Research Article

Louk-Man Issaka, Mamadou Abdoul Diop*, and Hasna Hmoyed

Results on nonlocal stochastic integro-differential equations driven by a fractional Brownian motion

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Abstract: This paper deals with the existence of mild solutions for a class of non-local stochastic integro-differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Discussions are based on resolvent operators in the sense of Grimmer, stochastic analysis theory and fixed-point criteria. As a final point, an example is given to illustrate the effectiveness of the obtained theory.

Keywords: resolvent operators, C_0 -semigroup, stochastic functional integro-differential equations, nonlocal condition, fractional Brownian motion, mild solutions

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1 Introduction

The fractional Brownian motion (fBm) is one of the natural generalizations of the Brownian motion. It is a family of centered and continuous Gaussian processes with Hurst parameter $H \in (0, 1)$. It is reduced to the standard Brownian motion if $H = \frac{1}{2}$. But if $H \neq \frac{1}{2}$, fBm is different from a Markov process and martingale; therefore, the classical stochastic analysis is not possible to be used. The fractional Brownian motion was introduced by Kolmogorov [1] in 1940 and has very important properties such as self-similarity and non-stationary. Mandelbrot and Van Ness [2] made it famous by introducing it into financial models and studying its properties. These proprieties allow fBm to be used in several domains such as telecommunication, biology, finance, and engineering. For that it is beneficial and important to investigate stochastic differential equations driven by an fBm. Recently, stochastic partial functional differential equations driven by a fractional Brownian motion have drawn the interest of many researchers (see [3–10]). For example, under the global Lipschitz condition, Caraballo et al. [11] showed the existence, uniqueness and stability of mild solutions for stochastic partial differential equations (SPDEs) with finite delays driven by an fBm; under the global Lipschitz condition, Boufoussi and Hajji [12] considered the existence and uniqueness of mild solutions to neutral SPDEs with finite delays driven by an fBm; Boufoussi et al. [13] obtained the existence and uniqueness result of mild solution to a class of time-dependent stochastic functional differential equations driven by an fBm; Ren et al. [9] proved the

* **Corresponding author: Mamadou Abdoul Diop**, Université Gaston Berger de Saint-Louis, Unité de Formation et de Recherche de Sciences Appliquées et de Technologies (UFR SAT), Département de Mathématiques, B.P234, Saint-Louis, Sénégal, e-mail: mamadou-abdoul.diop@ugb.edu.sn

Louk-Man Issaka: Université Gaston Berger de Saint-Louis, Unité de Formation et de Recherche de Sciences Appliquées et de Technologies (UFR SAT), Département de Mathématiques, B.P234, Saint-Louis, Sénégal, e-mail: issaka.louk-man@ugb.edu.sn

Hasna Hmoyed: Université Gaston Berger de Saint-Louis, Unité de Formation et de Recherche de Sciences Appliquées et de Technologies (UFR SAT), Département de Mathématiques, B.P234, Saint-Louis, Sénégal, e-mail: hasnaahmedou@yahoo.fr

existence and uniqueness of the mild solution for a class of time-dependent stochastic evolution equations with finite delay driven by a standard cylindrical Wiener process and an independent cylindrical fractional Brownian motion.

For more details on the fractional Brownian motion, see [11,12,14–16] and references therein.

In addition, the theory of nonlocal evolution equations has become an important area of investigation in recent years due to their applications to various problems arising in physics, biology, aerospace and medicine. Nonlocal conditions are known to give a better description of real models than classical initial ones, e.g., the condition

$$Y(s) + \sum_{i=1}^M c_i Y(\tau_i + s) = \phi(s),$$

allows taking additional measurements instead of solely initial datum. The first result and physical significance for nonlocal problems are given by Byszewski's work [17]. It developed greater interest in various nonlocal issues related to differential equations and stochastic differential equations. Many of the basic results for nonlocal problems have been obtained, see [18–24] and references therein for more comments and citations. Recently, some authors have drawn attention to the Cauchy problems driven by differential equations. One can see the studies of Balachandran *et al.* [25], Balasubramaniam and Park [26], Balasubramaniam *et al.* [27], Deng [28–31], Liang and Xiao [32] and references therein.

Motivated by the previously mentioned problems, in this paper, we will extend some such results of mild solutions for the following nonlocal integro-differential stochastic equations driven by a fractional Brownian motion of the following form:

$$\begin{cases} dY(t) = \left(AY(t) + \int_0^t \Gamma(t-s)Y(s)ds + F(t, Y(t)) \right) dt + \sigma(t)dB^H(t), & t \in J := [0, b], \\ Y_0(t) = Y(0) + G(Y), \end{cases} \quad (1)$$

where A is the infinitesimal generator of a strongly continuous semi-group $\{T(t), t \geq 0\}$ in $\mathbb{X}$; $\Gamma: D(\Gamma) \rightarrow \mathbb{X}$ is a closed linear operator with domain $D(\Gamma) \supset D(A)$ independent of t . $B^H = \{B^H(t), t \in J\}$ is an fBm with Hurst index $H \in (\frac{1}{2}, 1)$ on a real separable Hilbert space $\mathbb{V}$. Let $(\Omega, \mathcal{F}_b, \mathbb{P})$ be a probability space with a normal filtration $\{\mathcal{F}_t\}_{t \in [0, b]}$. The process $\{Y(t)\}_{t \in [0, b]}$ takes values in the real separable Hilbert space $\mathbb{X}$. F , σ and G are appropriate functions satisfying some hypotheses. Y_0 is an $\mathcal{F}_0$ -measurable random variable independent of B^H with finite second moment.

The aim of our paper is to study the solvability of (1) and present the results on the existence of mild solutions of (1) based on the Krasnoselskii-Schaefer-type fixed point theorem combined with the theory of resolvent operator for integro-differential equations in the sense of Grimmer. We know that many existence results of stochastic differential equations with nonlocal conditions are under the compact assumptions on nonlocal terms. In this paper, we are interested in weakening these hypotheses regarding nonlocal terms.

The remainder of this paper is organized as follows. In Section 2, we recall briefly the notations, concepts and basic results about the Wiener process and deterministic integro-differential equations. The main results in Section 3 are devoted to the study of the existence and uniqueness of mild solutions for system (1) with their proofs. An example is given in Section 4 to illustrate the obtained results. Section 5 concludes the paper and presents future work.

2 Preliminaries

We will present in this section, some notations, definitions and preliminaries, which play an important role in obtaining the main results of this paper.

2.1 Wiener process

Let $\mathbb{V}$ and $\mathbb{X}$ be two real separable Hilbert spaces and $(\Omega, \mathcal{F}_b, \mathbb{P})$ be a complete probability space with a normal filtration $\mathcal{F}_{t \in [0, b]}$. We denote by $\mathcal{F}_b$ the predictable σ -field on $\Omega_b := [0, b] \times \Omega$. Space $\mathbb{X}$ is equipped with a Borel σ -field $\mathcal{B}(\mathbb{X})$.

Introduce the following Banach spaces:

$$\begin{aligned} L(\mathbb{V}, \mathbb{X}) &:= \{h : \mathbb{V} \rightarrow \mathbb{X} / \text{is a bounded linear operator}\}, \\ L^2(\Omega, \mathcal{F}_b, \mathbb{X}) &:= \{f : \Omega \rightarrow \mathbb{X} / f \text{ is } \mathcal{F}_b\text{-measurable square integrable random variable}\}, \\ C(J, L^2(\Omega, \mathcal{F}_b, \mathbb{X})) &:= \{Y : J \rightarrow L^2(\Omega, \mathcal{F}_b, \mathbb{X}) / Y \text{ is a continuous mapping from } J \text{ into } L^2(\Omega, \mathcal{F}_b, \mathbb{X}) \\ &\quad \text{such that } \sup_{t \in J} \mathbb{E} \|Y(t)\|^2 < \infty\}, \end{aligned}$$

$$\mathbf{C} := \{Y : J \times \Omega \rightarrow \mathbb{X} / Y \in C(J, L^2(\Omega, \mathcal{F}_b, \mathbb{X})) \text{ is an } \mathcal{F}_t\text{-adapted stochastic process}\}.$$

For $Y \in \mathbf{C}$, define the norm $\|Y\|_{\mathbf{C}}^2 = (\sup_{t \in J} \mathbb{E} \|Y(t)\|^2)$. It is clear that $(\mathbf{C}, \|\cdot\|_{\mathbf{C}})$ is a Banach space.

Before continuing, let us give the definition of one-dimensional fBm.

Definition 2.1. [3,4] A one-dimensional fBm with Hurst parameter $0 < H < 1$ is a centered Gaussian process $B^H = B^H(t)$, $t \in \mathbb{R}$ with the covariance function

$$R_H(s, t) = \mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \in \mathbb{R}.$$

For $H = \frac{1}{2}$, the fractional Brownian motion is then a standard Brownian motion.

In this paper, we assume that $H \in (\frac{1}{2}, 1)$.

For $\frac{1}{2} < H < 1$, fBm $\beta^H(t)$ can be represented by finite interval, i.e.,

$$\beta^H(t) = \int_0^t K^H(t, s) dW(s),$$

where $W = \{W(t), t \in J\}$ is a Wiener process and

$$K_H(t, s) = c_H \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where c_H is a non-negative constant with respect to H .

Denote by ϵ the linear space of step functions on J of the form

$$\phi(t) = \sum_{i=1}^{n-1} a_i I_{[t_i, t_{i+1}]}(t),$$

where $0 = t_1 < t_2 < \dots < t_n = b$, $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and X the closure of ϵ with respect to the scalar product

$$\langle I_{[0, t]}, I_{[0, s]} \rangle = R^H(t, s).$$

The Wiener integral of ϕ ($\phi \in \epsilon$) with respect to β^H is given by

$$\int_0^b \phi(s) d\beta^H(s) = \sum_{i=1}^{n-1} a_i (\beta^H(t_{i+1}) - \beta^H(t_i)).$$

Moreover, the mapping

$$\phi \rightarrow \int_0^b \phi(s) d\beta^H(s)$$

is an isometry between ϵ and the linear space $\text{span} \{\beta^H(t), t \in J\}$ viewed as a subspace of $L^2(\Omega)$, which can be extended to an isometry between $\mathcal{X}$ and the first Wiener chaos of the fBm $\overline{\text{span}}^{L^2(\Omega)} \{\beta^H(t), t \in J\}$. The image on an element $h \in \mathcal{X}$ by this isometry is called the Wiener integral of h with respect to β^H .

For any $\tau \in [0, b]$, consider the linear operator $K_\tau^* : \epsilon \rightarrow L^2[0, b]$ given by

$$(K_H^* \phi)(s) = \int_s^\tau \phi(t) \frac{\partial K(t, s)}{\partial t} dt.$$

The operator K_b^* induces an isometry between ϵ and $L^2[0, b]$ that can be extended to $\mathcal{X}$.

We have the following relation between Wiener integral with respect to fBm and Itô integral with respect to the Wiener process:

$$\int_0^b h(s) d\beta^H(s) = \int_0^b (K_b^* h)(s) dW(s), \quad h \in \mathbb{X},$$

iff $K_b^* h \in L^2[0, b]$.

For $t \in [0, b]$, $\int_0^t h(s) d\beta^H(s)$ is defined by

$$\int_0^t h(s) d\beta^H(s) = \int_0^t h(s) I_{[0, t]}(s) d\beta^H(s).$$

Moreover, we have

$$\int_0^t h(s) d\beta^H(s) = \int_0^t (K_t^* h)(s) dW(s), \quad t \in [0, b], h I_{[0, t]} \in \mathcal{X},$$

iff $K_b^* h \in L^2[0, b]$.

Define $L_{\mathcal{X}}^2[0, b]$ by

$$L_{\mathcal{X}}^2[0, b] = \{h \in \mathcal{X}, K_b^* h \in L^2[0, b]\}.$$

For $H > \frac{1}{2}$, we have (see [11])

$$L_{\mathbb{H}}^1[0, b] \subset L_{\mathbb{X}}^2[0, b].$$

Next, we define the infinite dimensional fBm and give the definition of the corresponding stochastic integral.

Let $Q \in \mathcal{L}(V, V)$ be a non-negative self-adjoint trace class operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{tr} Q = \sum_{n=1}^\infty \lambda_n < \infty$, where $\lambda_n (n = 1, 2, \dots)$ is a nonnegative real number. We define the infinite-dimensional fBm on $\mathbb{X}$ with covariance Q as

$$B_Q^H(t) = \sum_{n=1}^\infty \beta_n^H(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^\infty \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are real, independent one-dimensional fBm. Define the space $L_Q^0(\mathbb{V}, \mathbb{X})$ by

$$L_Q^0(\mathbb{V}, \mathbb{X}) = \{\xi : \mathbb{V} \rightarrow \mathbb{X} | \xi \text{ is a } Q\text{-Hilbert-Schmidt operator}\}. \quad (2)$$

Note that $\xi \in L(\mathbb{V}, \mathbb{X})$ is called a Q -Hilbert-Schmidt operator, if

$$\|\xi\|_{L_Q^0(\mathbb{V}, \mathbb{X})} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \xi e_n\|^2 < \infty.$$

The space $L_Q^0(\mathbb{V}, \mathbb{X})$ equipped with the inner product

$$\langle \xi, \psi \rangle_{L_Q^0(\mathbb{V}, \mathbb{X})} = \sum_{n=1}^{\infty} \langle \xi e_n, \psi e_n \rangle$$

is a separable Hilbert space.

Definition 2.2. [3,11,14] Let $\Lambda: [0, b] \rightarrow L_Q^0(\mathbb{V}, \mathbb{X})$ such that

$$\sum_{n=1}^{\infty} \|K_b^*(\Lambda Q^{\frac{1}{2}}) e_n\|_{L^2([0, b], \mathbb{X})} < \infty. \quad (3)$$

Then its stochastic integral with respect to the fBm B^H is defined as follows:

$$\int_0^t \Lambda(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \Lambda(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t (K_b^*(\Lambda Q^{\frac{1}{2}}) e_n)(s) dW(s), \quad t \in [0, b].$$

Note that if

$$\sum_{n=1}^{\infty} \|\Lambda Q^{\frac{1}{2}} e_n\|_{L^{\frac{1}{H}}([0, b], \mathbb{X})} < \infty, \quad (4)$$

then particularly (3) holds, which follows immediately from (4).

Lemma 2.1. [11,14] If $\Lambda: [0, b] \rightarrow L_Q^0(\mathbb{V}, \mathbb{X})$ satisfies (4), then, for any $0 \leq s < t \leq b$, we have

$$\mathbb{E} \left\| \int_s^t \Lambda(\tau) dB^H(\tau) \right\|_{\mathbb{X}}^2 \leq C_H(t-s)^{2H-1} \sum_{n=1}^{\infty} \int_s^t \|\Lambda(\tau) Q^{\frac{1}{2}} e_n\|_{\mathbb{X}}^2 d\tau,$$

where C_H is a constant depending on H . If, in addition,

$$\sum_{n=1}^{\infty} \|\Lambda(t) Q^{\frac{1}{2}} e_n\|_{\mathbb{X}} \text{ is uniformly convergent for } t \in [0, b],$$

then

$$\mathbb{E} \left\| \int_s^t \Lambda(s\tau) dB^H(\tau) \right\|_{\mathbb{X}}^2 \leq C_H(t-s)^{2H-1} \int_0^t \|\Lambda(\tau)\|_{L_Q^0(\mathbb{V}, \mathbb{X})}^2 d\tau.$$

2.2 Integro-differential equations

In this subsection, we recall some knowledge on partial integro-differential equations and the related resolvent operators. Let $\mathbb{V}$ and $\mathbb{X}$ be two Banach spaces such that $\|z\|_{\mathbb{X}} := \|Az\| + \|z\|$, for $z \in \mathbb{X}$. A and $\Gamma(t)$ are closed linear operators on $\mathbb{X}$. Let $C(\mathbb{R}^+; \mathbb{X})$, $\mathcal{L}(\mathbb{X})$ stand for the space of all continuous functions from $\mathbb{R}^+$ into $\mathbb{X}$, the set of all bounded linear operators from $\mathbb{X}$ to $\mathbb{V}$, respectively. In what follows, we suppose the following assumptions:

- (H1) The operator A is the infinitesimal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on $\mathbb{X}$.
 (H2) For all $t \geq 0$, $\Gamma(t)$ is the closed linear operator from $D(A)$ to $\mathbb{X}$ and $\Gamma(t) \in \mathcal{L}(\mathbb{V}, \mathbb{X})$. For any $z \in \mathbb{X}$, the map $t \mapsto \Gamma(t)z$ is bounded, differentiable and the derivative $t \mapsto \Gamma'(t)z$ is bounded uniformly continuous on $\mathbb{R}^+$.

We consider the following Cauchy problem:

$$\begin{aligned} z'(t) &= Az(t) + \int_0^t \Gamma(t-s)z(s)ds \quad \text{for } t \geq 0, \\ z(0) &= z_0 \in \mathbb{X}. \end{aligned} \quad (5)$$

Theorem 2.2. [33] Suppose that (H1)–(H2) are verified. Then there exists a unique resolvent operator for the Cauchy problem (5).

Definition 2.3. [33] A resolvent operator for Eq. (5) is a bounded linear operator valued function $\mathcal{R}(t) \in \mathcal{L}(\mathbb{X})$ for $t \geq 0$, having the following properties:

- (i) $\mathcal{R}(0) = \mathbf{I}$ and $\|\mathcal{R}(t)\| \leq ye^{\delta t}$ for some constants $y > 0$ and $\delta \in \mathbb{R}$.
- (ii) For each $x \in \mathbb{X}$, $\mathcal{R}(t)x$ is strongly continuous for $t \geq 0$.
- (iii) $\mathcal{R}(t) \in \mathcal{L}(\mathbb{X})$ for $t \geq 0$. For $x \in \mathbb{X}$, $\mathcal{R}(\cdot)x \in C^1(\mathbb{R}^+; \mathbb{X}) \cap C(\mathbb{R}^+; \mathbb{X})$ and

$$\mathcal{R}'(t)x = A\mathcal{R}(t)x + \int_0^t \Gamma(t-s)\mathcal{R}(s)xds = \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)\Gamma(s)xds \quad \text{for } t \geq 0.$$

In the following, we give some results for the existence of solutions for the following integro-differential equation:

$$\begin{cases} z'(t) = Az(t) + \int_0^t \Gamma(t-s)z(s)ds + \mu(t) & \text{for } t \geq 0, \\ z(0) = z_0 \in \mathbb{X}, \end{cases} \quad (6)$$

where $\mu : \mathbb{R}^+ \rightarrow \mathbb{X}$ is a continuous function.

Definition 2.4. [33] A continuous function $z : \mathbb{R}^+ \rightarrow \mathbb{X}$ is said to be a strict solution of Eq. (6) if $z \in C^1(\mathbb{R}^+; \mathbb{X}) \cap C(\mathbb{R}^+, \mathbb{X})$ and z satisfies Eq. (6).

Theorem 2.3. [33] Assume that (H1)–(H2) hold. If z is a strict solution of Eq. (6), then

$$z(t) = \mathcal{R}(t)z_0 + \int_0^t \mathcal{R}(t-s)\mu(s)ds \quad \text{for } t \geq 0.$$

Lemma 2.4. [34] Assume that (H1)–(H2) hold. The resolvent operator $(\mathcal{R}(t))_{t \geq 0}$ is compact for $t > 0$ if and only if the semigroup $(T(t))_{t \geq 0}$ is compact for $t > 0$.

Lemma 2.5. [29] Assume that (H1)–(H2) hold. If the resolvent operator $(\mathcal{R}(t))_{t \geq 0}$ is compact for $t > 0$, then it is norm continuous (or continuous in the uniform operator topology) for $t > 0$.

Lemma 2.6. [29] Let Assumptions (H1)–(H2) be satisfied. Then, there exists a constant $L = L(b)$ such that

$$\|\mathcal{R}(t+\epsilon) - \mathcal{R}(\epsilon)\mathcal{R}(t)\|_{\mathcal{L}(\mathbb{X})} \leq L\epsilon \quad \text{for } 0 \leq \epsilon \leq t \leq b.$$

Now, we give the definition of mild solution for (1).

Definition 2.5. A $\mathbb{X}$ -valued stochastic process $Y \in \mathbf{C}$ is said to be a mild solution of system (1) if $Y(0) + G(Y) = Y_0$ and for any $t \in J$, it satisfies the following integral equation:

$$Y(t) = \mathcal{R}(t)[Y_0 - G(Y)] + \int_0^t \mathcal{R}(t-s)F(s, Y(s))ds + \int_0^t \mathcal{R}(t-s)\sigma(s)dB^H(s), \quad P - a.s.$$

Before continuing our development, let us recall the fixed point lemma, we will use in the rest of the work.

Lemma 2.7. [35, Krasnoselskii's fixed point theorem] Let $\mathbb{X}$ be a Banach space, $V \subset \mathbb{X}$ be a bounded closed and convex subset. Assume that $\mathbb{F}_1, \mathbb{F}_2 : V \rightarrow \mathbb{X}$ are two maps satisfying

- (i) $\mathbb{F}_1x + \mathbb{F}_2y \in V$ for $\forall x, y \in V$;
- (ii) $\mathbb{F}_1$ is a contraction;
- (iii) $\mathbb{F}_2$ is completely continuous.

Then, the equation $\mathbb{F}_1x + \mathbb{F}_2x = x$ has a solution on V .

3 Existence of mild solutions

This part is devoted to state and prove our main results. We define the operator $\mathbb{F}$ on $\mathbf{C}$ by

$$(\mathbb{F}Y)(t) = \mathcal{R}(t)[Y_0 - G(Y)] + \int_0^t \mathcal{R}(t-s)F(s, Y(s))ds + \int_0^t \mathcal{R}(t-s)\sigma(s)dB^H(s), \quad P - a.s.$$

To go ahead, the following assumptions are important:

(H3) There is a positive constant M such that $\sup_{0 \leq s, t \leq b} \|\mathcal{R}(t)\| \leq M$.

(H4) The mapping $F : J \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ is measurable from $(\Omega_b \times \mathbb{X}, \mathcal{P}_b \times \mathcal{B})$ into $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Moreover, it has linear growth in the variable y uniformly into t ; that is, there exists a positive constant $c_1 > 0$ such that

$$\|F(t, \omega, y)\| \leq c_1(1 + \|y\|), \quad \forall y \in \mathbb{X}, \forall t \in J, \text{ almost all } \omega \in \Omega.$$

(H5) There exists a constant $L_1 > 0$ such that

$$\|F(t, \omega, y) - F(t, \omega, z)\| \leq L_1\|y - z\|, \quad \forall z, y \in \mathbb{X}, \forall t \in J, \text{ almost all } \omega \in \Omega.$$

(H6) The function $\sigma : J \rightarrow L_Q^0(\mathbb{V}, \mathbb{X})$ is measurable and there exists a positive constant $c_2 > 0$ such that

- (i) $\sup_{0 \leq s \leq b} \|\sigma(s)\|_{L_Q^0(\mathbb{V}, \mathbb{X})}^2 \leq c_2$,
- (ii) $\sum_{n=1}^{\infty} \|\sigma Q^{\frac{1}{2}} e_n\|_{L^{\frac{1}{2}}([0, b], \mathbb{X})} < \infty$,
- (iii) $\sum_{n=1}^{\infty} \|\sigma Q^{\frac{1}{2}} e_n\|_{\mathbb{X}}$ is uniformly convergent for $t \in [0, b]$.

(H7) There exists a constant $L_2 > 0$ such that $G : \mathbf{C} \rightarrow \mathbb{X}$ satisfies

$$\|G(Y_1) - G(Y_2)\|^2 \leq L_2\|Y_1 - Y_2\|_{\mathbf{C}}^2.$$

(H8) There exists a constant $c_3 > 0$ such that

$$\|G(Y)\| \leq c_3(1 + \|Y\|), \quad \forall Y \in \mathbf{C}, \text{ almost all } \omega \in \Omega.$$

Lemma 3.1. Assume that hypotheses **(H1)**, **(H2)**, **(H3)**, **(H4)**, **(H6)** and **(H8)** are satisfied. For any $Y \in \mathbf{C}$, $t \mapsto (\mathbb{F}Y)(t)$ is continuous on the interval $[0, b]$ in the L^2 -sense.

Proof. Let $0 \leq t_1 \leq t_2 \leq b$. Then, for any $Y \in \mathbf{C}$, we have

$$\begin{aligned} \mathbb{E} \|(\mathbb{F}Y)(t_2) - (\mathbb{F}Y)(t_1)\|^2 &\leq 3\mathbb{E} \|(\mathcal{R}(t_2) - \mathcal{R}(t_1))[Y_0 - G(Y)]\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_0^{t_2} \mathcal{R}(t_2 - s)F(s, Y(s))ds - \int_0^{t_1} \mathcal{R}(t_1 - s)F(s, Y(s))ds \right\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_0^{t_2} \mathcal{R}(t_2 - s)\sigma(s)dB^H(s) - \int_0^{t_1} \mathcal{R}(t_1 - s)\sigma(s)dB^H(s) \right\|^2 \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = 3\mathbb{E} \|(\mathcal{R}(t_2) - \mathcal{R}(t_1))[Y_0 - G(Y)]\|^2, \quad (7)$$

$$I_2 = 3\mathbb{E} \left\| \int_0^{t_2} \mathcal{R}(t_2 - s)F(s, Y(s))ds - \int_0^{t_1} \mathcal{R}(t_1 - s)F(s, Y(s))ds \right\|^2, \quad (8)$$

$$I_3 = 3\mathbb{E} \left\| \int_0^{t_2} \mathcal{R}(t_2 - s)\sigma(s)dB^H(s) - \int_0^{t_1} \mathcal{R}(t_1 - s)\sigma(s)dB^H(s) \right\|^2. \quad (9)$$

Therefore, we only need to check that I_i tends to zero when $t_2 \rightarrow t_1$, $i = 1, 2, 3$. For I_1 , by using strong continuity of $\mathcal{R}(t)$, we have

$$\lim_{t_2 \rightarrow t_1} (\mathcal{R}(t_2) - \mathcal{R}(t_1))[Y_0 - G(Y)] = 0.$$

By using **(H8)** and Definition 2.3, we obtain

$$\|(\mathcal{R}(t_2) - \mathcal{R}(t_1))[Y_0 - G(Y)]\| \leq 2M(\|Y_0\| + \|G(Y)\|) \leq 2M(\|Y_0\| + c_3(1 + \|Y\|)) \in L^2(\Omega).$$

Thanks to the property of Lebesgue integral, we obtain

$$\lim_{t_2 \rightarrow t_1} I_1 = 0.$$

For I_2 , we can get by direct calculations

$$I_2 \leq 6\mathbb{E} \left\| \int_0^{t_1} (\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s))F(s, Y(s))ds \right\|^2 + 6\mathbb{E} \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2 - s)F(s, Y(s))ds \right\|^2 := I_{21} + I_{22}.$$

For I_{21} , we have

$$I_{21} \leq 6\mathbb{E} \int_0^{t_1} \|(\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s))F(s, Y(s))\|^2 ds.$$

Exploiting properties (i) and (ii) of Definition 2.3, for each $s, t_1, t_2 \in [0, b]$, we have

$$\lim_{t_2 \rightarrow t_1} (\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s))F(s, Y(s)) = 0$$

and

$$\|\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s)\|^2 F(s, Y(s)) \leq 2M^2 \|F(s, Y(s))\|^2 \leq 2M^2 c_1^2 (1 + \|Y\|)^2 \leq 4M^2 c_1^2 (1 + \|Y\|)^2.$$

Then, by the Lebesgue majorant theorem, we conclude that

$$\lim_{t_2 \rightarrow t_1} I_{21} = 0.$$

Now, we have

$$I_{22} = 6\mathbb{E} \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2 - s) F(s, Y(s)) ds \right\|^2.$$

By (H3), Definition 2.3 and Hölder's inequality, we obtain

$$I_{22} \leq M^2 c_1^2 (t_2 - t_1) \int_{t_1}^{t_2} \mathbb{E}(1 + \|Y(s)\|^2) ds, \quad (10)$$

$$\leq 2M^2 c_1^2 (t_2 - t_1) \int_{t_1}^{t_2} (1 + \mathbb{E}\|Y(s)\|^2) ds, \quad (11)$$

and then

$$\lim_{t_2 \rightarrow t_1} I_{22} = 0.$$

As a result, $\lim_{t_2 \rightarrow t_1} I_2 = 0$.

For I_3 , we have

$$I_3 \leq 6\mathbb{E} \left\| \int_0^{t_1} (\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s)) \sigma(s) dB^H(s) \right\|^2 + 6\mathbb{E} \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2 - s) \sigma(s) dB^H(s) \right\|^2 \leq I_{31} + I_{32}.$$

By Lemma 2.1, we obtain

$$I_{31} \leq C_H t_1^{2H-1} \int_0^{t_1} \|(\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s)) \sigma(s)\|_{L_Q^0(\mathbb{V}, \mathbb{X})}^2 ds.$$

Using continuity of $(\mathcal{R}(t))_{t \geq 0}$ together with (H6) we obtain

$$\lim_{t_2 \rightarrow t_1} (\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s)) \sigma(s) = 0,$$

and, since

$$\|(\mathcal{R}(t_2 - s) - \mathcal{R}(t_1 - s)) \sigma(s)\|^2 \leq 2N^2 c_2^2 (e^{t_2 \delta} + e^{t_1 \delta}),$$

the Lebesgue majorant theorem implies

$$\lim_{t_2 \rightarrow t_1} I_{31} = 0.$$

Also by Lemma 2.1, we obtain

$$I_{32} \leq C_H (t_2 - t_1) \int_{t_1}^{t_2} \|\mathcal{R}(t_2 - s) \sigma(s)\|_{L_Q^0(\mathbb{V}, \mathbb{X})}^2 ds \leq C_H (t_2 - t_1) M^2 c_2^2,$$

hence

$$\lim_{t_2 \rightarrow t_1} I_3 = 0.$$

The aforementioned arguments show that $\lim_{t_2 \rightarrow t_1} \mathbb{E} \|(\mathbb{F}Y)(t_2) - (\mathbb{F}Y)(t_1)\|^2 = 0$. Therefore, we conclude that the function $t \mapsto (\mathbb{F}Y)(t)$ is continuous on $[0, b]$ in the L^2 -sense. $\square$

Lemma 3.2. Assume that hypotheses **(H1)**, **(H2)**, **(H3)**, **(H4)**, **(H6)** and **(H8)** are satisfied. Then, operator $\mathbb{F}$ sends $\mathbf{C}$ into itself.

Proof. For any $Y \in \mathbf{C}$, we have

$$\begin{aligned} \mathbb{E} \|(\mathbb{F}Y)(t)\|^2 &\leq 3\mathbb{E} \|\mathcal{R}(t)[Y_0 - G(Y)]\|^2 + 3\mathbb{E} \left\| \int_0^t \mathcal{R}(t-s)F(s, Y(s))ds \right\|^2 + 3\mathbb{E} \left\| \int_0^t \mathcal{R}(t-s)\sigma(s)dB^H(s) \right\|^2 \\ &:= P_1 + P_2 + P_3. \end{aligned}$$

For P_1 , by **(H8)**, we have

$$P_1 \leq 6M^2\mathbb{E}(\|Y_0\|^2 + \|G(Y)\|^2) \leq 6M^2[\mathbb{E}\|Y_0\|^2 + 2c_2^2(1 + \|Y\|_{\mathbf{C}}^2)].$$

By application of **(H3)**, Definition 2.3 and Hölder inequality, we get

$$P_2 \leq 3M^2\mathbb{E} \left(\int_0^t \|F(s, Y(s))\|ds \right)^2 \leq 3(M_{\mathbf{C}_1})^2 t \mathbb{E} \left(\int_0^t (1 + \|Y(s)\|)^2 ds \right) \leq 6(M_{\mathbf{C}_1})^2 b(1 + \|Y\|_{\mathbf{C}}^2).$$

By using **(H6)** and Lemma 2.1, we show that

$$P_3 \leq 3C_H t^{2H-1} \int_0^t \|\mathcal{R}(t-s)\sigma(s)\|_{L_Q^0(\mathbb{V}, \mathbb{X})}^2 ds \leq 3C_H M^2 b^{2H} c_2.$$

Therefore, when we gather all these estimates, we obtain $\|\mathbb{F}Y\|_{\mathbf{C}}^2 = \sup_{t \in J} \mathbb{E} \|(\mathbb{F}Y)(t)\|^2 < \infty$. By Lemma 3.1, $(\mathbb{F}Y)(t)$ is continuous on $[0, b]$ and so $\mathbb{F}$ maps $\mathbf{C}$ into $\mathbf{C}$. This puts an end on the proof. $\square$

Theorem 3.3. Assume that **(H1)**–**(H8)** are satisfied. Then, system (1) has a unique mild solution on $\mathbf{C}$, provided that

$$2M^2(L_2 + L_1^2 b^2) < 1. \quad (12)$$

Proof. We show that $\mathbb{F}$ is a contraction mapping. For any $Y_1, Y_2 \in \mathbf{C}$, by **(H4)**, **(H6)** and Definition 2.3, we have

$$\begin{aligned} \mathbb{E} \|(\mathbb{F}Y_1)(t) - (\mathbb{F}Y_2)(t)\|^2 &\leq 2\mathbb{E} \|\mathcal{R}(t)[G(Y_1) - G(Y_2)]\|^2 + 2\mathbb{E} \left\| \int_0^t \mathcal{R}(t-s)[F(s, Y_1) - F(s, Y_2)]ds \right\|^2 \\ &\leq 2M^2\mathbb{E} \|G(Y_1) - G(Y_2)\|^2 + 2M^2\mathbb{E} \left(\int_0^t \|F(s, Y_1) - F(s, Y_2)\|ds \right)^2 \\ &\leq 2M^2 L_2 \|Y_1 - Y_2\|_{\mathbf{C}}^2 + 2M^2 b^2 \mathbb{E} \left(\int_0^t \|F(s, Y_1) - F(s, Y_2)\|ds \right)^2 \\ &\leq 2M^2(L_2 + L_1^2 b^2) \|Y_1 - Y_2\|_{\mathbf{C}}^2. \end{aligned}$$

Then,

$$\|(\mathbb{F}Y_1)(t) - (\mathbb{F}Y_2)(t)\|_{\mathbf{C}}^2 \leq 2M^2(L_2 + L_1^2 b^2) \|Y_1 - Y_2\|_{\mathbf{C}}^2.$$

It follows from (12) that $\mathbb{F}$ is a contraction mapping. According to the contraction principle, we know that the operator $\mathbb{F}$ has a unique fixed point Y in $\mathbf{C}$, which is a mild solution of system (1). The proof is complete. $\square$

In what follows, we will use the Krasnoselskii fixed point theorem to establish the existence result for stochastic system (1). First, we introduced this additional condition.

(H9) The resolvent operator $\mathcal{R}(t)$ is compact for $t > 0$.

Theorem 3.4. Assume that hypotheses (H1), (H2), (H3), (H6), (H7), (H8) and (H9) are satisfied. Then, system (1) has a mild solution on $\mathbf{C}$ provided

$$M^2 L_2 + 12M^2 c_3^2 + 6M^2 c_1^2 b^2 < 1. \quad (13)$$

Proof. For any $r > 0$ such that

$$r^2 \geq \frac{6M^2 \mathbb{E} \|Y_0\|^2 + 12M^2 c_3^2 + 3M^2 C_H b^{2H-1} c_2^2 + 6M^2 c_1^2 b^2}{1 - 12M^2 c_3^2 - 6M^2 c_1^2 b^2}, \quad (14)$$

let $\mathbf{D}_r = \{Y \in \mathbf{C} : \|Y\|_{\mathbf{C}} \leq r\}$. Then, $\mathbf{D}_r \subset \mathbf{C}$ is a bounded closed and convex subset.

We define two operators $\mathbb{F}_1$ and $\mathbb{F}_2^*$ on $\mathbf{D}_r$ as follows:

$$\begin{aligned} (\mathbb{F}_1 Y)(t) &= \mathcal{R}(t)[Y_0 - G(Y)] + \int_0^t \mathcal{R}(t-s)\sigma(s)dB^H(s), \quad t \in [0, b], \\ (\mathbb{F}_2^* Y)(t) &= \int_0^t \mathcal{R}(t-s)F(s, Y(s))ds, \quad t \in [0, b]. \end{aligned}$$

In what follows, we will prove that the operators $\mathbb{F}_1$ and $\mathbb{F}_2^*$ satisfy all the conditions of Lemma 2.7. To this end, we subdivide our proof into several steps.

Step 1. For any $Y, Z \in \mathbf{D}_r$, $\mathbb{F}_1 Z + \mathbb{F}_2^* Y \in \mathbf{D}_r$.

$$\begin{aligned} \mathbb{E} \|(\mathbb{F}_1 Z)(t) + (\mathbb{F}_2^* Y)(t)\|^2 &\leq 3\mathbb{E} \|\mathcal{R}(t)[Z_0 - G(Z)]\|^2 + 3\mathbb{E} \left\| \int_0^t \mathcal{R}(t-s)\sigma(s)dB^H(s) \right\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_0^t \mathcal{R}(t-s)F(s, Y(s))ds \right\|^2 \\ &\leq 6M^2(\mathbb{E} \|Z_0\|^2 + \mathbb{E} \|G(Z)\|^2) + 3C_H t^{2H-1} \int_0^t \|\mathcal{R}(t-s)\sigma(s)\|^2 ds \\ &\quad + 3M^2 \mathbb{E} \left(\int_0^t \|F(s, Y(s))\| ds \right)^2 \\ &\leq 6M^2(\mathbb{E} \|Z_0\|^2 + 2c_3^2(1+r^2)) + 3M^2 C_H b^{2H-1} c_2^2 + 6M^2 c_1^2 b^2(1+r^2). \end{aligned}$$

By (14), it follows that

$$\|(\mathbb{F}_1 Z)(t) + (\mathbb{F}_2^* Y)(t)\|_{\mathbf{C}}^2 \leq 6M^2 \mathbb{E} \|Z_0\|^2 + 12M^2 c_3^2 + 3M^2 C_H b^{2H-1} c_2^2 + 6M^2 c_1^2 b^2 + [12M^2 c_3^2 + 6M^2 c_1^2 b^2]r^2 \leq r^2.$$

Thus, $\mathbb{F}_1 Z + \mathbb{F}_2^* Y \in \mathbf{D}_r$.

Step 2. $\mathbb{F}_1$ is contraction.

For any $Y_1, Y_2 \in \mathbf{C}$, according to (H7), we have

$$\mathbb{E}\|(\mathbb{F}_1 Y_1)(t) - (\mathbb{F}_1 Y_2)(t)\|^2 = \mathbb{E}\|\mathcal{R}(t)[G(Y_1) - G(Y_2)]\| \leq M^2 L_2 \|Y_1 - Y_2\|_{\mathbf{C}}^2.$$

Hence,

$$\|(\mathbb{F}_1 Y_1)(t) - (\mathbb{F}_1 Y_2)(t)\|_{\mathbf{C}}^2 \leq M^2 L_2 \|Y_1 - Y_2\|_{\mathbf{C}}^2.$$

In virtue of (13), $\mathbb{F}_1$ is a contraction on $\mathbf{D}_r$.

Step 3. $\mathbb{F}_2$ is completely continuous.

We subdivide this step into three claims.

Claim 1. $\{\mathbb{F}_2^* Y / Y \in \mathbf{D}_r\}$ is uniformly bounded.

For any $Y \in \mathbf{D}_r$, by (H4), (13) and Hölder's inequality, one has

$$\sup_{t \in J} \mathbb{E}\|(\mathbb{F}_2^* Y)(t)\|^2 \leq M^2 c_1^2 \sup_{t \in J} \mathbb{E} \left(\int_0^t (1 + \|Y(s)\|) ds \right)^2 \leq 2M^2 c_1^2 b^2 (1 + r^2) \leq r^2,$$

which implies that $\{\mathbb{F}_2^* Y / Y \in \mathbf{D}_r\}$ is uniformly bounded.

Claim 2. $\{\mathbb{F}_2^* Y / Y \in \mathbf{D}_r\}$ is an equicontinuous set.

Let $Y \in \mathbf{D}_r$, and $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned} \mathbb{E}\|(\mathbb{F}_2^* Y)(t_2) - (\mathbb{F}_2^* Y)(t_1)\|^2 &= \mathbb{E} \left\| \int_0^{t_1-\epsilon} [\mathcal{R}(t_2-s) - \mathcal{R}(t_1-s)] F(s, Y(s)) ds \right. \\ &\quad \left. + \int_{t_1-\epsilon}^{t_1} [\mathcal{R}(t_2-s) - \mathcal{R}(t_1-s)] F(s, Y(s)) ds + \int_{t_1}^{t_2} \mathcal{R}(t_2-s) F(s, Y(s)) ds \right\|^2 \\ &\leq 3\mathbb{E} \left\| \int_0^{t_1-\epsilon} [\mathcal{R}(t_2-s) - \mathcal{R}(t_1-s)] F(s, Y(s)) ds \right\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_{t_1-\epsilon}^{t_1} [\mathcal{R}(t_2-s) - \mathcal{R}(t_1-s)] F(s, Y(s)) ds \right\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2-s) F(s, Y(s)) ds \right\|^2 \\ &\leq 3b \int_0^{t_1-\epsilon} \|\mathcal{R}(t_2-s) - \mathcal{R}(t_1-s)\|_{\mathcal{L}(\mathbb{X})}^2 C_1^2 (1+r)^2 ds \\ &\quad + 3b \int_{t_1-\epsilon}^{t_1} \|\mathcal{R}(t_2-s) - \mathcal{R}(t_1-s)\|_{\mathcal{L}(\mathbb{X})}^2 C_1^2 (1+r)^2 ds \\ &\quad + 3b \int_{t_1}^{t_2} \|\mathcal{R}(t_2-s)\|_{\mathcal{L}(\mathbb{X})}^2 C_1^2 (1+r)^2 ds. \end{aligned} \tag{15}$$

By (H9), we deduce that the right hand side of (15) tends to zero independently of $Y \in \mathbf{D}_r$ as $t_2 \rightarrow t_1$. Therefore, $\{\mathbb{F}_2^* Y / Y \in \mathbf{D}_r\}$ is equicontinuous.

Claim 3. For any $t \in [0, b]$, the set $V(t) = \{(\mathbb{F}_2^* Y)(t) : Y \in \mathbf{D}_r\}$ is relatively compact.

The case $t = 0$ is obvious. So we consider $t \in (0, b]$. Let $0 < t \leq b$ be fixed and ϵ be real number $\epsilon \in (0, t)$ and for $Y \in \mathbf{D}_r$, we define the operators

$$(\mathbb{F}_2^{*\epsilon} Y)(t) = \mathcal{R}(\epsilon) \int_0^{t-\epsilon} \mathcal{R}(t-s-\epsilon) F(s, Y(s)) ds$$

and

$$(\tilde{\mathbb{F}}_2^\epsilon Y)(t) = \int_0^{t-\epsilon} \mathcal{R}(t-s) F(s, Y(s)) ds.$$

By Lemma 2.6 and the compactness of the resolvent operator $\mathcal{R}(\epsilon)$, the set $\mathbb{F}_\epsilon^*(t) = \{(\mathbb{F}_2^{*\epsilon} Y)(t) : Y \in \mathbf{D}_r\}$ is relatively compact in $\mathbb{X}$, for every ϵ , $0 < \epsilon < t$. Moreover, also by Lemma 2.6 and Hölder's inequality, for each $Y \in \mathbf{D}_r$, we obtain,

$$\begin{aligned} \mathbb{E} \|(\mathbb{F}_2^{*\epsilon} Y)(t) - (\tilde{\mathbb{F}}_2^\epsilon Y)(t)\|^2 &= \left\| \mathcal{R}(\epsilon) \int_0^{t-\epsilon} \mathcal{R}(t-s-\epsilon) F(s, Y(s)) ds - \int_0^{t-\epsilon} \mathcal{R}(t-s) F(s, Y(s)) ds \right\|^2 \\ &\leq b \int_0^{t-\epsilon} \|\mathcal{R}(\epsilon) \mathcal{R}(t-s-\epsilon) - \mathcal{R}(t-s)\|_{\mathcal{L}(\mathbb{X})}^2 \mathbb{E} \|F(s, Y(s))\|^2 ds \\ &\leq b(L\epsilon)^2 \int_0^{t-\epsilon} \mathbb{E} \|F(s, Y(s))\|^2 ds \\ &\leq b(L\epsilon)^2(t-\epsilon) [c_1^2(1+r)^2] \\ &\leq b(L\epsilon)^2(t-\epsilon) [c_1^2(2+2r)]. \end{aligned}$$

So the set $\tilde{\mathbb{F}}_\epsilon(t) = \{(\tilde{\mathbb{F}}_2^\epsilon Y)(t) : Y \in \mathbf{D}_r\}$ is precompact in $\mathbb{X}$ by using the total boundedness.

Applying this idea again, we obtain

$$\begin{aligned} \mathbb{E} \|(\mathbb{F}_2^* Y)(t) - (\tilde{\mathbb{F}}_2^\epsilon Y)(t)\|^2 &= \mathbb{E} \left\| \int_0^t \mathcal{R}(t-s) F(s, Y(s)) ds - \int_0^{t-\epsilon} \mathcal{R}(t-s) F(s, Y(s)) ds \right\|^2 \\ &\leq \int_{t-\epsilon}^t \|\mathcal{R}(t-s)\| \mathbb{E} \|F(s, Y(s))\|^2 ds \\ &\leq M^2 \int_{t-\epsilon}^t \mathbb{E} \|F(s, Y(s))\|^2 ds \\ &\leq M^2 c_1^2 (1+r)^2 \epsilon \xrightarrow{\epsilon \rightarrow 0} 0, \end{aligned}$$

and there are precompact sets $\{(\mathbb{F}_2^\epsilon Y)(t) : Y \in \mathbf{D}_r\}$. Thus, the set $\{(\mathbb{F}_2^\epsilon Y)(t) : Y \in \mathbf{D}_r\}$ precompact in $\mathbb{X}$.

By Claims 1–3 and Arzela-Ascoli theorem, we conclude that $\mathbb{F}_2^*$ is completely continuous. According to Lemma 2.7, $\mathbb{F}_1 + \mathbb{F}_2^*$ has a fixed point on $\mathbf{D}_r$. Consequently, system (1) has a mild solution. The proof is complete. $\square$

4 An example

We consider the following problem nonlocal stochastic integro-differential equation:

$$\begin{cases} \frac{\partial}{\partial t} u(t, y) = \frac{\partial^2}{\partial y^2} u(t, y) + \int_0^t O(t-s) u(s, y) ds + F(t, u(t, y)) + \sigma(t) \frac{dB^H(t)}{dt}, & t \in [0, 1], y \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & t \in [0, 1], \\ u(0, y) + \sum_{i=1}^m a_i(y) u(t_i, y) = u_0(y), & y \in [0, \pi], \end{cases} \quad (16)$$

where $0 < t_1 < t_2 < \dots < t_m < b = 1$, $B^H(t)$ denotes a cylindrical fBm defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, $a_i \in L^2([0, \pi])$. And $O: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function. Let $\mathbb{X} = \mathbb{V} = L^2([0, \pi])$ with the norm $\|\cdot\|$. The operator A is defined by $Au(z) = \frac{\partial^2 u}{\partial z^2}$ with the domain

$$D(A) = \{u \in H, u, u'' \text{ are absolutely continuous, } u'' \in H \text{ and } u(0) = u(\pi) = 0\}.$$

Note that there exists a complete orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of eigenvectors of A with $e_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz)$, $n = 0, 1, \dots$, and A generates a strongly continuous semigroup $\{T(t), t \geq 0\}$, which is compact, analytic and self-adjoint [35,36]. Thus, Assumptions **(H1)**, **(H2)** and **(H9)** are satisfied. We choose a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$, $\alpha_n \geq 0$. Define an operator $Q: \mathbb{V} \rightarrow \mathbb{V}$ by $Qe_n = \alpha_n e_n$ and assume that

$$\text{tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\alpha_n} < \infty.$$

Define the process $B^H(t)$ by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\alpha_n} \beta_n^H(t) e_n,$$

where $\{\beta_n^H\}_{n \in \mathbb{N}}$ is a sequence of mutually independent one-dimensional fBm.

Let $\Gamma: D(A) \subset \mathbb{X} \rightarrow \mathbb{X}$ be the operator defined by $\Gamma(t)(y) = O(t)Ay$ for $t \geq 0, y \in D(A)$ and

$$u(t)(z) = u(t, z), \quad F(t, u(t))(z) = F(t, u(t, z)), \quad G(u) = \sum_{i=1}^m a_i(y) u(t_i, y), \quad h_i = \sup_{y \in [0, \pi]} \|a_i(y)\|^2; \quad \sigma = I.$$

Then, system (16) can be written in the abstract form

$$\begin{cases} dY(t) = (AY(t) + F(t, Y(t)) + \int_0^t \Gamma(t-s)Y(s)ds)dt + \sigma(t)dB^H(t), & t \in J := [0, b], \\ Y_0(t) = Y(0) + G(Y). \end{cases} \quad (17)$$

Moreover, we suppose $O: \mathbb{R}^+ \rightarrow \mathbb{R}$ is bounded and C^1 function such that O' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied and hence, by Theorem 2.2, Eq. (5) has a resolvent operator $(\mathcal{R}(t))_{t \geq 0}$ on $\mathcal{L}(\mathbb{X})$.

Define

$$F(t, u(t))(z) = \frac{e^{-t}|u(t, z)|}{(1 + e^t)(1 + |u(t, z)|)},$$

one can see that F satisfies **(H4)**. Moreover,

$$\|F(t, u(t)) - F(t, v(t))\| = \frac{e^{-t}||u(t, z)| - |v(t, z)||}{(1 + e^t)(1 + |u(t, z)|)(1 + |v(t, z)|)} \leq \frac{e^{-t}}{1 + e^t} |u(t, z) - v(t, z)| \leq \frac{1}{2} |u(t, z) - v(t, z)|.$$

Hence, **(H5)** is satisfied. Assume now that **(H6)**, **(H7)**, **(H8)** and (12) are satisfied. By Theorem 3.3, system (17) has a mild solution on $[0, b]$.

5 Conclusion

Existence results of nonlocal stochastic integro-differential equations driven by a fractional Brownian motion have been investigated. First, by using the contraction principle, the existence and uniqueness of mild solutions are given. Next, the existence of mild solutions is investigated based on Krasnoselskii's fixed point theorem. Finally, the obtained theoretical results have been verified by an illustrative example. As further direction, we will investigate the existence results of nonlocal stochastic integro-differential equations driven by a fractional Brownian motion via Kuratowski measure of noncompactness.

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