Research Article

Jinxing Zhao* and Guixin Deng

Remark on subgroup intersection graph of finite abelian groups

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Abstract: Let G be a finite group. The subgroup intersection graph $\Gamma(G)$ of G is a graph whose vertices are non-identity elements of G and two distinct vertices x and y are adjacent if and only if $|\langle x \rangle \cap \langle y \rangle| > 1$, where $\langle x \rangle$ is the cyclic subgroup of G generated by x . In this paper, we show that two finite abelian groups are isomorphic if and only if their subgroup intersection graphs are isomorphic.

Keywords: subgroup intersection graph, finite abelian group, isomorphic

MSC 2020: 05C25

1 Introduction

There are many papers on assigning a graph to a group. The most famous example is the Cayley graphs, whose vertices are elements of groups and adjacency relations are defined by subsets of the groups, see [1,2]. For example, B. Csákány and G. Pollák [3] defined the intersection graph of a group G , whose vertices are the proper non-trivial subgroups of G , and two vertices H_1 and H_2 are adjacent if $H_1 \neq H_2$ and they have a non-trivial intersection. In [4], B. Zelinka continued the work on intersection graphs of finite abelian groups. R. Shen classified finite groups with disconnected intersection graphs in [5].

The authors of [6] introduced the power graph of group G , whose vertices set is G and two vertices x and y are adjacent if $x \neq y$ and one is a power of the other. P. J. Cameron and S. Ghosh [7] proved that finite abelian groups with isomorphic power graphs are isomorphic. They also conjectured that G_1 and G_2 have the same number of elements of each order if their power graphs are isomorphic.

In this paper, we mainly study the subgroup intersection graph of finitely generated abelian groups, which is denoted by $\Gamma(G)$. This graph was defined by T. T. Chelvam and M. Sattanathan in [8] as follows: the vertices are the non-identity elements of G , and two vertices x and y are adjacent if and only if $x \neq y$ and $|\langle x \rangle \cap \langle y \rangle| > 1$, where $\langle x \rangle$ is the cyclic subgroup of G generated by x . They characterized some fundamental properties of $\Gamma(G)$ in [8]. In particular, they determined that when $\Gamma(G)$ is complete or Eulerian. In [9], T. T. Chelvam and M. Sattanathan obtained all planar $\Gamma(G)$ and unicycle $\Gamma(G)$ for abelian groups. Our main result shows that two finite abelian groups with isomorphic subgroup intersection graphs are isomorphic. We also give example to show that this result is not valid for non-abelian groups and infinite abelian groups.

Figure 1 shows the subgroup intersection graph of the cyclic group of order 6.

This paper is organized as follows. In Section 2, we introduce notation and terminology of graphs and groups and review some of the standard facts of finite generated abelian group. In Section 3, we determine

* **Corresponding author: Jinxing Zhao**, School of Mathematical Sciences, Inner Mongolia University, Hohhot, 010021, People's Republic of China, e-mail: zhjx@imu.edu.cn

Guixin Deng: College of Mathematics and Statistics, Nanning Normal University, Nanning, 530001, People's Republic of China, e-mail: dengguixin@live.com

Figure 1: Subgroup intersection graph of cyclic group of order 6.

the structure of subgroup intersection graphs on finite abelian p -groups and show that two finite abelian p -groups are isomorphic if their subgroup intersection graphs are isomorphic. We also give an example to show that the result is not valid for non-abelian groups. In Section 4, we present and prove our main result.

2 Preliminaries

We follow the notations of graph used in [10]. We will write $V(\Gamma)$ and $E(\Gamma)$ for the set of vertices and the set of edges of Γ . For $x, y \in V(\Gamma)$, the notation $(x, y) \in E(\Gamma)$ means that x and y are adjacent. The induced subgraph of Γ on a vertex set $S \subset V(\Gamma)$, denoted by $\Gamma[S]$, is the subgraph with vertex set S and $(x, y) \in E(\Gamma[S])$ if and only if $(x, y) \in E(\Gamma)$. Let $\{\Gamma_i\}_{i \in I}$ be a collection of graphs. The union graph of Γ_i , denoted by $\cup_{i \in I} \Gamma_i$, is the graph whose vertex set and edge set are disjoint union of $V(\Gamma_i)$ and $E(\Gamma_i)$, respectively. Hence, each graph is the union of its connected components. For simplicity of notation, we write $\Gamma = n\Gamma_1$ if Γ is the union of n copies of Γ_1 .

Next, we review some results of the structure of finite generated abelian groups. For convenience, we repeat the relevant material from Chapter I of [11] without proof.

Throughout this paper, a group G is always written multiplicatively. The identity of G will be denoted by 1_G and the subset of non-identity elements of G will be denoted by G^* . The order of an element $g \in G$, denoted by $o(g)$, is the minimal positive integer k such that $g^k = 1_G$. If no such k exists, then the order of g is said to be infinite. An element g is called torsion if $o(g) < \infty$. If G is a finite group, the exponent of G is the least common multiple of order of all elements of G . We will denote by G_{tor} the set of all torsion elements of a group G . If G is abelian, then G_{tor} is a subgroup of G . If G is abelian and p is a prime, we denote by $G(p)$ the subgroup of G of all elements whose order is a power of p .

Theorem 2.1. [11, Chapter I, Theorem 8.1] *Let G be a torsion abelian group. Then G is the direct sum of its subgroups $G(p)$ for all primes p such that $G(p) \neq 0$.*

A finitely generated abelian torsion group is finite. We next describe the structure of finite abelian p -group. A finite p -group G is said to be of type $(p^{r_1}, \dots, p^{r_s})$ if G is isomorphic to the product of cyclic groups of orders p^{r_i} , $i = 1, \dots, s$.

Theorem 2.2. [11, Chapter I, Theorem 8.2] *Every finite abelian p -group is isomorphic to a product of cyclic p -groups. If it is of type $(p^{r_1}, \dots, p^{r_s})$ with*

$$1 \leq r_1 \leq \dots \leq r_s,$$

then the sequence of integer $(r_1, \dots, r_s)$ is uniquely determined.

Theorem 2.3. [11, Chapter I, Theorem 8.5] *Let G be a finitely generated abelian group. Then the torsion subgroup G_{tor} is finite, $\frac{G}{G_{\text{tor}}}$ is free and there exists a free subgroup H of G such that G is the direct sum of G_{tor} and H .*

The rank of the free group $H \simeq \frac{G}{G_{\text{tor}}}$ is also called the rank of G . Theorems 2.1–2.3 completely describe the structure of finitely generated abelian groups.

3 The structure of $\Gamma(G)$ for a finite p -group

We fixed a prime p in this section. In [8], T. T. Chelvam and M. Sattanathan show that the subgroup intersection graph of every finite p -group is a union of complete graphs. We give a total description of finite abelian p -groups.

Lemma 3.1. *Let G be a finite abelian p -group of exponent p^e . Then*

$$\Gamma(G) \simeq \cup_{i=0}^{e-1} C_i K_{(p-1)(\sum_{j=0}^i m_j)},$$

where $m_i = |\{x \in G | x^{p^i} = 1_G\}|$ and $c_i = \frac{1}{p-1} \left(\frac{m_{i+1}}{m_i} - \frac{m_{i+2}}{m_{i+1}} \right)$. In particular, $c_{e-1} > 0$.

Proof. Let $x \in G$ be an element with order p and let $H = \{1_G, x, x^2, \dots, x^{p-1}\}$ be the cyclic group generated by x . Since z is adjacent to some identity element of H if and only if $H \subset \langle z \rangle$, so the connected component containing x , say L_x , is a complete graph, whose vertices are $\{z \in G : \exists k \geq 0, z^{p^k} \in \{x, \dots, x^{p-1}\}\}$.

Let $n_k = |\{z \in G : z^{p^k} = x\}|$. Then there exists $i > 0$ such that $n_j = m_j$ for $j \leq i$ and $n_j = 0$ for $j > i$. Note that $|\{z \in G : z^{p^j} = x^r\}| = n_i$ for given $r \in \{1, \dots, p-1\}$. It follows that $|L_x| = (p-1)(1 + m_1 + \dots + m_i)$ vertices and there are exactly $(p-1)m_t$ vertices in L_x with order p^t for $t \in \{1, \dots, i+1\}$. Therefore, the number of elements in G with order p^{i+1} is

$$m_{i+1} - m_i = (p-1)(c_i + \dots + c_{e-1})m_i,$$

and we obtain that $c_i = \frac{1}{p-1} \left(\frac{m_{i+1}}{m_i} - \frac{m_{i+2}}{m_{i+1}} \right)$. This completes the proof. $\square$

Theorem 3.1. $\Gamma(G_1) \simeq \Gamma(G_2)$ if and only if $G_1 \simeq G_2$ for any two abelian p -groups.

Proof. Suppose that the type of G_1 is $(p^{r_1}, \dots, p^{r_s})$ and the type of G_2 is $(p^{t_1}, \dots, p^{t_l})$, where $1 \leq r_1 \leq \dots \leq r_s$, $1 \leq t_1 \leq \dots \leq t_l$. Then we have $m_i = |\{x \in G_1 | x^{p^i} = 1_{G_1}\}| = p^{\sum_{j=1}^s \min\{i, r_j\}}$, $n_i = |\{x \in G_2 | x^{p^i} = 1_{G_2}\}| = p^{\sum_{j=1}^l \min\{i, t_j\}}$, $c_i = \frac{1}{p-1} \left(\frac{m_{i+1}}{m_i} - \frac{m_{i+2}}{m_{i+1}} \right)$ and $d_i = \frac{1}{p-1} \left(\frac{n_{i+1}}{n_i} - \frac{n_{i+2}}{n_{i+1}} \right)$. Note that $c_{r_s-1} > 0$ and $d_{t_l-1} > 0$. So $r_s = t_l = e$ and $c_i = d_i$ for any $0 \leq i \leq e-1$. By a direct computation, we obtain

$$\frac{m_{i+1}}{m_i} = \frac{n_{i+1}}{n_i}, \quad i = 0, 1, \dots, e-1.$$

Combining with that $\frac{m_{i+1}}{m_i} = p^{|\{j: r_j = i+1, 1 \leq j \leq s\}|}$ and $\frac{n_{i+1}}{n_i} = p^{|\{j: t_j = i+1, 1 \leq j \leq l\}|}$, one has $s = l$, $r_i = t_i$ and $G_1 \simeq G_2$. $\square$

To the best of our knowledge, it is not a simple task to determine the number of solutions $x^{p^k} = a$ in non-abelian groups, see [12, 13]. In fact, Theorem 3.1 is not true for non-abelian groups. Let G_1 be the finite abelian p -group of type (p, p, p) and let G_2 be the non-abelian group with presentation

$$\{a, b, c | a^p = b^p = c^p = 1, a^{-1}b^{-1}ab = 1, ac = ca, bc = cb\}.$$

Then both G_1 and G_2 have exponent p , but

$$\Gamma(G_1) \simeq \Gamma(G_2) \simeq (p^2 + p + 1)K_{p-1}.$$

4 Main results

An independent set S of a graph Γ is a subset of $V(\Gamma)$ such that u and v are not adjacent for any $u, v \in S$. The independent number of Γ denoted by $\beta(\Gamma)$ is equal to $\max\{|S|\}$, where S runs over all independent sets of Γ .

Lemma 4.1. *Let G be a finite group. Then $\beta(\Gamma(G))$ is equal to the number of subgroups of prime order of G . An element of G is contained in an independent set of maximal size if and only if $o(x)$ is a power of a prime.*

Proof. Let $A = \{H_i\}_{i \in I}$ be the collection of all cyclic subgroups of prime order of G and let X be a maximal independent set of $\Gamma(G)$. We claim that: for any given $H \in A$, there exists exactly one $x \in X$ such that $H \subseteq \langle x \rangle$. Suppose that $H \not\subseteq \langle x \rangle$ for any $x \in X$. Choose a non-identity element $z \in H$. Since H is a cyclic group of prime order, one has $|\langle z \rangle \cap \langle x \rangle| = |H \cap \langle x \rangle| > 1$ if and only if $H \subseteq \langle x \rangle$. So z is not adjacent to any point in X and $X \cup \{z\}$ is also an independent set. This contradicts to the choice of X . If $H \subseteq \langle x \rangle$ and $H \subseteq \langle y \rangle$ for $x \neq y$, then $H \subseteq \langle x \rangle \cap \langle y \rangle$ and x and y are adjacent. This proves the claim.

We define a map $\sigma : A \rightarrow X$ such that $\sigma(H) = x$ if $H \subseteq \langle x \rangle$. By the claim, σ is well-defined and surjective. Therefore, $\beta(\Gamma(G)) \leq |A|$. On the other hand, choose $y_i \neq 0$ in H_i , then $\langle y_i \rangle = H_i$ and $Y = \{y_i\}_{i \in I}$ is an independent set with $|Y| = \beta(\Gamma(G))$. This shows that $\beta(\Gamma(G)) = |A|$. If $x \in G$ and $o(x)$ is a power of a prime, then there exists unique $y \in Y$ such that $\langle y \rangle \subseteq \langle x \rangle$. Thus, $Y \cup \{x\} \setminus \{y\}$ is also an independent set with size $\beta(\Gamma(G))$.

Now suppose that $a \in X$ and $o(a)$ is not a power of a prime. Then $\langle a \rangle$ contains at least two distinct subgroups of prime order. So $\sigma : A \rightarrow X$ is not injective in this case and $|X| < |A|$. The proof is finished. $\square$

Lemma 4.2. *Let G be a group and let V_1, V_2 be the set of non-identity elements of finite and infinite order of G , respectively. Let Γ_i be the induced subgraph on V_i . Then $\Gamma(G) = \Gamma_1 \cup \Gamma_2$ and each connected component of Γ_2 is a complete graph of infinite order.*

Proof. If $x, y \in G$ with $o(x) < \infty$ and $o(y) = \infty$, then each element in $\langle x \rangle$ has finite order and each non-identity element in $\langle y \rangle$ has infinite order. So x and y are not adjacent and thus $\Gamma = \Gamma_1 \cup \Gamma_2$.

Now suppose $o(x), o(y), o(z)$ are all infinite and $(x, y), (x, z) \in E(\Gamma(G))$. Then exist non-zero integers m, n, k, l such that $x^n = y^m$ and $x^k = z^l$. Hence, $y^{mk} = x^{nk} = z^{ln}$ and $(y, z) \in E(\Gamma(G))$. It follows that the connected component containing x is a complete graph. This completes the proof. $\square$

Lemma 4.3. *Suppose that G is a finitely generated abelian group of rank r . Then*

- (1) *If $r = 1$, then $\Gamma(G) = \Gamma(G_{\text{tor}}) \cup K_{\infty}$;*
- (2) *If $r \geq 2$, then $\Gamma(G)$ is the union graph of $\Gamma(G_{\text{tor}})$ and infinite countable copies of K_{∞} .*

Proof. Keeping the same notations in Lemma 4.2. Since G is countable, $\Gamma(G)$ is a union of $\Gamma_1 = \Gamma(G_{\text{tor}})$ and some copies of K_{∞} by Lemma 4.2. It is easy to prove that $\Gamma_2 = K_{\infty}$ if $r = 1$, and there are infinite connected component in Γ_2 if $r \geq 2$. $\square$

Theorem 4.1. *Suppose that G_1, G_2 are finitely generated abelian groups of rank r_1, r_2 , respectively. Then $\Gamma(G_1) \simeq \Gamma(G_2)$ if and only if $G_{1\text{tor}} \simeq G_{2\text{tor}}$ and one of the following conditions is satisfied.*

- (1) $r_1 = r_2 \leq 1$;
- (2) $\min\{r_1, r_2\} \geq 2$.

Proof. It suffices to deal with the case that $G_i = G_{i\text{tor}}$ by Lemma 4.3. Assume that $\Gamma_i = \Gamma(G_i)$ and $\varphi : \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism of graphs. Hence, $|G_1| = |G_2| = n$. Let p be a prime divisor of n . Since the induced subgraph of Γ_i on the subset $G_i(p)^*$ is isomorphic to $\Gamma(G_i(p))$, it suffices to prove that $\varphi(G_1(p)^*) = G_2(p)^*$.

Suppose $x_1, x_2 \in G_1(p)^*$. By Lemma 4.1, the order of $y_i = \varphi(x_i)$ is also a power of a prime, say p_i . We claim that $p_1 = p_2$. If x_1 and x_2 are adjacent, then y_1 and y_2 are also adjacent. Hence, both $o(y_1)$ and $o(y_2)$ are

powers of a same prime. Now assume that x_1 and x_2 are not adjacent and $p_1 \neq p_2$. Then $o(y_1 y_2) = o(y_1) o(y_2)$ and $\langle y_i \rangle \subset \langle y_1 y_2 \rangle$. It follows that $(y_1 y_2, y_1), (y_1 y_2, y_2) \in E(\Gamma_2)$ and $p_1 p_2 \nmid o(y_1 y_2)$. By Lemma 4.1, the order of $x = \varphi^{-1}(y_1 y_2)$ is not a power of a prime. Since φ is an isomorphism of graphs, we have $(x, x_1), (x, x_2) \in E(\Gamma_1)$. Hence, the cyclic group generated by x contains two distinct subgroups of order p , which is impossible. So $p_1 = p_2$.

Let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorization. According to the aforementioned arguments, there exists a permutation τ of $\{1, \dots, r\}$ such that $\varphi(G_1(p_i)^*) = G_2(p_{\tau(i)})^*$. Hence, we obtain $|G_1(p_i)| = p_i^{e_i} = |G_2(p_{\tau(i)})| = p_{\tau(i)}^{e_{\tau(i)}}$ and $\tau(i) = i$. It follows that $\varphi(G_1(p)^*) = G_2(p)^*$. So $\Gamma(G_1(p)) \simeq \Gamma(G_2(p))$. By Theorem 3.1, we have $G_1(p) \simeq G_2(p)$, and thus $G_1 \simeq G_2$. $\square$

Corollary 4.1. *Let G_1, G_2 be finite abelian groups. Then $\Gamma(G_1) \simeq \Gamma(G_2)$ if and only if $G_1 \simeq G_2$.*

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