{\alpha_0} + \frac{1}{\sigma(2-\sigma m_0)\rho} + \frac{1}{2} \mathcal{E}_w(T) + \left(\frac{1}{2} + \frac{1}{2-\sigma m_0} \right) \delta L |\Omega| (t-T) \\
& + \frac{1}{2-\sigma m_0} \langle \varepsilon(T) w_t(T), w(T) \rangle.
\end{aligned} \tag{3.26}$$

Then we have

$$\mathcal{E}_w(t) \leq \frac{C_M}{t-T} + \phi_T^t((u_0^1(T), v_0^1(T)), (u_0^2(T), v_0^2(T))). \tag{3.27}$$

3.3. Asymptotically compact

Theorem 3.1. Under the assumption (1.2)-(1.10), for any fixed $t \in \mathbb{R}$, bounded sequence $\{x_n\}_{n=1}^\infty \subset X_{\tau_n}$ and any $\{\tau_n\}_{n=1}^\infty \subset \mathbb{R}^{-t}$, with $\tau_n \rightarrow -\infty$ as $n \rightarrow \infty$, sequence $\{U(t, \tau_n)x_n\}_{n=1}^\infty$ has a convergent subsequence.

Proof. For any fixed $\varepsilon > 0$, we first choose some proper δ such that $(\frac{1}{2} + \frac{1}{2-\sigma m_0})\delta L |\Omega| \leq \frac{\varepsilon}{4}$, then for some fixed t , let $T < t$ such that $t-T$ so large that

$$\frac{C_M}{t-T} < \frac{\varepsilon}{2}.$$

Hence, thanks to Theorem 2.10, we only need to verify that $\phi_T^t \in \mathcal{C}(B_t)$, for each fixed T .

Let (u_n, u_{n_t}, η_n^t) be the solution corresponding to initial data $(u_0^n, v_0^n, \eta_0^n) \in B_T$ for the problem (2.2). From (3.4), $\|u_n\|_1^2 + \varepsilon(\xi)\|u_{n_t}\|^2 + \|\eta_n^t\|_{\mu,1}^2$ is bounded, where the bound depends on the T , furthermore, $\|u_n\|_1^2$ is bounded. Moreover, by (1.2), (1.3), for fixed T , $\xi \in [T, t]$, $\varepsilon(\xi)$ is bounded, hence $\|u_{n_t}\|^2$ is bounded.

According to Alaoglu Theorem, without loss of generality, we assume that (at most by passing to subsequence)

- (i) $u_n \rightarrow u$ *-weakly in $L^\infty(T, t; H_0^1(\Omega))$;
- (ii) $u_{n_t} \rightarrow u_t$ *-weakly in $L^\infty(T, t; L^2(\Omega))$;
- (iii) $u_n \rightarrow u$ strong in $L^{p+1}(T, t; L^{p+1}(\Omega))$;
- (iv) $u_n(T) \rightarrow u(T)$ and $u_n(t) \rightarrow u(t)$ strong in $L^2(\Omega)$.

Here we use the compact emdedding $H_0^1 \hookrightarrow L^{p+1}$ ($p < 5$).

Now, we will deal with each term in (3.25) one by one.

Firstly, from Lemma 3.2, (i)-(iv) we get

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_\Omega \varepsilon(t)(u_{n_t}(t) - u_{m_t}(t))(u_n(t) - u_m(t)) dx = 0, \tag{3.28}$$

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_T^t \int_\Omega L(u_{n_t}(s) - u_{m_t}(s))(u_n(s) - u_m(s)) dx ds = 0, \tag{3.29}$$

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} C \left(\int_T^t \int_{\Omega} a(x) |u_n(s) - u_m(s)|^{p+1} dx ds \right)^{\frac{1}{p+1}} = 0, \quad (3.30)$$

where $C = \frac{(C' \|\Omega\| \|a\|_{L^\infty} (t-T))^{\frac{p}{p+1}} + (C' C_T)^{\frac{p}{p+1}}}{(t-T)(2-\sigma m_0)}$, and

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_T^t \int_{\Omega} (f(u_n) - f(u_m))(u_n(s) - u_m(s)) dx ds = 0. \quad (3.31)$$

Similar to the proof of the Theorem 5.4 in [1], we have

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_T^t \int_{\Omega} (f(u_n) - f(u_m))(u_{n_t}(s) - u_{m_t}(s)) dx ds = 0. \quad (3.32)$$

At the same time, for each fixed t , $|\int_s^t \int_{\Omega} (u_{n_t}(\xi) - u_{m_t}(\xi))(f(u_n(\xi)) - f(u_m(\xi))) dx d\xi|$ is bounded, then by the Lebesgue dominated convergence theorem we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_T^t \int_s^t \int_{\Omega} (u_{n_t}(\xi) - u_{m_t}(\xi))(f(u_n(\xi)) - f(u_m(\xi))) dx d\xi ds \\ &= \int_T^t \left(\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_s^t \int_{\Omega} (u_{n_t}(\xi) - u_{m_t}(\xi))(f(u_n(\xi)) - f(u_m(\xi))) dx d\xi \right) ds \\ &= 0. \end{aligned} \quad (3.33)$$

Hence, collecting all (3.28)-(3.33), we get that $\phi_T^t \in \mathcal{C}(B_t)$, so the proof is completed. $\square$

3.3 Existence of the time-dependent global attractor

Theorem 3.2. Under the conditions (1.2)-(1.10), the process $U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t$ generated by problem (1.1) has an invariant time-dependent global attractor $\mathcal{U} = \{A_t\}_{t \in \mathbb{R}}$.

Proof. From Lemma 3.2, Theorem 3.1 and Theorem 2.9, we know that there exists a unique time-dependent global attractor $\mathcal{U} = \{A_t\}_{t \in \mathbb{R}}$. Furthermore, by virtue of the strong continuity of the process stated in Lemma 3.1, we can obtain that $\mathcal{U}$ is invariant. $\square$

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