Research Article

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Weak solutions and optimal controls of stochastic fractional reaction-diffusion systems

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Abstract: The aim of this paper is to investigate a class of nonlinear stochastic reaction-diffusion systems involving fractional Laplacian in a bounded domain. First, the existence and uniqueness of weak solutions are proved by using Galérkin's method. Second, the existence of optimal controls for the corresponding stochastic optimal control problem is obtained. Finally, several examples are provided to demonstrate the theoretical results.

Keywords: stochastic system, fractional Laplacian, weak solution, optimal control

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1 Introduction

In this paper, we discuss a class of stochastic fractional reaction-diffusion systems in Ω :

$$\begin{cases} du + (-\Delta)^\alpha u dt = f(u) dt + g dt + \sigma dW, & t \in (0, T], x \in \Omega, \\ u(t) = 0, & t \in [0, T], x \in \mathbb{R}^d \setminus \Omega, \\ u(0) = u_0, & x \in \Omega, \end{cases} \quad (1)$$

where Ω is a smooth bounded domain contained in $\mathbb{R}^d$, $\alpha \in (0, 1)$, $T \in (0, +\infty)$, u is a vector-valued function, σ is an operator-valued function and $\{W(t)\}_{t \in [0, T]}$ is the space-time noise.

Different from Laplacian, the fractional Laplacian is a nonlocal linear operator. A natural question arises: Whether we can extend the results of Laplacian problems to the fractional Laplacian ones or not? Unfortunately, these extensions are not always true, see [1–3]. In particular, Devillanova and Carlo Marano have studied the following fractional differential equation in [3]:

$$u^{(2)}(t) + 2\xi\omega^{2-\beta}u^{(\beta)}(t) + \omega^2u^{(0)}(t) = 0.$$

Devillanova and Carlo Marano have delicately compared the nonfractional cases ($\beta = 0, 1$) and the fractional case ($0 < \beta < 1$) through the mathematical analysis and experimental data. Furthermore, the authors have dealt with the fractional derivative by using the Laplace transform and inversion procedure and some rich results have been obtained. In the last decade, there have been many significant investigations on fractional Laplacian problems in deterministic cases, see [4–6] and references therein. But the corresponding stochastic cases need to be further studied. It is well known that $(-\Delta)^\alpha \rightarrow (-\Delta)$,

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as $\alpha \rightarrow 1$. When $\alpha = 1$, system (1) becomes the standard stochastic reaction-diffusion system. In [7], Ahmed studied on the situation of $\alpha = 1$ and $g \equiv 0$, and the well-posedness of weak solution is obtained for $p \in [1, +\infty)$. We extend this result to the fractional Laplacian case and obtain the existence and uniqueness of weak solutions when $p \in [1, 2_{\alpha}^*/2)$.

Recently, there have been several papers on stochastic fractional system. In [8], Durga and Muthukumar have considered a type of stochastic time-fractional system as follows:

$$\begin{cases} D_t^\alpha u = \Delta u + f + \sigma \frac{dW}{dt}, & t \in (0, T], \\ u(0) = u_0, \end{cases}$$

where D_t^α is the α th Caputo type of fractional derivative. Based on fractional calculus, the existence and regularity of mild solutions are obtained, whereas the existence of optimal controls for the corresponding optimal problem is proved by Balder's method. In [9], Bezdek has considered the stochastic fractional reaction-diffusion system on a circle

$$\begin{cases} du - \mathcal{L}u dt = \sigma(u) dW, & t \in (0, +\infty), \\ u(0) = u_0, \end{cases}$$

where $\mathcal{L}$ is a general fractional derivative operator and σ is a locally Lipschitz continuous mapping. Based on the assumption of the growth condition

$$|\sigma(x)| \leq a|x|^\gamma + b,$$

where $\gamma \geq 1$, Bezdek proved the existence of mild solutions. In [10], Wang investigated the following stochastic fractional reaction-diffusion system:

$$\begin{cases} d\varphi + (-\Delta)^\alpha \varphi dt = f(\varphi) dt + g dt + c \varphi dW, & t > \tau, x \in \Omega, \\ \varphi(t) = 0, & t > \tau, x \in \mathbb{R}^d \setminus \Omega, \\ \varphi(\tau) = \varphi_\tau, & x \in \Omega, \end{cases} \quad (2)$$

where φ is the real valued function and $c \in (0, +\infty)$. Under suitable assumptions on f , the existence and uniqueness of solutions are achieved. Wang et al. [11] keep discussing system (2) on the whole $\mathbb{R}^d$. But only the existence of solutions is obtained.

Motivated by the aforementioned results, we focus our attention on the properties of weak solutions of system (1). These solutions satisfy (1) in the weak sense with probability 1. Additionally, we wish to extend the optimal control theory of deterministic linear control problems in [12,13] to stochastic control problems. We consider the stochastic control problem of (1) as follows:

$$\begin{cases} du + (-\Delta)^\alpha u dt = f(u) dt + g dt + \mathcal{A}v dt + \sigma dW, & t \in (0, T], \\ u(0) = u_0 \end{cases}$$

with the cost function

$$J(v) = E \left\{ \int_0^T \| \mathcal{N}u(v) - z_d \|^2 dt + (\mathcal{K}v, v)_{\bar{V}} \right\},$$

where $v = v(t, x)$ is a stochastic control and $\mathcal{A}$ is a given operator which is called a controller.

The framework of this paper is organized as follows. In Section 2, the proper functional spaces and basic concepts are presented. In Section 3, the assumptions on the nonlinear term and noise are stated, then the existence and uniqueness of weak solutions are obtained by using Galrkin's method along with a uniform estimate. In Section 4, the existence of optimal controls is proved. In Section 5, some sufficient conditions and examples are shown to illustrate our results.

2 Preliminaries

Stochastic fractional reaction-diffusion system (1) involves both stochastic term and fractional Laplacian. It is necessary to employ the theories of stochastic partial differential equations (see [14,15]) and fractional calculus (see [16,17]) to study this type of system.

Let $\mathcal{S}$ be the Schwartz space, the fractional Laplacian for $\varphi \in \mathcal{S}$ is defined by

$$(-\Delta)^\alpha \varphi(x) = C(d, \alpha) \text{P.V.} \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x - y|^{d+2\alpha}} dy, \quad x \in \mathbb{R}^d,$$

where $\alpha \in (0, 1)$ and

$$\frac{1}{C(d, \alpha)} = \int_{\mathbb{R}^d} \frac{1 - \cos \xi_1}{|\xi|^{d+2\alpha}} d\xi.$$

In this paper, we denote $(-\Delta)^\alpha u = ((-\Delta)^\alpha u^1, (-\Delta)^\alpha u^2, \dots, (-\Delta)^\alpha u^N)$.

Let $L^p(\mathbb{R}^d, \mathbb{R}^N)$ be the space of all p th integrable functions. For any $u \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and $1 \leq p < +\infty$, the norm is defined by

$$\|u\|_{L^p(\mathbb{R}^d, \mathbb{R}^N)} = \left(\sum_{i=1}^N \int_{\mathbb{R}^d} |u^i(x)|^p dx \right)^{\frac{1}{p}}.$$

Since fractional Laplacian is a nonlocal operator, let

$$L^p \triangleq \{u \in L^p(\mathbb{R}^d, \mathbb{R}^N) : u^i = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Omega, i = 1, 2, \dots, N\}.$$

Noting that, for a given function $f \in L^p(\Omega, \mathbb{R}^N)$, it is only defined on Ω , we say $f \in L^p$ means that we extend the domain of f to $\mathbb{R}^d$ by setting

$$f(x) = \mathbf{0}, \quad x \in \mathbb{R}^d \setminus \Omega.$$

The function space $H^\alpha(\mathbb{R}^d, \mathbb{R}^N)$ is given by

$$H^\alpha(\mathbb{R}^d, \mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^d, \mathbb{R}^N) : \sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u^i(x) - u^i(y)|^2}{|x - y|^{d+2\alpha}} dx dy < +\infty \right\}.$$

Let

$$X_0 \triangleq \{u \in H^\alpha(\mathbb{R}^d, \mathbb{R}^N) : u^i = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega, i = 1, 2, \dots, N\}$$

be endowed with the norm

$$\|u\|_{X_0} = \left(\sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u^i(x) - u^i(y)|^2}{|x - y|^{d+2\alpha}} dx dy \right)^{\frac{1}{2}}.$$

It is easily seen that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space under the inner product

$$(u, v)_{X_0} = \sum_{i=1}^N \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u^i(x) - u^i(y))(v^i(x) - v^i(y))}{|x - y|^{d+2\alpha}} dx dy.$$

Based on [16, Proposition 1.18] and [17, Corollary 7.2], we get the following.

Lemma 2.1. *Let $\alpha \in (0, 1)$, then for any $u \in X_0$,*

$$\langle (-\Delta)^\alpha u, u \rangle_{X_0^*, X_0} = \frac{1}{2} C(d, \alpha) \|u\|_{X_0}^2. \quad (3)$$

Lemma 2.1 extends Proposition 1.18 of [16] directly, and they have similar proofs, so we omit the proof of Lemma 2.1. Setting $p_\alpha^* = \frac{dp}{d-\alpha p}$ for $p \in (0, +\infty)$, we state the following.

Lemma 2.2. *Let $\alpha \in (0, 1)$ and $2\alpha < d$. If $p \in [1, 2_\alpha^*/2]$, then*

$$X_0 \hookrightarrow L^{2p} \hookrightarrow L^2.$$

Thus, there exists $c_0 > 0$ such that

$$\|u\|_{L^2} \leq c_0 \|u\|_{X_0} \quad (4)$$

Moreover, the embedding $X_0 \hookrightarrow L^{2p}$ is compact when $p \in [1, 2_\alpha^*/2)$.

We work in a complete probability space $(\Sigma, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. Let Y be a separable Hilbert space, $\sigma(t)$ be a bounded linear operator from Y to L^2 for any $t \in [0, T]$ and W be a Y -valued Q -Wiener process defined on $(\Sigma, \mathcal{F}, P, \mathcal{F}_{t \in [0, T]})$. We recall a result of [15] as follows.

Lemma 2.3. [15, Proposition 4.1] *Let $Q \in \mathcal{L}(Y)$ be a symmetric nonnegative operator with $\text{Tr } Q < +\infty$, then for $\forall t \in [0, T]$ and $\forall y \in Y$:*

(1) *W is a Gaussian process on Y and*

$$E(W(t), y)_Y = 0, \quad E(W(t), y)_Y^2 = t(Qy, y).$$

(2) *W has the following expression:*

$$W(t) = \sum_{j=1}^{+\infty} \sqrt{\lambda_j} \beta_j(t) e_j,$$

where $\{e_j\}$ is an orthonormal basis of Y , $\{\lambda_j\}$ is the sequence of eigenvalues of Q and $\{\beta_j(t)\}$ is a sequence of Brownian motions which are independent from each other on probability space $(\Sigma, \mathcal{F}, P, \mathcal{F}_{t \in [0, T]})$.

Next, we introduce the space of random processes. For any $p \in [1, +\infty)$, let

$$L_p^{\mathcal{F}}([0, T], X) = \left\{ u \text{ is an } \mathcal{F}_{t \in [0, T]} \text{ adapted random process: } E \int_0^T \|u(t)\|_X^p dt < \infty \right\}$$

and define the norm

$$\|u\|_{L_p^{\mathcal{F}}([0, T], X)} = \left(E \int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

While $p = +\infty$, let

$$L_\infty^{\mathcal{F}}([0, T], X) = \left\{ u \text{ is an } \mathcal{F}_{t \in [0, T]} \text{ adapted random process: } \text{ess sup}_{t \in [0, T]} E \|u(t)\|_X^2 < \infty \right\}$$

and endow u with the following norm:

$$\|u\|_{L_\infty^{\mathcal{F}}([0, T], X)} = \left(\text{ess sup}_{t \in [0, T]} E \|u(t)\|_X^2 \right)^{\frac{1}{2}}.$$

$(L_p^{\mathcal{F}}([0, T], X), \|\cdot\|_{L_p^{\mathcal{F}}([0, T], X)})$ is a Banach space for any $p \in [1, +\infty]$. Moreover, it is a reflexive Banach space for any $p \in [1, +\infty)$. In this paper, we confine X to be L^2 , X_0 or L^{2p} .

We denote $L_2^{\mathcal{F}_0}(\Sigma, X)$ by the space of all $\mathcal{F}_0$ measurable random variables with bounded second moments. Let

$$C_T^1([0, T]) = \{\phi \in C^1([0, T]) : \phi \text{ vanished at } T\},$$

then $C_T^1([0, T])$ is a sub-space of $C^1([0, T])$. We give the concept of weak solutions for system (1) as follows.

Definition 2.4. An $\mathbb{R}^N$ -valued stochastic process $u \in L_{\infty}^{\mathcal{F}}([0, T], L^2) \cap L_2^{\mathcal{F}}([0, T], X_0) \cap L_{2p}^{\mathcal{F}}([0, T], L^{2p})$ is called a weak solution of system (1), if u satisfies

$$\begin{aligned} & - (u_0, \phi(0)v)_{L^2} - \int_0^T (u(t), \dot{\phi}v)_{L^2} dt + \int_0^T \langle (-\Delta)^{\alpha} u(t), v \rangle_{X_0^*, X_0} \phi(t) dt \\ & = \int_0^T \langle f(u(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt + \int_0^T (g(t), \phi(t)v)_{L^2} dt + \int_0^T (\phi(t)v, \sigma(t) dW(t)), \quad \text{a.s.,} \end{aligned} \quad (5)$$

for any $v \in X_0$ and $\phi \in C_T^1([0, T])$.

3 The existence and uniqueness of weak solutions

From [18, proposition 9] and [19, proposition 4], we know that there exists a sequence of eigenfunctions of fractional Laplacian which constructs an orthonormal basis of L^2 and an orthogonal basis of X_0 . This result ensures that we can use Galérkin's method to obtain the existence of weak solutions of (1).

First, due to the theory of semigroups of nonlinear operators (see [20, Chapter 4]), we assume that $f: L^{2p} \rightarrow (L^{2p})^*$ is m -dissipative. We also impose the following conditions on f and the operator-valued function $\sigma(t)$.

(H1): There exist $c_1 \in [0, +\infty)$ and $c_2 \in (0, +\infty)$, such that

$$\langle f(u), u \rangle_{(L^{2p})^*, L^{2p}} \leq c_1 \|u\|_{L^2}^2 - c_2 \|u\|_{L^{2p}}^{2p}.$$

(H2): There exist $c_3 \in (0, +\infty)$ and $c_4 \in (0, +\infty)$, such that

$$\|f(u)\|_{(L^{2p})^*} \leq c_3 + c_4 \|u\|_{L^{2p}}^{2p-1}.$$

(H3): $f(u)$ is lower semi-continuous and

$$\langle f(u) - f(v), u - v \rangle_{(L^{2p})^*, L^{2p}} \leq 0.$$

(H4): There exists an $M \in (0, +\infty)$, such that

$$\|\sigma(t)\| \leq M, \quad \forall t \in [0, T], \quad \text{uniformly in } x \in \Omega.$$

Next, we give an important result on the nonlinear term f .

Lemma 3.1. Let $p \in [1, +\infty)$ and $\{u_n\}$ be bounded in $L_{2p}^{\mathcal{F}}([0, T], L^{2p})$. If (H2)–(H3) hold, then for any $v \in X_0$ and $\phi \in C_T^1([0, T])$, there exists $\tilde{u} \in L_{2p}^{\mathcal{F}}([0, T], L^{2p})$ such that

$$\int_0^T \langle f(u_n(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \rightarrow \int_0^T \langle f(\tilde{u}(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt, \quad \text{a.s.}$$

Proof. Since $\{u_n\}$ is bounded in $L_{2p}^{\mathcal{F}}([0, T], L^{2p})$, there exists $\tilde{u} \in L_{2p}^{\mathcal{F}}([0, T], L^{2p})$ such that

$$u_n \rightarrow \tilde{u} \text{ weakly in } L_{2p}^{\mathcal{F}}([0, T], L^{2p}). \quad (6)$$

We need to show that for any $\xi \in L_\infty(\Sigma)$,

$$\mathbb{E} \left\{ \xi \int_0^T \langle f(u_n(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \right\} \rightarrow \mathbb{E} \left\{ \xi \int_0^T \langle f(\tilde{u}(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \right\}.$$

In virtue of (H2) and basic inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, $p > 0$, we obtain

$$\|f(u_n)\|_{(L_{2p}^{\mathcal{F}}([0, T], L^{2p}))^*} \leq \left\{ \mathbb{E} \int_0^T (c_3 + c_4 \|u_n\|_{L^{2p}}^{2p-1})^{2p/(2p-1)} dt \right\}^{(2p-1)/2p} \leq a_1 + a_2 \|u_n\|_{L_{2p}^{\mathcal{F}}([0, T], L^{2p})}^{2p-1}, \quad (7)$$

where a_1 and a_2 are constants dependent on $\{c_3, c_4, p\}$. This inequality implies that $\{f(u_n)\}$ is bounded in $(L_{2p}^{\mathcal{F}}([0, T], L^{2p}))^* = L_{2p/(2p-1)}^{\mathcal{F}}([0, T], L^{2p/(2p-1)})$. We also use $\{f(u_n)\}$ as the subsequence if necessary, thus there exists $h \in (L_{2p}^{\mathcal{F}}([0, T], L^{2p}))^*$ such that

$$f(u_n) \rightarrow h \text{ weakly } * \text{ in } (L_{2p}^{\mathcal{F}}([0, T], L^{2p}))^*.$$

In addition, by Hölder inequality and (7), we get

$$\mathbb{E} \left\{ \xi \int_0^T \langle f(u_n), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \right\} \leq a_3 + a_4 \|u_n\|_{L_{2p}^{\mathcal{F}}([0, T], L^{2p})}^{2p-1},$$

where a_3 and a_4 are constants dependent on $\{a_1, a_2, \|\xi\|_{L_\infty(\Sigma)}, \|\phi\|_{L^{1/(2p-1)}([0, T])}, \|v\|_{X_0}\}$. So

$$\mathbb{E} \left\{ \xi \int_0^T \langle f(u_n(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \right\} \rightarrow \mathbb{E} \left\{ \xi \int_0^T \langle h(t), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \right\}$$

holds by the Dominated Convergence Theorem.

Now let us verify

$$h = f(\tilde{u}), \quad \text{a.s.}$$

From (6), the sequence $\{u_n\}$ is weakly convergence to $\tilde{u}$ in $L_{2p}^{\mathcal{F}}([0, T], L^{2p})$, further by the Mazur theorem, there exists a sequence $\{\zeta_n\}$ which is a suitable convex combination of $\{u_n\}$ and $\{\zeta_n\}$ that converges strongly to $\tilde{u}$, that is,

$$\zeta_n \rightarrow \tilde{u} \text{ in } L_{2p}^{\mathcal{F}}([0, T], L^{2p}).$$

On the other hand, from assumption (H3), we can deduce that for $\forall \psi \in L^{2p}$,

$$\int_0^T \langle f(\psi) - h, \psi - \tilde{u} \rangle_{(L^{2p})^*, L^{2p}} dt = \int_0^T \langle f(\psi) - f(\zeta_n) + f(\zeta_n) - h, \psi - \zeta_n + \zeta_n - \tilde{u} \rangle_{(L^{2p})^*, L^{2p}} dt \leq 0, \quad \text{a.s.}$$

Furthermore, since f is lower semi-continuous, choosing $\psi = \tilde{u} + \epsilon\mu$ for any $\epsilon \in (0, +\infty)$ and $\mu \in L_{2p}^{\mathcal{F}}([0, T], L^{2p})$, we derive that

$$\int_0^T \langle f(\tilde{u}) - h, \mu \rangle_{(L^{2p})^*, L^{2p}} dt \leq \lim_{\epsilon \rightarrow 0} \int_0^T \langle f(\tilde{u} + \epsilon\mu) - h, \mu \rangle_{(L^{2p})^*, L^{2p}} dt \leq 0, \quad \text{a.s.} \quad (8)$$

In view of the arbitrariness of μ , inequality (8) is available if and only if $h = f(\tilde{u})$, a.s. $\square$

The main result of this section is given as follows.

Theorem 3.2. Let $\alpha \in (0, 1)$, $2\alpha < d$ and $p \in [1, 2_\alpha^*/2)$. If (H1)–(H4) hold, then system (1) has unique weak solution u , restrict to any $g \in L_2^{\mathcal{F}}([0, T], L^2)$ and $u_0 \in L_2^{\mathcal{F}_0}(\Sigma, L^2)$.

Proof. We start with the uniqueness of weak solutions and then we prove the existence of weak solutions by using the Galrkin method. This proof is divided into four steps.

Step 1. The uniqueness of weak solutions.

Assume that $\hat{u}, \check{u} \in L^\infty_{\mathcal{F}}([0, T], L^2) \cap L^2_{\mathcal{F}}([0, T], X_0) \cap L^{2p}_{\mathcal{F}}([0, T], L^{2p})$ are the weak solutions with initial states $\hat{u}_0, \check{u}_0$ and $\hat{g}, \check{g}$, respectively. Since $\hat{u}$ and $\check{u}$ satisfy system (1) in the weak sense, from integrating by parts, we deduce that

$$\begin{aligned} & \frac{1}{2} \|\hat{u} - \check{u}\|_{L^2}^2 + \int_0^t ((-\Delta)^\alpha (\hat{u} - \check{u}), \hat{u} - \check{u})_{X_0^*, X_0} ds \\ &= \frac{1}{2} \|\hat{u}_0 - \check{u}_0\|_{L^2}^2 + \int_0^t (f(\hat{u}) - f(\check{u}), \hat{u} - \check{u})_{(L^{2p})^*, L^{2p}} ds + \int_0^t (\hat{g} - \check{g}, \hat{u} - \check{u})_{L^2} ds + \int_0^t (\sigma^* \hat{u} - \sigma^* \check{u}, dW). \end{aligned}$$

Due to assumption (H3), (3), (4) and Young's inequality, we get that for arbitrarily $\varepsilon > 0$,

$$\begin{aligned} & \frac{1}{2} \|\hat{u} - \check{u}\|_{L^2}^2 + \frac{1}{2} C(d, \alpha) \int_0^t \|\hat{u} - \check{u}\|_{X_0}^2 ds \\ & \leq \frac{1}{2} \|\hat{u}_0 - \check{u}_0\|_{L^2}^2 + c_0 \varepsilon \int_0^t \|\hat{u} - \check{u}\|_{X_0}^2 ds + c_\varepsilon \int_0^t \|\hat{g} - \check{g}\|_{L^2}^2 ds + \int_0^t (\sigma^* \hat{u} - \sigma^* \check{u}, dW), \end{aligned}$$

where $c_\varepsilon = \varepsilon^{-1}$. Choosing ε such that $c_0 \varepsilon = C(d, \alpha)/4$, then taking the expectation we obtain

$$\mathbb{E} \|\hat{u} - \check{u}\|_{L^2}^2 + \frac{1}{2} C(d, \alpha) \mathbb{E} \int_0^t \|\hat{u} - \check{u}\|_{X_0}^2 ds \leq \mathbb{E} \|\hat{u}_0 - \check{u}_0\|_{L^2}^2 + 2c_\varepsilon \mathbb{E} \int_0^t \|\hat{g} - \check{g}\|_{L^2}^2 ds,$$

where the expectation of $\int_0^t (\sigma^* \hat{u} - \sigma^* \check{u}, dW)$ equals zero. Setting $\hat{u}_0 = \check{u}_0, \hat{g} = \check{g}$, it is easy to see that $\hat{u} = \check{u}$.

Step 2. The existence of solutions for the finite dimensional truncated system.

In view of [18], we let $\{\bar{e}_i\}$ be an orthonormal sequence of eigenfunctions of fractional Laplacian and $\{\bar{e}_i\}$ is also an orthonormal basis of L^2 and an orthogonal basis of X_0 . As in Lemma 2.3, $\{e_i\}$ is an orthonormal basis of Y and $B_n \triangleq (\beta_1, \beta_2, \dots, \beta_n)'$ is an n dimensional Brownian motion. We consider the truncation of system (1):

$$\begin{cases} du_n + (-\Delta)^\alpha u_n dt = f(u_n) dt + g_n dt + \sigma dW_n, & t \in (0, T], x \in \Omega, \\ u_n(t) = 0, & t \in [0, T], x \in \mathbb{R}^d \setminus \Omega, \\ u_n(0) = \sum_{j=1}^n (u_0, \bar{e}_j)_{L^2} \bar{e}_j \triangleq \sum_{j=1}^n \vartheta_n^j(0) \bar{e}_j, & x \in \Omega, \end{cases} \quad (9)$$

where $u_n(t) \triangleq \sum_{j=1}^n \vartheta_n^j(t) \bar{e}_j$, $g_n(t) \triangleq \sum_{j=1}^n (g(t), \bar{e}_j)_{L^2} \bar{e}_j$ and $W_n(t) \triangleq \sum_{j=1}^n (W(t), e_j)_Y e_j$. Consequently, let $\mathcal{L}$ be an $n \times n$ matrix, F and G be n -vectors and Θ be an $n \times n$ matrix, whose elements are given by:

$$\begin{aligned} L_{ij} &= \langle (-\Delta)^\alpha \bar{e}_i, \bar{e}_j \rangle_{X_0^*, X_0}, \\ F_i(\vartheta_n) &= \left\langle f\left(\sum_{j=1}^n \vartheta_n^j \bar{e}_j\right), \bar{e}_i \right\rangle_{(L^{2p})^*, L^{2p}}, \\ G_i(t) &= (g(t), \bar{e}_i)_{L^2}, \\ \theta_{ij}(t) &= \sqrt{\lambda_j} (\bar{e}_i, \sigma(t) e_j)_{L^2}, \end{aligned}$$

where $i, j = 1, 2, \dots, n$. Next, we study the following n dimensional stochastic system:

$$d\vartheta_n = \mathcal{L} \vartheta_n dt + F(\vartheta_n) dt + G dt + \Theta dB_n, \quad t \in [0, T]. \quad (10)$$

Because f is m -dissipative, F is m -dissipative. Hence, we can use the linear interpolation methods and Crandall-Liggett's theory to show the existence of solutions of system (10). Let $\Pi_k = \{0 = t_k^0 < t_k^1 < \dots, < t_k^k = T\}$ be the k th uniform partition of $[0, T]$ with $|\Pi_k| \rightarrow 0$, as $k \rightarrow +\infty$. Denote $\delta_k \triangleq |\Pi_k|$, then the sequence of approximate solutions $\{\vartheta_k(t)\}$ is given by

$$\vartheta_k(t_k^i) = (I - \delta_k F)^{-1}[\vartheta_k(t_k^{i-1}) + \delta_k \mathcal{L}\vartheta_k(t_k^{i-1}) + \delta_k G(t_k^{i-1}) + \Theta(t_k^{i-1})(B_n(t_k^i) - B_n(t_k^{i-1}))],$$

where $i = 1, 2, \dots, k$. According to [20, Theorem 4.7], there exists $\vartheta \in C^{\mathcal{F}}([0, T], L^2)$ such that

$$\vartheta_k \rightarrow \vartheta, \quad \text{uniformly on } [0, T], \quad \text{as } k \rightarrow \infty,$$

that is, the process

$$\vartheta = (\vartheta^1, \vartheta^2, \dots, \vartheta^n)'$$

is a solution of (10). Thus, $u_n = \sum_{j=1}^n \vartheta^j \bar{e}_j$ is a solution of (9).

Step 3. A uniform estimate.

Integrating by parts on both sides of (9), we obtain

$$\begin{aligned} & \frac{1}{2}(u_n(t), u_n(t))_{L^2} + \int_0^t \langle (-\Delta)^\alpha u_n(s), u_n(s) \rangle_{X_0^*, X_0} ds \\ &= \frac{1}{2}(u_n(0), u_n(0))_{L^2} + \int_0^t \langle f(u_n(s), u_n(s)) \rangle_{(L^{2p})^*, L^{2p}} ds + \int_0^t (g_n(s), u_n(s))_{L^2} ds + \int_0^t (u_n(s), \sigma(s) dW_n(s)). \end{aligned}$$

Consequently, similar to the calculation of Step 1, we have

$$\begin{aligned} & \mathbb{E}\|u_n(t)\|_{L^2}^2 + \frac{1}{2}C(d, \alpha) \int_0^t \mathbb{E}\|u_n(s)\|_{X_0}^2 ds + 2c_2 \int_0^t \mathbb{E}\|u_n(s)\|_{L^{2p}}^{2p} ds \\ & \leq \mathbb{E}\|u_n(0)\|_{L^2}^2 + 2c_1 \int_0^t \mathbb{E}\|u_n(s)\|_{L^2}^2 ds + 2c_\varepsilon \int_0^t \mathbb{E}\|g_n(s)\|_{L^2}^2 ds \end{aligned} \quad (11)$$

by using (H1), (3), (4) and Young's inequality. Since $u_0 \in L_2^{\mathcal{F}_0}(\Sigma, L^2)$ and $g \in L_2^{\mathcal{F}}([0, T], L^2)$, using (11) and Gronwall inequality, we get that

$$\mathbb{E}\|u_n(t)\|_{L^2}^2 \leq ce^{2c_1 T},$$

where $c = c(\|u_0\|_{L_2^{\mathcal{F}_0}(\Sigma, L^2)}, \|g\|_{L_2^{\mathcal{F}}([0, T], L^2)}, T)$. Hence using (11) once more, we deduce that $\{u_n\}$ is bounded in $L_\infty^{\mathcal{F}}([0, T], L^2) \cap L_2^{\mathcal{F}}([0, T], X_0) \cap L_{2p}^{\mathcal{F}}([0, T], L^{2p})$. So there exists a subsequence of $\{u_n\}$, which is also denoted by $\{u_n\}$ and an element $\tilde{u}$, such that

$$\begin{aligned} u_n &\rightharpoonup \tilde{u} \text{ weakly } * \text{ in } L_\infty^{\mathcal{F}}([0, T], L^2), \\ u_n &\rightharpoonup \tilde{u} \text{ weakly in } L_2^{\mathcal{F}}([0, T], X_0), \\ u_n &\rightharpoonup \tilde{u} \text{ weakly in } L_{2p}^{\mathcal{F}}([0, T], L^{2p}). \end{aligned} \quad (12)$$

Step 4. The existence of weak solutions for (1).

We assert that $\tilde{u}$ is a weak solution of (1), that is, we need to show $\tilde{u}$ satisfies formula (5). Taking any $\phi \in C_T^1([0, T])$ and $\xi \in L_\infty(\Sigma)$, by (9), we can compute that

$$\begin{aligned}
0 &= \mathbb{E}\{\xi(u_n(0), \phi(0)\bar{e}_i)\} + \mathbb{E}\left\{\xi \int_0^T (u_n(t), \dot{\phi}\bar{e}_i)_{L^2} dt\right\} - \mathbb{E}\left\{\xi \int_0^T \langle (-\Delta)^\alpha u_n(t), \bar{e}_i \rangle_{X_0^*, X_0} \phi(t) dt\right\} \\
&\quad + \mathbb{E}\left\{\xi \int_0^T \langle f(u_n(t)), \phi(t)\bar{e}_i \rangle_{(L^{2p})^*, L^{2p}} dt\right\} + \mathbb{E}\left\{\xi \int_0^T (g_n(t), \phi(t)\bar{e}_i)_{L^2} dt\right\} + \mathbb{E}\left\{\xi \int_0^T (\phi(t)\bar{e}_i, \sigma(t)dW_n(t))\right\} \\
&\triangleq I_1 + I_2 - I_3 + I_4 + I_5 + I_6,
\end{aligned}$$

where $i = 1, 2, \dots, n$. Following the fact that $\phi\bar{e}_i$ is the deterministic function, we will use (12) and the Dominated Convergence Theorem to prove the assertion.

Start with $I_1 = \mathbb{E}\{\xi(u_n(0), \phi(0)\bar{e}_i)\}$. Noting that $u_n(0)$ is the n -dimensional truncation of u_0 , thus we derive

$$u_n(0) \rightarrow u_0 \quad \text{in } L_2^{\mathcal{F}_0}(\Sigma, L^2).$$

Moreover, since there exists $b_1 \in (0, +\infty)$ such that

$$\mathbb{E}\{\xi(u_n(0), \phi(0)\bar{e}_i)\} \leq b_1 \|u_0\|_{L_2^{\mathcal{F}_0}(\Sigma, L^2)}^2,$$

where $b_1 = b_1(\|\xi\|_{L_\infty(\Sigma)}, \phi(0))$, then we have

$$\mathbb{E}\{\xi(u_n(0), \phi(0)\bar{e}_i)\} \rightarrow \mathbb{E}\{\xi(u_0, \phi(0)\bar{e}_i)\}, \quad n \rightarrow \infty.$$

We further consider $I_2 = \mathbb{E}\left\{\xi \int_0^T (u_n(t), \dot{\phi}\bar{e}_i)_{L^2} dt\right\}$. By the Hölder inequality, there exists $b_2 \in (0, +\infty)$ such that

$$\mathbb{E}\left\{\xi \int_0^T (u_n(t), \dot{\phi}\bar{e}_i)_{L^2} dt\right\} \leq \|\xi\|_{L_\infty(\Sigma)} \int_0^T \left(\mathbb{E}\|u_n(t)\|_{L^2}^2\right)^{\frac{1}{2}} \left(\mathbb{E}\|\dot{\phi}(t)\bar{e}_i\|_{L^2}^2\right)^{\frac{1}{2}} dt \leq b_2 \|u_n\|_{L_\infty^{\mathcal{F}}([0, T], L^2)},$$

where $b_2 = b_2(\|\xi\|_{L_\infty(\Sigma)}, \|\dot{\phi}\|_{L^1([0, T])})$. Combining (12) with the Dominated Convergence Theorem, we get

$$\mathbb{E}\left\{\xi \int_0^T (u_n(t), \dot{\phi}\bar{e}_i)_{L^2} dt\right\} \rightarrow \mathbb{E}\left\{\xi \int_0^T (\tilde{u}(t), \dot{\phi}\bar{e}_i)_{L^2} dt\right\}, \quad n \rightarrow \infty.$$

Consider $I_3 = \mathbb{E}\left\{\xi \int_0^T \langle (-\Delta)^\alpha u_n(t), \bar{e}_i \rangle_{X_0^*, X_0} \phi(t) dt\right\}$. Recalling that $p \in [1, 2_a^*/2)$ and $\phi\bar{e}_i \in L^2([0, T], X_0)$, we use Lemma 2.2 and (12) to obtain

$$\begin{aligned}
\int_0^T \langle (-\Delta)^\alpha u_n(t), \bar{e}_i \rangle_{X_0^*, X_0} \phi(t) dt &= \frac{C(d, \alpha)}{2} \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u_n^i(t, x) - u_n^i(t, y))(\phi(t)\bar{e}_i(x) - \phi(t)\bar{e}_i(y))}{|x - y|^{d+2\alpha}} dx dy dt \\
&\rightarrow \frac{C(d, \alpha)}{2} \int_0^T \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(\tilde{u}^i(t, x) - \tilde{u}^i(t, y))(\phi(t)\bar{e}_i(x) - \phi(t)\bar{e}_i(y))}{|x - y|^{d+2\alpha}} dx dy dt.
\end{aligned}$$

Therefore, there exists $b_3 \in (0, +\infty)$ such that

$$\mathbb{E}\left\{\xi \int_0^T \langle (-\Delta)^\alpha u_n(t), \bar{e}_i \rangle_{X_0^*, X_0} \phi(t) dt\right\} \leq b_3 \|\xi\|_{L_\infty(\Sigma)}.$$

Thus, we conclude that

$$\mathbb{E}\left\{\xi \int_0^T \langle (-\Delta)^\alpha u_n(t), \bar{e}_i \rangle_{X_0^*, X_0} \phi(t) dt\right\} \rightarrow \mathbb{E}\left\{\xi \int_0^T \langle (-\Delta)^\alpha \tilde{u}(t), \bar{e}_i \rangle_{X_0^*, X_0} \phi(t) dt\right\}.$$

We consider the fourth term I_4 . Since $\{u_n\}$ is bounded in $L^{\mathcal{F}}_{2p}([0, T], L^{2p})$, Lemma 3.1 implies that

$$\mathbb{E} \left\{ \xi \int_0^T \langle f(u_n(t)), \phi(t) \bar{e}_i \rangle_{(L^{2p})^*, L^{2p}} dt \right\} \rightarrow \mathbb{E} \left\{ \xi \int_0^T \langle f(\tilde{u}(t)), \phi(t) \bar{e}_i \rangle_{(L^{2p})^*, L^{2p}} dt \right\},$$

as $n \rightarrow \infty$.

We further consider $I_5 = \mathbb{E} \left\{ \xi \int_0^T (g_n(t), \phi(t) \bar{e}_i)_{L^2} dt \right\}$. As g_n is the n -dimensional truncation of g , it is clear that

$$g_n \rightarrow g \text{ in } L^{\mathcal{F}}_2([0, T], L^2).$$

Using the Dominated Convergence Theorem, we get

$$\mathbb{E} \left\{ \xi \int_0^T (g_n(t), \phi(t) \bar{e}_i)_{L^2} dt \right\} \rightarrow \mathbb{E} \left\{ \xi \int_0^T (g(t), \phi(t) \bar{e}_i)_{L^2} dt \right\}.$$

For $I_6 = \mathbb{E} \left\{ \xi \int_0^T (\phi(t) \bar{e}_i, \sigma(t) dW_n(t)) \right\}$, the fact that W_n is the n -dimensional truncation of W leads to

$$W_n \rightarrow W \text{ in } Y.$$

From assumption (H4), we deduce

$$\mathbb{E} \left\{ \xi \int_0^T (\phi(t) \sigma^*(t) \bar{e}_i, dW_n(t)) \right\} \rightarrow \mathbb{E} \left\{ \xi \int_0^T (\phi(t) \sigma^*(t) \bar{e}_i, dW(t)) \right\}.$$

To sum up, because $\{\bar{e}_i\}$ is a basis of X_0 , for any $v \in X_0$, $\phi \in C_T^1([0, T])$ and $\xi \in L_{\infty}(\Sigma)$, we get

$$\begin{aligned} & -\mathbb{E} \{ \xi \langle \tilde{u}_0, \phi(0)v \rangle \} - \mathbb{E} \left\{ \xi \int_0^T (\tilde{u}(t), \dot{\phi}v)_{L^2} dt \right\} + \mathbb{E} \left\{ \xi \int_0^T \langle (-\Delta)^{\alpha} \tilde{u}(t), v \rangle_{X_0^*, X_0} \phi(t) dt \right\} \\ & = \mathbb{E} \left\{ \xi \int_0^T \langle f(\tilde{u}(t)), \phi(t)v \rangle_{(L^{2p})^*, L^{2p}} dt \right\} + \mathbb{E} \left\{ \xi \int_0^T (g(t), \phi(t)v)_{L^2} dt \right\} + \mathbb{E} \left\{ \xi \int_0^T (\phi(t) \sigma^*(t)v, dW(t)) \right\}. \end{aligned}$$

In other words, $\tilde{u}$ satisfies formula (5). □

4 The existence of optimal controls

Let $\tilde{U}$ be a real Hilbert space, $\mathcal{U} = L^{\mathcal{F}}_{\infty}([0, T], \tilde{U})$ be a control functional space and $\mathcal{A}$ be a bounded linear operator from $\tilde{U}$ to L^2 . We consider the following stochastic control problem:

$$\begin{cases} du + (-\Delta)^{\alpha} u dt = f(u) dt + g dt + \mathcal{A} v dt + \sigma dW, & t \in (0, T], \\ u(0) = u_0, \end{cases} \quad (13)$$

where $u_0 \in L^{\mathcal{F}_0}_2(\Sigma, L^2)$, $g \in L^{\mathcal{F}}_2([0, T], L^2)$, $v \in \mathcal{U}$ and $\mathcal{A}v \in L^{\mathcal{F}}_2([0, T], L^2)$. We know that there exists unique weak solution of (13) by Theorem 3.2. So we are allowed to define the solution map as follows:

$$\begin{aligned} \Psi: \mathcal{U} & \rightarrow L^{\mathcal{F}}_{\infty}([0, T]; L^2) \cap L^{\mathcal{F}}_2([0, T]; X_0) \cap L^{\mathcal{F}}_{2p}([0, T]; L^{2p}); \\ & v \mapsto u(v), \end{aligned}$$

where $u(v)$ is the state of control problem (13). The observed state is denoted by $z(v) = \mathcal{N}u(v)$ and the desired state is denoted by $z_d \in L^{\mathcal{F}}_2([0, T], L^2)$. We consider the cost function with the following form:

$$J(v) = \mathbb{E} \left\{ \int_0^T \|\mathcal{N}u(v) - z_d\|_{L^2}^2 dt + (\mathcal{K}v, v)_{\tilde{U}} \right\}, \quad (14)$$

where $\mathcal{N} \in \mathcal{L}(L_2^{\mathcal{F}}([0, T], X_0); L^2)$ and $\mathcal{K} \in \mathcal{L}(\tilde{U}, \tilde{U})$, the operator $\mathcal{K}$ satisfies

$$(\mathcal{K}v, v)_{\tilde{U}} = (v, \mathcal{K}v)_{\tilde{U}} \geq k\|v(t)\|_{\tilde{U}}^2,$$

where $k \in [0, +\infty)$ being a constant. Let $U_{\text{ad}} \subset \mathcal{U}$ be an admissible set. We call $v_0 \in U_{\text{ad}}$ be the optimal control of $J(v)$ if

$$J(v_0) = \min_{v \in U_{\text{ad}}} J(v).$$

Thus, we have the following result.

Theorem 4.1. *Suppose all assumptions in Theorem 3.2 hold and U_{ad} is a compact subset of $\mathcal{U}$, then stochastic control problem (13) with (14) has at least one optimal control $v_0 \in U_{\text{ad}}$.*

Proof. Since U_{ad} is compact, it suffices to prove that Ψ is continuous and J is lower semicontinuous. Let $\{v_k\} \subset U_{\text{ad}}$ and

$$v_k \rightarrow \bar{v} \text{ in } U_{\text{ad}}.$$

Let $\{u_k\}$, $\bar{u}$ denote the corresponding weak solutions of problem (13) with the same initial state u_0 and g . The proof is divided into two steps.

Step 1. Ψ is continuous.

Since $\{u_k\}$ and $\bar{u}$ are weak solutions of (13), $u_k - \bar{u}$ satisfies

$$du_k - d\bar{u} + (-\Delta)^\alpha u_k - (-\Delta)^\alpha \bar{u} dt = f(u_k) dt - f(\bar{u}) dt + \mathcal{A}v_k dt - \mathcal{A}\bar{v} dt$$

in the weak sense. From integrating by parts, (H3) and Hölder inequality, we get

$$\begin{aligned} & \mathbb{E}\|u_k - \bar{u}\|_{L^2}^2 + C(d, \alpha) \int_0^t \mathbb{E}\|u_k - \bar{u}\|_{X_0}^2 ds \\ & \leq 2 \mathbb{E} \int_0^t \langle f(u_k) - f(\bar{u}), u_k - \bar{u} \rangle_{(L^{2p})^*, L^{2p}} ds + 2 \mathbb{E} \int_0^t (\mathcal{A}v_k - \mathcal{A}\bar{v}, u_k - \bar{u})_{L^2} ds \\ & \leq 2 \mathbb{E} \int_0^t (\mathcal{A}v_k - \mathcal{A}\bar{v}, u_k - \bar{u})_{L^2} ds \\ & \leq 2 \left(\mathbb{E} \int_0^t \|\mathcal{A}(v_k - \bar{v})\|_{L^2}^{\frac{2p}{p-1}} ds \right)^{\frac{2p-1}{2p}} \left(\mathbb{E} \int_0^t \|u_k - \bar{u}\|_{L^2}^{2p} ds \right)^{\frac{1}{2p}}. \end{aligned} \quad (15)$$

On the other hand, since weak solutions satisfy inequality (11), $\{u_k\}$ is bounded in $L_\infty^{\mathcal{F}}([0, T]; L^2) \cap L_2^{\mathcal{F}}([0, T]; X_0) \cap L_{2p}^{\mathcal{F}}([0, T]; L^{2p})$. Therefore, there exists a constant $M > 0$ such that

$$\max \{\|\bar{u}\|_{L_{2p}^{\mathcal{F}}([0, T]; L^{2p})}, \|u_k\|_{L_{2p}^{\mathcal{F}}([0, T]; L^{2p})}, k \in \mathbb{N}\} \leq M.$$

Thus, from inequality (15) and Lemma 2.2, there exist $c > 0$ such that

$$\mathbb{E}\|u_k - \bar{u}\|_{L^2}^2 + C(d, \alpha) \int_0^t \mathbb{E}\|u_k - \bar{u}\|_{X_0}^2 ds \leq 4Mc \left(\mathbb{E} \int_0^t \|\mathcal{A}(v_k - \bar{v})\|_{L^2}^{\frac{2p}{p-1}} ds \right)^{\frac{2p-1}{2p}}.$$

Taking to the limit, we obtain

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \left\{ \mathbb{E} \|u_k - \bar{u}\|_{L^2}^2 + C(d, \alpha) \int_0^t \mathbb{E} \|u_k - \bar{u}\|_{X_0}^2 ds \right\} = 0.$$

Furthermore, we deduce that

$$u_k \rightarrow \bar{u} \text{ in } L_{\infty}^{\mathcal{F}}([0, T]; L^2) \cap L_2^{\mathcal{F}}([0, T]; X_0).$$

This proves that Ψ is continuous.

Step 2. J is lower semi-continuous.

First, write

$$J(v) = \mathbb{E} \left\{ \int_0^T \|\mathcal{N}u(t, v) - z_d(t)\|_{L^2}^2 dt \right\} + \mathbb{E} \{(\mathcal{K}v(t), v(t))_{\tilde{U}}\} \triangleq J_1(v) + J_2(v)$$

and

$$J_1(\bar{v}) = \mathbb{E} \left\{ \int_0^T \|\mathcal{N}\bar{u}(t, v) - z_d(t)\|_{L^2}^2 - \|\mathcal{N}u_k(t, v) - z_d(t)\|_{L^2}^2 dt \right\} + \mathbb{E} \left\{ \int_0^T \|\mathcal{N}u_k(t, v) - z_d(t)\|_{L^2}^2 dt \right\}.$$

Since $\mathcal{N} \in \mathcal{L}(L_2^{\mathcal{F}}([0, T], X_0), L^2)$, for any $\epsilon > 0$, there exists N_{ϵ} , when $k > N_{\epsilon}$,

$$\|\mathcal{N}\bar{u}(t, v) - z_d(t)\|_{L^2}^2 - \|\mathcal{N}u_k(t, v) - z_d(t)\|_{L^2}^2| < \epsilon.$$

Thus, we get

$$J_1(\bar{v}) \leq T\epsilon + \mathbb{E} \left\{ \int_0^T \|\mathcal{N}u_k(t, v) - z_d(t)\|_{L^2}^2 dt \right\} = T\epsilon + J_1(v_k)$$

if $k > N_{\epsilon}$. So we arrive at

$$J_1(\bar{v}) \leq \lim_{k \rightarrow \infty} J_1(v_k)$$

by the arbitrariness of ϵ .

Finally, according to $v_k \rightarrow \bar{v}$ in $\mathcal{U}$, we deduce $v_k(t) \rightarrow \bar{v}(t)$, a.e., for all $t \in [0, T]$. Hence, $\{v_k(t)\}$ is a bounded subset of $\tilde{U}$. Since $\mathcal{K} \in \mathcal{L}(\tilde{U}, \tilde{U})$,

$$\lim_{k \rightarrow \infty} (\mathcal{K}v_k(t), v_k(t))_{\tilde{U}} \leq (\mathcal{K}\bar{v}(t), \bar{v}(t))_{\tilde{U}}, \quad \text{a.s.}$$

Due to the Fatou lemma, we only need to show

$$\lim_{k \rightarrow \infty} \mathbb{E} \{(\mathcal{K}v_k(t), v_k(t))_{\tilde{U}}\} < \infty. \quad (16)$$

Following

$$(\mathcal{K}v_k(t), v_k(t))_{\tilde{U}} \leq \|\mathcal{K}v_k(t)\|_{\tilde{U}} \|v_k(t)\|_{\tilde{U}} \leq \|\mathcal{K}\| \|v_k(t)\|_{\tilde{U}}^2, \quad \text{a.s.,}$$

inequality (16) holds. Thus, we get that

$$J_2(\bar{v}) \leq \lim_{k \rightarrow \infty} J_2(v_k).$$

To sum up, J attains its minimum at $\bar{v}$. □

5 Examples

In this section, we give some sufficient conditions and two examples to illustrate the results. Assume that

(A1): $f \in C(\mathbb{R}^N \times \bar{\Omega} \times [0, T], \mathbb{R}^N)$;

(A2): There exists an $M \in (0, +\infty)$ such that

$$|f(\xi, x, t)| \leq M|\xi|^{2p-1}, \quad \forall (\xi, x, t) \in \mathbb{R}^N \times \Omega \times [0, T];$$

(A3): There exists a $\theta \in (0, +\infty)$ such that

$$f(\xi, x, t)\xi \leq -\theta|\xi|^{2p}, \quad \forall (\xi, x, t) \in \mathbb{R}^N \times \Omega \times [0, T];$$

(A4): For any $\xi, \eta \in \mathbb{R}^N$, $x \in \Omega$ and $t \in [0, T]$,

$$(f(\xi, x, t) - f(\eta, x, t))(\xi - \eta) \leq 0.$$

Thus from (A2) and (A3), we obtain

$$\begin{cases} \langle f(u), u \rangle_{(L^{2p})^*, L^{2p}} \leq -\theta \|u\|_{L^{2p}}^{2p}, \\ \|f(u)\|_{(L^{2p})^*} \leq M \|u\|_{L^{2p}}^{2p-1}, \end{cases}$$

then f satisfies (H1) and (H2). Moreover by (A1) and (A4), we get

$$\langle f(u) - f(v), u - v \rangle_{(L^{2p})^*, L^{2p}} \leq 0,$$

that is, f satisfies (H3). So we deduce the following result.

Corollary 5.1. *Let $\alpha \in (0, 1)$, $2\alpha < d$ and $p \in [1, 2_\alpha^*/2)$. If assumptions (A1)–(A4) and (H4) hold, then system (1) has unique weak solution u for any $g \in L_2^{\mathcal{F}}([0, T], L^2)$ and $u_0 \in L_2^{\mathcal{F}_0}(\Sigma, L^2)$.*

Remark 5.2. Under the assumptions in Corollary 5.1, the result of Section 4 remains valid.

Next, two examples of optimal control problems are provided as follows. These examples are inspired by [12].

Example 5.3. Let $\Omega = B_0$ be a unit ball in $\mathbb{R}^3$ and $\tilde{U} = L^2(B_0)$. We consider the control problem as follows:

$$\begin{cases} du + (-\Delta)^\alpha u dt = -\psi u^{2p-1} dt + v dt + (1+t)^{k_0} dW, & t \in (0, T], x \in B_0, \\ u(t) = 0, & t \in [0, T], x \in \mathbb{R}^3 \setminus B_0, \\ u(0) = a_0 e^{-b_0|x|}, & x \in B_0, \end{cases} \quad (17)$$

where k_0 , a_0 and b_0 are strictly positive constants, $\psi = \psi(x, t) \geq 0$ is a deterministic function and $\psi \in L^\infty([0, T] \times B_0)$. The cost function is

$$J(v) = \mathbb{E} \left\{ \int_0^T \|u(v) - z_d(t)\|_{L^2}^2 dt \right\}.$$

We check that $f = -\psi u^{2p-1}$ satisfies (A1)–(A4), $\sigma = (1+t)^{k_0}$ satisfies (H4) and $u_0 \in L_2^{\mathcal{F}_0}(\Sigma, L^2)$. $\mathcal{A} \in \mathcal{L}(\tilde{U}, L^2)$ is an identity, $\mathcal{N}$ is an injection from X_0 into L^2 and $\mathcal{K}$ is a null operator. Therefore, using Corollary 5.1 and Remark 5.2, we obtain that optimal control problem (17) has at least one optimal control.

Example 5.4. Let $\Omega = B_0$ be a unit ball in $\mathbb{R}^3$, $C = \partial B_0$ and $\tilde{U} = L^2(C)$. Consider the following control problem:

$$\begin{cases} du + (-\Delta)^\alpha u dt = -\gamma_0 |u|^{2p-2} u dt + \mathcal{A} v dt + \frac{1}{1+t} dW, & t \in (0, T], x \in B_0, \\ u(t) = 0, & t \in [0, T], x \in \mathbb{R}^3 \setminus B_0, \\ u(0) = a_0 e^{-b_0 |x|}, & x \in B_0, \end{cases} \quad (18)$$

where γ_0 , a_0 and b_0 are strictly positive constants and $\mathcal{A}$ is given by

$$(\mathcal{A}v, u) = \int_C uv dS.$$

The cost function J is chosen to be

$$J(v) = \mathbb{E} \left\{ \int_0^T \left\| \frac{\partial}{\partial n} u(v) - z_d(t) \right\|_{L^2(C)}^2 dt + \lambda \int_C |v(t)|^2 dS \right\},$$

where λ is a positive parameter.

It is easy to see that $f = -\gamma_0 |u|^{2p-2} u$ satisfies (A1)–(A4) and $\sigma = \frac{1}{1+t}$ satisfies (H4) and $u_0 \in L_2^{\mathcal{F}_0}(\Sigma, L^2)$. By simple calculations, we compute that the observed state $\frac{\partial}{\partial n} u(v) \in L_2^{\mathcal{F}}([0, T], L^2(C))$. So we get the existence of optimal controls for optimal control problem (18) by Corollary 5.1 and Remark 5.2.

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References

- [1] W. X. Chen, *Direct methods on fractional equations*, Discrete Contin. Dyn. Syst. **39** (2019), no. 3, 1269–1310.
- [2] S. Dipierro, O. Savin, and E. Valdinoci, *All functions are locally s -harmonic up to a small error*, J. Eur. Math. Soc. **19** (2017), no. 4, 957–966.
- [3] G. Devillanova and G. Carlo Marano, *A free fractional viscous oscillator as a forced standard damped vibration*, Fract. Calc. Appl. Anal. **19** (2016), no. 2, 319–356.
- [4] G. Molica Bisci and V. D. Rădulescu, *Ground state solutions of scalar field fractional for Schrödinger equations*, Calc. Var. Partial Differ. Equ. **54** (2015), no. 3, 2985–3008.
- [5] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$* , J. Differential Equations **255** (2013), no. 8, 2340–2362.
- [6] X. J. Chang and Z. Q. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity*, Nonlinearity **26** (2013), no. 2, 479–494.
- [7] N. U. Ahmed, *Weak solutions of stochastic reaction diffusion equations and their optimal control*, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), no. 6, 1011–1029.
- [8] N. Durga and P. Muthukumar, *Optimal control of fractional reaction diffusion equations with Poisson jumps*, J. Anal. **27** (2019), no. 2, 605–621.
- [9] P. Bezdek, *Existence and blow-up of solutions to the fractional stochastic heat equations*, Stoch. Partial Differ. Equ. Anal. Comput. **6** (2018), 73–108.
- [10] B. X. Wang, *Asymptotic behavior of non-autonomous fractional stochastic reaction diffusions equations*, Nonlinear Anal. **158** (2017), 60–82.
- [11] D. S. Li, B. X. Wang, and X. H. Wang, *Random dynamics of fractional stochastic reaction diffusion equations on $\mathbb{R}^n$ without uniqueness*, J. Math. Phys. **60** (2019), 072704, DOI: 10.1063/1.5063840.
- [12] J. L. Lions, *Some aspects of the optimal control of distributed parameter system*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1972.
- [13] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York, 1971.
- [14] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang, *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach*, Springer, New York, 2nd ed., 2010.

- [15] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.
- [16] G. Molica Bisci, V. D. Rădulescu, and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Cambridge University Press, Cambridge, 2016.
- [17] E. Di Nezza, P. Giampiero, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.
- [18] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), no. 5, 2105–2137.
- [19] R. Servadei and E. Valdinoci, *A Brézis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal. **12** (2013), no. 6, 2445–2464.
- [20] N. U. Ahmed, *Optimization and identification of systems governed by evolution equations on Banach space*, Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Harlow; co-published in the United States with John Wiley and Sons, Inc., New York, 1988.