Research Article

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Some inequalities for star duality of the radial Blaschke-Minkowski homomorphisms

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Abstract: In 2006, Schuster introduced the radial Blaschke-Minkowski homomorphisms. In this article, associating with the star duality of star bodies and dual quermassintegrals, we establish Brunn-Minkowski inequalities and monotonic inequality for the radial Blaschke-Minkowski homomorphisms. In addition, we consider its Shephard-type problems and give a positive form and a negative answer, respectively.

Keywords: radial Blaschke-Minkowski homomorphism, star duality, Brunn-Minkowski inequality, monotonic inequality, Shephard-type problem, dual quermassintegral

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1 Introduction and main results

For a compact star-shaped (about the origin) K in n -dimensional Euclidean space $\mathbb{R}^n$, the radial function of K , $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty)$, is defined by [1]

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, then K will be called a star body (with respect to the origin). Let S_o^n and S_{os}^n denote the set of star bodies in $\mathbb{R}^n$ with respect to the origin and origin-symmetric. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

The intersection body was introduced by Lutwak [2]. For each $K \in S_o^n$, $n \geq 2$, the intersection body, IK , of K is an origin-symmetric star body whose radial function is defined by

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp),$$

for all $u \in S^{n-1}$. Here, $u^\perp$ is the $(n-1)$ -dimensional hyperplane orthogonal to u and V_{n-1} denotes the $(n-1)$ -dimensional volume.

In 2006, based on the properties of intersection bodies, Schuster [3] introduced the radial Blaschke-Minkowski homomorphism, which is more general than intersection body operator as follows.

Definition 1.A. A map $\Psi : S_o^n \rightarrow S_o^n$ is called a radial Blaschke-Minkowski homomorphism, if it satisfies the following conditions:

- (1) Ψ is continuous with respect to radial metric;

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- (2) For all $K, L \in S_o^n$, $\Psi(K \hat{+} L) = \Psi K \tilde{+} \Psi L$, where $K \hat{+} L$ denotes the radial Blaschke sum of K and L , and $\Psi K \tilde{+} \Psi L$ denotes the radial Minkowski sum of ΨK and ΨL ;
- (3) For all $K \in S_o^n$ and every $\varphi \in SO(n)$, $\Psi(\varphi K) = \varphi \Psi(K)$, where $SO(n)$ denotes the group of rotations in n -dimensions.

Furthermore, Schuster [3] introduced the mixed radial Blaschke-Minkowski homomorphism. Let $\Psi : S_o^n \rightarrow S_o^n$ be a radial Blaschke-Minkowski homomorphism, then there is a continuous operator

$$\Psi : \underbrace{S_o^n \times \cdots \times S_o^n}_{n-1} \rightarrow S_o^n,$$

symmetric in its arguments such that for $K_1, \dots, K_r \in S_o^n$ and $\lambda_1, \dots, \lambda_r \geq 0$,

$$\Psi(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum_{i_1, \dots, i_{n-1}} \lambda_{i_1} \dots \lambda_{i_{n-1}} \Psi(K_{i_1}, \dots, K_{i_{n-1}}),$$

where “ $\tilde{+}$ ” denotes the radial Minkowski addition. For measure [3] $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$, there is

$$\rho(\Psi(K_1, \dots, K_{n-1}), \cdot) = \rho(K_1, \cdot) \dots \rho(K_{n-1}, \cdot) * \mu. \quad (1.1)$$

Clearly, the aforementioned continuous operator generalizes the notion of radial Blaschke-Minkowski homomorphism, we call $\Psi : \underbrace{S_o^n \times \cdots \times S_o^n}_{n-1} \rightarrow S_o^n$ the mixed radial Blaschke-Minkowski homomorphism and

write $\Psi(K_1, K_2, \dots, K_{n-1})$. If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, we write $\Psi_i(K, L)$ for $\Psi(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{L, \dots, L}_i)$ and call $\Psi_i(K, L)$ the i -mixed radial Blaschke-Minkowski homomorphism of K and L . If $L = B$, we write $\Psi_i K$ for $\Psi_i(K, B)$. In particular, denote $\Psi_0 K$ by ΨK .

By (1.1) and allow i is any real, Wang, Chen and Zhang [4] defined the i th radial Blaschke-Minkowski homomorphisms as follows.

Definition 1.B. For $K \in S_o^n$, real i satisfies $0 \leq i < n - 1$, the i th radial Blaschke-Minkowski homomorphism, $\Psi_i K$, of K is given by

$$\rho(\Psi_i K, \cdot) = \rho(K, \cdot)^{n-i-1} * \mu,$$

where $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$.

Regarding the studies of the (mixed) radial Blaschke-Minkowski homomorphisms, many results have been found in the previous studies, see, e.g., [5–7, 9–18, 20].

In 1999, Moszyńska [19] introduced the star duality of star bodies. For $K \in S_o^n$, the star duality, K° , of K is given by

$$\rho(K^\circ, u) = \frac{1}{\rho(K, u)}, \quad (1.2)$$

for every $u \in S^{n-1}$. With the emergence of this notion, a number of characterizations and inequalities were established about star bodies, see [20–26].

We use $\Psi_j^\circ K$ to denote the star duality of the j th radial Blaschke-Minkowski homomorphism $\Psi_j K$. The purpose of this article is to establish Brunn-Minkowski inequalities, monotonic inequality and consider Shephard-type problems for star duality of radial Blaschke-Minkowski homomorphisms. First, associated with the L_p radial combination, we give the following Brunn-Minkowski inequality.

Theorem 1.1. For $j \in \mathbb{R}$, let Ψ_j be the j th radial Blaschke-Minkowski homomorphism. If reals i and j satisfy $0 \leq i, j < n - 1$, and $K, L \in S_o^n$, then for $p > 0$,

$$\widetilde{W}_i(\Psi_j^\circ(K \tilde{+}_p L))^{-\frac{p}{(n-j-1)(n-i)}} \geq \widetilde{W}_i(\Psi_j^\circ K)^{-\frac{p}{(n-j-1)(n-i)}} + \widetilde{W}_i(\Psi_j^\circ L)^{-\frac{p}{(n-j-1)(n-i)}}; \quad (1.3)$$

for $j - n + 1 < p < 0$,

$$\widetilde{W}_i(\Psi_j^\circ(K \widetilde{\mp}_p L))^{-\frac{p}{(n-j-1)(n-i)}} \leq \widetilde{W}_i(\Psi_j^\circ K)^{-\frac{p}{(n-j-1)(n-i)}} + \widetilde{W}_i(\Psi_j^\circ L)^{-\frac{p}{(n-j-1)(n-i)}}. \quad (1.4)$$

In each case, with equality if and only if K and L are dilates. Here, $\widetilde{W}_i(M)$ denotes the dual quermassintegrals of M and $K \widetilde{\mp}_p L$ denotes the L_p radial Minkowski sum of K and L .

In particular, let $i = 0$ and $p = 1$ in Theorem 1.1 and note that $\widetilde{W}_0(M) = V(M)$ (see (2.7)), we have as follows.

Corollary 1.1. For $j \in \mathbb{R}$, let Ψ_j be the j th radial Blaschke-Minkowski homomorphism. If $K, L \in S_o^n$ and $0 \leq j < n - 1$, then

$$V(\Psi_j^\circ(K \widetilde{\mp}_p L))^{-\frac{1}{n(n-j-1)}} \geq V(\Psi_j^\circ K)^{-\frac{1}{n(n-j-1)}} + V(\Psi_j^\circ L)^{-\frac{1}{n(n-j-1)}}, \quad (1.5)$$

with equality if and only if K and L are dilates.

Remark 1.1. Recall that Schuster [3] gave the notion of Blaschke-Minkowski homomorphism and proved the following Brunn-Minkowski inequality for polar duality of Blaschke-Minkowski homomorphisms: let Φ_j be the j th Blaschke-Minkowski homomorphism, if K and L are convex bodies, $0 \leq j \leq n - 3$, then

$$V(\Phi_j^*(K + L))^{-\frac{1}{n(n-j-1)}} \geq V(\Phi_j^* K)^{-\frac{1}{n(n-j-1)}} + V(\Phi_j^* L)^{-\frac{1}{n(n-j-1)}}, \quad (1.6)$$

with equality if and only if K and L are homothetic. Here, $K + L$ denotes the Minkowski sum of K and L , and $\Phi_j^* Q$ denotes the polar duality of $\Phi_j Q$.

Comparing (1.5) with (1.6), we can feel that inequality (1.5) is an analogue of inequality (1.6).

Note that the L_p harmonic radial sum is just the L_{-p} radial Minkowski sum of star bodies (see (2.1) and (2.2)). From this, if $p \leq -1$, we replace p by $-p$ in Theorem 1.1, then inequality (1.4) yields the following Brunn-Minkowski inequality for the L_p harmonic radial sum.

Corollary 1.2. For $j \in \mathbb{R}$, let Ψ_j be the j th radial Blaschke-Minkowski homomorphism. If $K, L \in S_o^n$, reals i and j satisfy $0 \leq i, j < n - 1$, then for $1 \leq p < n - j - 1$,

$$\widetilde{W}_i(\Psi_j^\circ(K \widetilde{\mp}_p L))^{-\frac{p}{(n-j-1)(n-i)}} \leq \widetilde{W}_i(\Psi_j^\circ K)^{-\frac{p}{(n-j-1)(n-i)}} + \widetilde{W}_i(\Psi_j^\circ L)^{-\frac{p}{(n-j-1)(n-i)}},$$

with equality if and only if K and L are dilates. Here, $K \widetilde{\mp}_p L$ denotes the L_p harmonic radial addition of K and L .

Next, according to the L_p harmonic Blaschke sum, another Brunn-Minkowski inequality for the star duality of radial Blaschke-Minkowski homomorphisms is established as follows.

Theorem 1.2. For $j \in \mathbb{R}$, let Ψ_j be the j th radial Blaschke-Minkowski homomorphism. If $K, L \in S_o^n$, and reals i and j satisfy $0 \leq i, j < n - 1$, then for $p > 0$,

$$\frac{\widetilde{W}_i(\Psi_j^\circ(K \mp_p L))^{-\frac{n+p}{(n-j-1)(n-i)}}}{V(K \mp_p L)} \geq \frac{\widetilde{W}_i(\Psi_j^\circ K)^{-\frac{n+p}{(n-j-1)(n-i)}}}{V(K)} + \frac{\widetilde{W}_i(\Psi_j^\circ L)^{-\frac{n+p}{(n-j-1)(n-i)}}}{V(L)}, \quad (1.7)$$

equality holds if and only if K and L are dilates. Here, $K \mp_p L$ denotes the L_p Blaschke sum of K and L .

Furthermore, we consider the Shephard-type problems for the star duality of radial Blaschke-Minkowski homomorphisms and get a positive form and a negative answer, respectively.

Theorem 1.3. Let Ψ_i be the i th radial Blaschke-Minkowski homomorphism, $L \in S_o^n$ and $0 \leq i < n - 1$. If $K \in \Psi_i S_o^n$ and $\Psi_i^\circ K \subseteq \Psi_i^\circ L$, then

$$\widetilde{W}_i(K^\circ) \leq \widetilde{W}_i(L^\circ), \quad (1.8)$$

equality holds if and only if $K = L$.

Theorem 1.4. Let Ψ_i be an even i th radial Blaschke-Minkowski homomorphism, $K \in S_o^n$ and $0 \leq i < n - 1$. If K is not origin-symmetric, then there exists $L \in S_o^n$, such that

$$\Psi_i^\circ K \subset \Psi_i^\circ L.$$

But

$$\widetilde{W}_i(K^\circ) > \widetilde{W}_i(L^\circ).$$

In this article, the proofs of Theorems 1.1–1.4 are given in Section 3. In addition, in Section 3 we also give a monotonic inequality of star duality of the radial Blaschke-Minkowski homomorphisms.

2 Preliminaries

2.1 Polar duality

If E is an arbitrary nonempty subset of $\mathbb{R}^n$, the polar duality [1,27], E^* , of E is defined by

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}.$$

2.2 Some L_p combinations

For $K, L \in S_o^n$, $\lambda, \mu \geq 0$ (both not zero) and real $p \neq 0$, the L_p radial Minkowski combination [27,28], $\lambda \circ K \tilde{+}_p \mu \circ L \in S_o^n$, of K and L is given by

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p, \quad (2.1)$$

where “ $\tilde{+}_p$ ” denotes the L_p radial Minkowski addition, $\lambda \circ K$ denotes the L_p radial scalar multiplication and $\lambda \circ K = \lambda^{\frac{1}{p}} K$. Obviously, if $p = 1$, we can get classical radial Minkowski combination.

If $K, L \in S_o^n$, $\lambda, \mu \geq 0$ (both not zero) and real $p \geq 1$, then the L_p harmonic radial combination [29], $\lambda \circ K \tilde{+}_{-p} \mu \circ L \in S_o^n$, of K and L is defined by

$$\rho(\lambda \circ K \tilde{+}_{-p} \mu \circ L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}, \quad (2.2)$$

where “ $\tilde{+}_{-p}$ ” denotes the L_p harmonic radial addition. Obviously, L_{-p} radial combination is the L_p harmonic radial combination.

Let $K, L \in S_o^n$, $\lambda, \mu \geq 0$ (both not zero) and real $p > 0$, the L_p harmonic Blaschke combination [30,31], $\lambda * K \tilde{+}_p \mu * L \in S_o^n$, of K and L is given by

$$\frac{\rho(\lambda * K \tilde{+}_p \mu * L, \cdot)^{n+p}}{V(\lambda * K \tilde{+}_p \mu * L)} = \lambda \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+p}}{V(L)}, \quad (2.3)$$

where “ $\tilde{+}_p$ ” denotes the L_p harmonic Blaschke addition, and $\lambda * K$ denotes the L_p harmonic Blaschke scalar multiplication and $\lambda * K = \lambda^{\frac{1}{p}} K$. When $p = 1$, (2.3) is the classical harmonic Blaschke combination.

2.3 Dual mixed quermassintegrals

Lutwak [32] gave the notion of dual mixed volume. For $K_1, K_2, \dots, K_n \in S_o^n$, the dual mixed volume, $\widetilde{V}(K_1, K_2, \dots, K_n)$, of $K_1, K_2, \dots, K_n$ is given by

$$\tilde{V}(K_1, K_2, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \dots \rho(K_n, u) du. \quad (2.4)$$

In (2.4), if $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = L$, then we write

$$\tilde{W}_i(K, L) = \tilde{V}(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B, \dots, B}_i, L) \quad (i = 0, 1, \dots, n-2).$$

For $K, L \in S^n_o$, if allow i is any real, then the dual mixed quermassintegral, $\tilde{W}_i(K, L)$, of K and L is defined by

$$\tilde{W}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i-1} \rho(L, u) du. \quad (2.5)$$

Taking $L = K$, then (2.5) yields the dual quermassintegral, $\tilde{W}_i(K)$, of K by

$$\tilde{W}_i(K) = \tilde{W}_i(K, K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} du. \quad (2.6)$$

Obviously, if $i = 0$, then

$$\tilde{W}_0(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n du = V(K). \quad (2.7)$$

The Minkowski inequality [32] for the dual mixed quermassintegrals can be showed as follows.

Theorem 2.A. For $K, L \in S^n_o$, i is any real. If $i < n-1$, then

$$\tilde{W}_i(K, L) \leq \tilde{W}_i(K)^{\frac{n-i-1}{n-i}} \tilde{W}_i(L)^{\frac{1}{n-i}}, \quad (2.8)$$

with equality if and only if K is a dilate of L . If $i > n-1$, inequality (2.8) is reverse. If $i = n-1$, (2.8) becomes an equality.

2.4 General i th radial Blaschke bodies

For $K, L \in S^n_o$, $0 \leq i < n-1$ and $\lambda, \mu \geq 0$ (both not zero), the i th radial Blaschke combination [4], $\lambda \cdot K \hat{+}_i \mu \cdot L \in S^n_o$, of K and L is defined by

$$\rho(\lambda \cdot K \hat{+}_i \mu \cdot L, \cdot)^{n-i-1} = \lambda \rho(K, \cdot)^{n-i-1} + \mu \rho(L, \cdot)^{n-i-1}. \quad (2.9)$$

Taking $i = 0$ in (2.9), then $\lambda \cdot K \hat{+}_0 \mu \cdot L$ is the radial Blaschke combination [1] $\lambda \cdot K \hat{+} \mu \cdot L$.

For $\tau \in [-1, 1]$, let

$$\lambda = f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad \mu = f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}, \quad (2.10)$$

and $L = -K$ in (2.9), writing

$$\hat{\nabla}_i^\tau K = f_1(\tau) \cdot K \hat{+}_i f_2(\tau) \cdot (-K), \quad (2.11)$$

and called $\hat{\nabla}_i^\tau K$ the general i th radial Blaschke body of K .

For the general i th radial Blaschke body [4], the following fact was also given.

Theorem 2.B. For $K \in S^n_o$, $i < n-1$. If $K \notin S^n_{os}$, then for $\tau \in [-1, 1]$,

$$\hat{\nabla}_i^\tau K \in S^n_{os} \Leftrightarrow \tau = 0.$$

3 Proofs of theorems

In this section, we will complete proofs of Theorems 1.1–1.4. In order to prove Theorem 1.1, we require the following lemmas.

Lemma 3.1. For $M, N \in S_o^n$, reals i, j satisfy $0 \leq i, j < n - 1$, then

$$\widetilde{W}_i(M^\circ, \Psi_j^\circ N) = \widetilde{W}_j(N^\circ, \Psi_i^\circ M). \quad (3.1)$$

Proof. From (2.5) and combined (1.2) with Definition 1.B, we have

$$\begin{aligned} \widetilde{W}_i(M^\circ, \Psi_j^\circ N) &= \frac{1}{n} \int_{S^{n-1}} \rho(M^\circ, u)^{n-i-1} \rho(\Psi_j^\circ N, u) du \\ &= \frac{1}{n} \int_{S^{n-1}} \frac{1}{\rho(M, u)^{n-i-1} \rho(\Psi_j N, u)} du \\ &= \frac{1}{n} \int_{S^{n-1}} \frac{1}{\rho(M, u)^{n-i-1} \rho(N, u)^{n-j-1} * \mu} du \\ &= \frac{1}{n} \int_{S^{n-1}} \frac{1}{\rho(\Psi_i M, u) \rho(N, u)^{n-j-1}} du \\ &= \frac{1}{n} \int_{S^{n-1}} \rho(N^\circ, u)^{n-j-1} \rho(\Psi_i^\circ M, u) du \\ &= \widetilde{W}_j(N^\circ, \Psi_i^\circ M). \end{aligned}$$

This is equality (3.1). □

Lemma 3.2. Let $K, L \in S_o^n$, real j satisfies $0 \leq j < n - 1$. If $p > 0$, then for any $Q \in S_o^n$,

$$\widetilde{W}_j((K \widetilde{\tau}_p L)^\circ, Q)^{-\frac{p}{n-j-1}} \geq \widetilde{W}_j(K^\circ, Q)^{-\frac{p}{n-j-1}} + \widetilde{W}_j(L^\circ, Q)^{-\frac{p}{n-j-1}}; \quad (3.2)$$

if $j - n + 1 < p < 0$, then for any $Q \in S_o^n$,

$$\widetilde{W}_j((K \widetilde{\tau}_p L)^\circ, Q)^{-\frac{p}{n-j-1}} \leq \widetilde{W}_j(K^\circ, Q)^{-\frac{p}{n-j-1}} + \widetilde{W}_j(L^\circ, Q)^{-\frac{p}{n-j-1}}. \quad (3.3)$$

In each case, equality holds if and only if K and L are dilates.

Proof. If $p > 0$, note that $0 \leq j < n - 1$, then $-\frac{n-j-1}{p} < 0$. By (2.5), (1.2), (2.1) and associated with the Minkowski integral inequality, we get

$$\begin{aligned} \widetilde{W}_j((K \widetilde{\tau}_p L)^\circ, Q)^{-\frac{p}{n-j-1}} &= \left(\frac{1}{n} \int_{S^{n-1}} \rho((K \widetilde{\tau}_p L)^\circ, u)^{n-j-1} \rho(Q, u) du \right)^{-\frac{p}{n-j-1}} \\ &= \left(\frac{1}{n} \int_{S^{n-1}} \rho(K \widetilde{\tau}_p L, u)^{-(n-j-1)} \rho(Q, u) du \right)^{-\frac{p}{n-j-1}} \\ &= \left(\frac{1}{n} \int_{S^{n-1}} (\rho(K, u)^p + \rho(L, u)^p)^{-\frac{n-j-1}{p}} \rho(Q, u) du \right)^{-\frac{p}{n-j-1}} \end{aligned}$$

$$\begin{aligned} &\geq \left(\frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{-(n-j-1)} \rho(Q, u) du \right)^{-\frac{p}{n-j-1}} + \left(\frac{1}{n} \int_{S^{n-1}} \rho(L, u)^{-(n-j-1)} \rho(Q, u) du \right)^{-\frac{p}{n-j-1}} \\ &= \widetilde{W}_j(K^\circ, Q)^{-\frac{p}{n-j-1}} + \widetilde{W}_j(L^\circ, Q)^{-\frac{p}{n-j-1}}. \end{aligned}$$

This is just inequality (3.2).

For $j - n + 1 < p < 0$, then $-\frac{n-j-1}{p} > 1$, similar to the aforementioned method, using the Minkowski integral inequality, we can obtain inequality (3.3).

According to equality condition of the Minkowski integral inequality, we know that equalities hold in (3.2) and (3.3) if and only if K and L are dilates. $\square$

Proof of Theorem 1.1. For $K, L \in S_o^n$, $0 \leq j < n - 1$ and $p > 0$, by (3.1) and (3.2), we obtain for any $Q \in S_o^n$,

$$\begin{aligned} \widetilde{W}_i(Q^\circ, \Psi_j(K \mp_p L))^{-\frac{p}{n-j-1}} &= \widetilde{W}_j((K \mp_p L)^\circ, \Psi_i^\circ Q)^{-\frac{p}{n-j-1}} \\ &\geq \widetilde{W}_j(K^\circ, \Psi_i^\circ Q)^{-\frac{p}{n-j-1}} + \widetilde{W}_j(L^\circ, \Psi_i^\circ Q)^{-\frac{p}{n-j-1}} \\ &= \widetilde{W}_i(Q^\circ, \Psi_j^\circ K)^{-\frac{p}{n-j-1}} + \widetilde{W}_i(Q^\circ, \Psi_j^\circ L)^{-\frac{p}{n-j-1}}. \end{aligned}$$

Since $0 \leq i < n - 1$, from (2.8) and note that $-\frac{p}{n-j-1} < 0$, we have

$$\widetilde{W}_i(Q^\circ, \Psi_j^\circ(K \mp_p L))^{-\frac{p}{n-j-1}} \geq \widetilde{W}_i(Q^\circ)^{-\frac{p(n-i-1)}{(n-j-1)(n-i)}} \left(\widetilde{W}_i(\Psi_j^\circ K)^{-\frac{p}{(n-j-1)(n-i)}} + \widetilde{W}_i(\Psi_j^\circ L)^{-\frac{p}{(n-j-1)(n-i)}} \right).$$

Taking $Q = \Psi_j(K \mp_p L)$, by (2.6) we get the desired inequality (1.3).

Similarly, if $j - n + 1 < p < 0$, then $-\frac{p}{n-j-1} > 1$, because of $0 \leq i < n - 1$, thus we can obtain inequality (1.4) by (2.8) and inequality (3.3).

According to equality conditions of Lemma 3.2 and (2.8), we see that equalities hold in (1.3) and (1.4) if and only if K and L are dilates. $\square$

Lemma 3.3. If $K, L \in S_o^n$, $p > 0$ and real j satisfies $0 \leq j < n - 1$, then for any $Q \in S_o^n$,

$$\frac{\widetilde{W}_j((K \mp_p L)^\circ, Q)^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} \geq \frac{\widetilde{W}_j(K^\circ, Q)^{-\frac{n+p}{n-j-1}}}{V(K)} + \frac{\widetilde{W}_j(L^\circ, Q)^{-\frac{n+p}{n-j-1}}}{V(L)}, \quad (3.4)$$

with equality if and only if K and L are dilates.

Proof. Since $p > 0$ and $0 < j < n - 1$, thus $-\frac{n-j-1}{n+p} < 0$. By (2.5), (1.2), (2.3) and combined with the Minkowski integral inequality, we obtain

$$\begin{aligned} \frac{\widetilde{W}_j((K \mp_p L)^\circ, Q)^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} &= \frac{\left(\frac{1}{n} \int_{S^{n-1}} \rho((K \mp_p L)^\circ, u)^{n-j-1} \rho(Q, u) du \right)^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} \\ &= \frac{\left(\frac{1}{n} \int_{S^{n-1}} \rho(K \mp_p L, u)^{-(n-j-1)} \rho(Q, u) du \right)^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} \\ &= \left(\frac{\frac{1}{n} \int_{S^{n-1}} (\rho(K \mp_p L, u)^{n+p})^{-\frac{n-j-1}{n+p}} \rho(Q, u) du}{V(K \mp_p L)^{-\frac{n-j-1}{n+p}}} \right)^{-\frac{n+p}{n-j-1}} \\ &= \left(\frac{1}{n} \int_{S^{n-1}} \left(\frac{\rho(K, u)^{n+p}}{V(K)} + \frac{\rho(L, u)^{n+p}}{V(L)} \right)^{-\frac{n-j-1}{n+p}} \rho(Q, u) du \right)^{-\frac{n+p}{n-j-1}} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{V(K)} \left(\frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{-(n-j-1)} \rho(Q, u) du \right)^{-\frac{n+p}{n-j-1}} + \frac{1}{V(L)} \left(\frac{1}{n} \int_{S^{n-1}} \rho(L, u)^{-(n-j-1)} \rho(Q, u) du \right)^{-\frac{n+p}{n-j-1}} \\
&= \frac{\widetilde{W}_j(K^\circ, Q)^{-\frac{n+p}{n-j-1}}}{V(K)} + \frac{\widetilde{W}_j(L^\circ, Q)^{-\frac{n+p}{n-j-1}}}{V(L)}.
\end{aligned}$$

This yields inequality (3.4).

According to the equality condition of the Minkowski integral inequality, we know that equality holds in (3.4) if and only if K and L are dilates. $\square$

Proof of Theorem 1.2. For $K, L \in S^n_o$, $0 \leq j < n-1$ and $p > 0$, by (3.1) and (3.4), we obtain for any $Q \in S^n_o$,

$$\begin{aligned}
\frac{\widetilde{W}_i(Q^\circ, \Psi_j^\circ(K \mp_p L))^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} &= \frac{\widetilde{W}_j((K \mp_p L)^\circ, \Psi_i^\circ Q)^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} \\
&\geq \frac{\widetilde{W}_j(K^\circ, \Psi_i^\circ Q)^{-\frac{n+p}{n-j-1}}}{V(K)} + \frac{\widetilde{W}_j(L^\circ, \Psi_i^\circ Q)^{-\frac{n+p}{n-j-1}}}{V(L)} \\
&= \frac{\widetilde{W}_i(Q^\circ, \Psi_j^\circ K)^{-\frac{n+p}{n-j-1}}}{V(K)} + \frac{\widetilde{W}_i(Q^\circ, \Psi_j^\circ L)^{-\frac{n+p}{n-j-1}}}{V(L)}.
\end{aligned}$$

By (2.8), for $0 \leq i < n-1$ and $-\frac{n+p}{n-j-1} < 0$, we have

$$\frac{\widetilde{W}_i(Q^\circ, \Psi_j^\circ(K \mp_p L))^{-\frac{n+p}{n-j-1}}}{V(K \mp_p L)} \geq \widetilde{W}_i(Q^\circ)^{-\frac{(n+p)(n-i-1)}{(n-i)(n-j-1)}} \left(\frac{\widetilde{W}_i(\Psi_j^\circ K)^{-\frac{n+p}{(n-j-1)(n-i)}}}{V(K)} + \frac{\widetilde{W}_i(\Psi_j^\circ L)^{-\frac{n+p}{(n-j-1)(n-i)}}}{V(L)} \right).$$

Let $Q = \Psi_j(K \mp_p L)$ in the aforementioned inequality, by (2.6), this yields inequality (1.7). Equality holds in (1.7) if and only if K and L are dilates. $\square$

Proof of Theorem 1.3. Since $\Psi_i^\circ K \subseteq \Psi_i^\circ L$ ($0 \leq i < n-1$), then $\rho(\Psi_i^\circ K, \cdot) \leq \rho(\Psi_i^\circ L, \cdot)$, thus using (2.5) we know for any $M \in S^n_o$ and $0 \leq j < n-1$,

$$\widetilde{W}_j(M^\circ, \Psi_i^\circ K) \leq \widetilde{W}_j(M^\circ, \Psi_i^\circ L).$$

This together with (3.1) yields

$$\widetilde{W}_i(K^\circ, \Psi_j^\circ M) \leq \widetilde{W}_i(L^\circ, \Psi_j^\circ M).$$

Because of $K \in \Psi_i S^n_o$, taking $\Psi_j M = K$, then by (2.6) and (2.8) we obtain

$$\widetilde{W}_i(K^\circ) \leq \widetilde{W}_i(L^\circ, K^\circ) \leq \widetilde{W}_i(L^\circ)^{\frac{n-i-1}{n-i}} \widetilde{W}_i(K^\circ)^{\frac{1}{n-i}},$$

i.e.,

$$\widetilde{W}_i(K^\circ) \leq \widetilde{W}_i(L^\circ).$$

This is just inequality (1.8).

According to the equality condition of inequality (2.8), we see that $\widetilde{W}_i(K^\circ) = \widetilde{W}_i(L^\circ)$ if and only if K° and L° are dilates. From this, let $K^\circ = cL^\circ$ ($c > 0$) and together (1.2) with $\widetilde{W}_i(K^\circ) = \widetilde{W}_i(L^\circ)$, we obtain $c = 1$. Therefore, equality holds in (1.8) if and only if $K = L$. $\square$

Lemma 3.4. If $K \in S^n_o$, $0 \leq i < n-1$ and $\tau \in [-1, 1]$, then

$$\widetilde{W}_i(\widehat{\nabla}_i^{\tau, \circ} K) \leq \widetilde{W}_i(K^\circ), \quad (3.5)$$

with equality for $\tau \in (-1, 1)$ if and only if K is origin-symmetric. For $\tau = \pm 1$, the aforementioned inequality becomes an equality. Here, $\widehat{\nabla}_i^{\tau, \circ} K$ denotes the star duality of $\widehat{\nabla}_i^T K$.

Proof. For $0 \leq i < n-1$, then $-\frac{n-i}{n-i-1} < 0$, thus from (1.2), (2.6), (2.9), (2.11) and associated with the Minkowski integral inequality, we have

$$\begin{aligned}
 (\widetilde{W}_i(\widehat{V}_i^{\tau, \circ} K))^{-\frac{n-i-1}{n-i}} &= \left(\frac{1}{n} \int_{S^{n-1}} \rho(\widehat{V}_i^{\tau, \circ} K, u)^{n-i} du \right)^{-\frac{n-i-1}{n-i}} \\
 &= \left(\frac{1}{n} \int_{S^{n-1}} \rho(\widehat{V}_i^{\tau} K, u)^{-(n-i)} du \right)^{-\frac{n-i-1}{n-i}} \\
 &= \left(\frac{1}{n} \int_{S^{n-1}} (\rho(f_1(\tau) \cdot K \cdot \widehat{\tau}_i f_2(\tau) \cdot (-K), u)^{n-i-1})^{-\frac{n-i}{n-i-1}} du \right)^{-\frac{n-i-1}{n-i}} \\
 &= \left(\frac{1}{n} \int_{S^{n-1}} (f_1(\tau) \rho(K, u)^{n-i-1} + f_2(\tau) \rho(-K, u)^{n-i-1})^{-\frac{n-i}{n-i-1}} du \right)^{-\frac{n-i-1}{n-i}} \\
 &\geq f_1(\tau) \left(\frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{-(n-i)} du \right)^{-\frac{n-i-1}{n-i}} + f_2(\tau) \left(\frac{1}{n} \int_{S^{n-1}} \rho(-K, u)^{-(n-i)} du \right)^{-\frac{n-i-1}{n-i}} \\
 &= f_1(\tau) \widetilde{W}_i(K^{\circ})^{-\frac{n-i-1}{n-i}} + f_2(\tau) \widetilde{W}_i((-K)^{\circ})^{-\frac{n-i-1}{n-i}}.
 \end{aligned} \tag{3.6}$$

Since $f_1(\tau) > 0$ and $f_2(\tau) > 0$ with $\tau \in (-1, 1)$, from the equality condition of the Minkowski integral inequality, we know that equality holds in (3.6) for $\tau \in (-1, 1)$ if and only if K and $-K$ are dilates, that is, K is origin-symmetric.

If $\tau = \pm 1$, then by $\widehat{V}_i^{\pm 1} K = \pm K$, we see (3.6) becomes an equality.

But by (2.6) and (1.2), we have

$$\begin{aligned}
 \widetilde{W}_i((-K)^{\circ}) &= \frac{1}{n} \int_{S^{n-1}} \rho(-K, u)^{-(n-i)} du \\
 &= \frac{1}{n} \int_{S^{n-1}} \rho(K, -u)^{-(n-i)} d(-u) \\
 &= \frac{1}{n} \int_{S^{n-1}} \rho(K^{\circ}, -u)^{n-i} d(-u) = \widetilde{W}_i(K^{\circ}).
 \end{aligned}$$

Note that (2.10) gives $f_1(\tau) + f_2(\tau) = 1$. These combined with (3.6) and $-\frac{n-i-1}{n-i} < 0$, we easily obtain

$$\widetilde{W}_i(\widehat{V}_i^{\tau, \circ} K) \leq \widetilde{W}_i(K^{\circ}).$$

Thus, inequality (3.5) is given.

By the equality condition of (3.6), we know that with equality in (3.5) for $\tau \in (-1, 1)$ if and only if K is origin-symmetric. For $\tau = \pm 1$, (3.5) becomes an equality. $\square$

Lemma 3.5. Let Ψ_i be an i th radial Blaschke-Minkowski homomorphism, $K \in S_o^n$ and $0 \leq i < n-1$, then for $c > 0$,

$$\Psi_i^{\circ}(cK) = c^{-(n-i-1)} \Psi_i^{\circ} K.$$

Proof. From (1.2), Definition 1.B and property of radial function ($\rho(cK, \cdot) = c\rho(K, \cdot)$, $c > 0$), we have for $c > 0$ and any $u \in S^{n-1}$,

$$\begin{aligned}\rho(\Psi_i^\circ(cK), u) &= \rho(\Psi_i(cK), u)^{-1} = \frac{1}{\rho(cK, u)^{n-i-1} * \mu} = \frac{1}{c^{n-i-1} \rho(K, u)^{n-i-1} * \mu} \\ &= c^{-(n-i-1)} \rho(\Psi_i^\circ K, u) = \rho(c^{-(n-i-1)} \Psi_i^\circ K, u),\end{aligned}$$

i.e.,

$$\Psi_i^\circ(cK) = c^{-(n-i-1)} \Psi_i^\circ K.$$

Hence, Lemma 3.5 is proven. $\square$

Lemma 3.6. Let Ψ_i be an even i th radial Blaschke-Minkowski homomorphism. If $K \in S_0^n$, $0 \leq i < n-1$ and $\tau \in [-1, 1]$, then

$$\Psi_i^\circ(\widehat{\nabla}_i^\tau K) = \Psi_i^\circ K.$$

Proof. Since Ψ_i is an even i th radial Blaschke-Minkowski homomorphism, thus for any $K \in S_0^n$, $\Psi_i(-K) = \Psi_i K$.

From this, associated with Definition 1.B, (1.2), (2.10) and (2.11), it follows that for any $u \in S^{n-1}$,

$$\begin{aligned}\rho(\Psi_i^\circ(\widehat{\nabla}_i^\tau K), u) &= \frac{1}{\rho(\Psi_i(\widehat{\nabla}_i^\tau K), u)} \\ &= \frac{1}{\rho(\widehat{\nabla}_i^\tau K, u)^{n-i-1} * \mu} \\ &= \frac{1}{(f_1(\tau)\rho(K, u)^{n-i-1} + f_2(\tau)\rho(-K, u)^{n-i-1}) * \mu} \\ &= \frac{1}{f_1(\tau)\rho(K, u)^{n-i-1} * \mu + f_2(\tau)\rho(-K, u)^{n-i-1} * \mu} \\ &= \frac{1}{f_1(\tau)\rho(\Psi_i K, u) + f_2(\tau)\rho(\Psi_i(-K), u)} \\ &= \frac{1}{\rho(\Psi_i K, u)} = \rho(\Psi_i^\circ K, u),\end{aligned}$$

i.e.,

$$\Psi_i^\circ(\widehat{\nabla}_i^\tau K) = \Psi_i^\circ K.$$

This gives proof of Lemma 3.6. $\square$

Proof of Theorem 1.4. Since K is not origin-symmetric, thus from Lemma 3.4 we know that

$$\widetilde{W}_i(\widehat{\nabla}_i^{\tau, \circ} K) < \widetilde{W}_i(K^\circ).$$

Note that for $0 < c < 1$, $c\widehat{\nabla}_i^\tau K \subseteq \widehat{\nabla}_i^\tau K$ implies $\widehat{\nabla}_i^{\tau, \circ} K \subset c^{-1}\widehat{\nabla}_i^{\tau, \circ} K$. Therefore, choose $0 < \varepsilon < 1$ such that

$$\widetilde{W}_i((1-\varepsilon)^{-1}\widehat{\nabla}_i^{\tau, \circ} K) < \widetilde{W}_i(K^\circ).$$

From this, let $L = (1-\varepsilon)\widehat{\nabla}_i^\tau K$, then $L \in S_0^n$ (Theorem 2.B gives that for $\tau = 0$, $L \in S_{os}^n$; for $\tau \in (-1, 1)$ and $\tau \neq 0$, $L \in S_0^n \setminus S_{os}^n$) and

$$\widetilde{W}_i(L^\circ) < \widetilde{W}_i(K^\circ).$$

But Lemma 3.5 gives that $\Psi_i^\circ((1-\varepsilon)K) = (1-\varepsilon)^{-(n-i-1)}\Psi_i^\circ K$, because of $0 \leq i < n-1$, then $(1-\varepsilon)^{-(n-i-1)} > 1$, these together with Lemma 3.6, we obtain that

$$\Psi_i^\circ L = \Psi_i^\circ((1-\varepsilon)\widehat{\nabla}_i^\tau K) = (1-\varepsilon)^{-(n-i-1)}\Psi_i^\circ(\widehat{\nabla}_i^\tau K) = (1-\varepsilon)^{-(n-i-1)}\Psi_i^\circ K \supset \Psi_i^\circ K.$$

Therefore, we complete the proof of Theorem 1.4. $\square$

Finally, we give a monotonic inequality of star duality of radial Blaschke-Minkowski homomorphisms.

Theorem 3.1. For $j \in \mathbb{R}$, let Ψ_j be the j th radial Blaschke-Minkowski homomorphism, $K, L \in S^n_o$ and reals i and j satisfy $0 \leq i, j < n - 1$. If for any $Q \in \Psi_i S^n_o$,

$$\widetilde{W}_j(K^\circ, Q^\circ) \leq \widetilde{W}_j(L^\circ, Q^\circ),$$

then

$$\widetilde{W}_i(\Psi_j^\circ K) \leq \widetilde{W}_i(\Psi_j^\circ L). \quad (3.7)$$

With equality if and only if $K = L$. Here, $\Psi_i S^n_o$ denotes the range of Ψ_i .

Proof. Since $\widetilde{W}_j(K^\circ, Q^\circ) \leq \widetilde{W}_j(L^\circ, Q^\circ)$ and $Q \in \Psi_i S^n_o$, thus let $Q = \Psi_i M$, we have

$$\widetilde{W}_j(K^\circ, \Psi_i^\circ M) \leq \widetilde{W}_j(L^\circ, \Psi_i^\circ M), \quad (3.8)$$

equality holds if and only if $K = L$.

(3.8) and (3.1) imply that

$$\widetilde{W}_i(M^\circ, \Psi_j^\circ K) \leq \widetilde{W}_i(M^\circ, \Psi_j^\circ L).$$

Taking $M = \Psi_j K$ in the aforementioned inequality, for $0 \leq i < n - 1$ and from (2.6) and (2.8) we have

$$\widetilde{W}_i(\Psi_j^\circ K) \leq \widetilde{W}_i(\Psi_j^\circ K, \Psi_j^\circ L) \leq \widetilde{W}_i(\Psi_j^\circ K)^{\frac{n-i-1}{n-i}} \widetilde{W}_i(\Psi_j^\circ L)^{\frac{1}{n-i}}, \quad (3.9)$$

i.e.,

$$\widetilde{W}_i(\Psi_j^\circ K) \leq \widetilde{W}_i(\Psi_j^\circ L).$$

This is the desired inequality (3.7).

According to the equality condition of (2.8), with equality in (3.9) if and only if $\Psi_j^\circ K$ and $\Psi_j^\circ L$ are dilates, thus combined with equality conditions of (3.8) and (3.9), we know that equality holds in (3.7) if and only if $K = L$. $\square$

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