Research Article

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Gradient estimates for a weighted nonlinear parabolic equation and applications

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Abstract: This paper derives elliptic gradient estimates for positive solutions to a nonlinear parabolic equation defined on a complete weighted Riemannian manifold. Applications of these estimates yield Liouville-type theorem, parabolic Harnack inequalities and bounds on weighted heat kernel on the lower boundedness assumption for Bakry-Émery curvature tensor.

Keywords: gradient estimates, Bakry-Émery tensor, parabolic equation, Liouville theorem, heat kernel, Harnack inequality

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1 Preliminaries

1.1 Introduction

We shall consider positive solutions to the nonlinear parabolic equation:

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)u(x, t) + p(x, t)u^\beta(x, t) + q(x, t)u(x, t) = 0, \quad (1.1)$$

on a weighted Riemannian manifold $(M^N, g, e^{-f}dv)$, otherwise known as a smooth metric measure space. Here $\beta \in \mathbb{R}$ and $p(x, t)$ and $q(x, t)$ are smooth functions at least C^1 in x and C^0 in t . Suppose $p(x, t)$ and $q(x, t)$ are zero, then (1.1) is the heat equation:

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)u(x, t) = 0, \quad (1.2)$$

whose minimal positive solution is known as the weighted heat kernel. The major aim of this paper is to establish improved elliptic gradient estimates of elliptic type on smooth solutions and then derive Liouville-type theorems, Harnack inequalities and heat kernel estimates, as applications of the obtained gradient estimates.

In recent years, gradient estimates for both elliptic and parabolic equations have become fundamental tools in geometric analysis. In their celebrated work [1], Li and Yau established parabolic gradient estimates on solutions to the linear heat equation on Riemannian manifolds having Ricci curvature bounded from below. They applied their results to get Harnack inequalities and various estimates on the heat kernel.

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Since then there has been a lot of work to improve, extend or generalize these results, for instance, see [2–12] and references therein. Historically, the technique of gradient estimates emanated from the study by Yau [13] (see also [14,15]), in which a gradient estimate for harmonic functions was first established using the maximum principle. This estimate was applied to obtain a Liouville theorem. A Liouville theorem says that a bounded positive solution to the harmonic equation is constant. Global elliptic gradient estimate was derived by Hamilton [9] for the heat equation on a closed manifold. Localized elliptic-type estimate was proved by Souplet and Zhang [11] for the heat equation by adding a logarithmic correction term. In the same spirit, we deal with local gradient estimates on weighted manifolds with Bakry-Émery tensor bounded from below. Thus, it is in order to give background information about this space.

1.2 Basics of weighted Riemannian manifold

Let (M^N, g) be an N -dimensional complete Riemannian manifold, dv volume element of M and f a smooth function on M . The triple $(M^N, g, e^{-f}dv)$ is called a weighted Riemannian manifold (smooth metric measure space). The space is endowed with weighted Ricci tensor and weighted Laplacian.

The m -weighted Ricci tensor otherwise known as the Bakry-Émery tensor in the literature is defined for some constant $m > 0$ as

$$\text{Ric}_f^m = \text{Ric}_f - \frac{1}{m} \nabla f \otimes \nabla f,$$

where $\text{Ric}_f = \text{Ric} + \nabla^2 f$, Ric is the Ricci tensor of (M, g) and ∇^2 is the Hessian operator with respect to g . Clearly, $\text{Ric}_f^m = \text{Ric}_f$ as $m \rightarrow \infty$. Note that the ∞ -Bakry-Émery tensor defines special solutions to the Hamilton Ricci flow [16]. These solutions are called gradient Ricci solitons $\text{Ric}_f = \lambda g$, for $\lambda \in \mathbb{R}$. For a detailed discussion on Ricci solitons, interested readers can see [17] for a survey on Ricci solitons.

The weighted Laplacian is defined as $\Delta_f := \Delta - \langle \nabla f, \nabla \cdot \rangle$, where Δ and ∇ are the usual Laplace-Beltrami and gradient operators, respectively.

The weighted Bochner formula for any $u \in C^\infty(M)$,

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla \Delta_f u, \nabla u \rangle + \text{Ric}_f(\nabla u, \nabla u) \quad (1.3)$$

gives a relation between the weighted Laplacian and weighted Ricci tensor.

The heat kernel on weighted manifold is the minimal positive solution to the heat equation (1.2) denoted as $H(x, y, t) = u(x, t)$, for each $y \in M$, satisfying the initial condition $\lim_{t \rightarrow 0} H(x, y, t) = \delta_{f,y}(x)$, where $\delta_{f,y}(x)$ is the weighted delta function given by

$$\int_M \psi(x) \delta_{f,y}(x) e^{-f} dv = \psi(y) \quad \text{for } \psi \in C_0^\infty(M).$$

Various Liouville-type theorems have been obtained for harmonic functions on weighted manifold, see for instance [18–22] and Cao and Zhou [23] and Wu and Wu [24,25] for related results. Specifically, we mention that Brighton [26] derived some elliptic-type gradient estimates for weighted harmonic functions when $\text{Ric}_f \geq 0$, using a power of u rather than $\log u$ in Yau [15]. Brighton's approach was refined by Munteanu and Wang [27] to show that positive weighted harmonic function of sub-exponential growth must be constant on the condition $\text{Ric}_f \geq 0$.

1.3 Motivations

The present work is an extension of results in [28], in which it was stated that equations of the form (1.1) arise in geometry and physics. For instance, the static form of (1.1) is equivalent to the weighted Yamabe

equation [29]. Hopefully, elliptic gradient estimates for Eq. (1.1) can be applied to get information that will be useful in solving the Yamabe problem on weighted manifolds [30].

Another motivation is coming from physical applications. Suppose f is a constant function on M and take either $p(x, t)$ or $q(x, t)$ to be zero, we have in particular

$$\left(\Delta - \frac{\partial}{\partial t}\right)u(x, t) + p(x, t)u^\beta(x, t) = 0, \quad (1.4)$$

which is a simple ecological model for population dynamics, where $u(x, t)$ is the population density at time t .

Recently, Zhu [31] derived local elliptic-type gradient estimates for (1.4) on (M^N, g) with Ricci tensor bounded below, where $\beta > 0$. He used his estimates to obtain some Liouville-type theorems for positive solutions, thereby generalizing result [11] according to the study by Souplet and Zhang. The present author in [28] has extended Zhu's results to weighted manifold for an arbitrary constant $\beta \in \mathbb{R}$. Later, Wu [30] obtained similar estimates on weighted manifolds and used those estimates to determine sufficient conditions on parameter which guarantee nonexistence of positive solutions to some cases of (1.1) viz a viz existence of Yamabe-type problems on weighted manifolds.

According to [27,32], one encounters some sort of difficulties in obtaining estimates of Li-Yau type for positive solutions to (1.2) when Ric_f is bounded below, even when growth condition is imposed on f . However, Wu [22] has recently proved local and global elliptic gradient estimates for positive solutions to (1.2) without any assumption on f . Thereby, extending Souplet-Zhang's result [11] to (1.2) with only Ric_f bounded from below. At present, therefore, we combine the approaches in [28] and [22,30] to derive elliptic gradient estimates for positive solutions of (1.1). Here, the estimates obtained do not require any condition on f .

1.4 Main result

A point x_0 is fixed on $(M^N, g, e^{-f}dv)$ and the distance function from x_0 to x with respect to g is denoted by $r(x)$ or $d(x, x_0)$. Define the space $Q_{R,T}$ by

$$Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty),$$

where $B(x_0, R)$ is the geodesic ball centered at x_0 with $R > 0$, $t_0 \in \mathbb{R}$ and $T > 0$. Denote by $\|\cdot\| = \sup_{Q_{R,T}}|\cdot|$, the norm with respect to g and $\mathbb{M} = \inf\{u(x, t) | (x, t) \in M \times [0, \infty)\}$. The main result concerning the localized elliptic gradient estimates is stated below.

Theorem 1.1. *Let $(M^N, g, e^{-f}dv)$ be an N -dimensional weighted Riemannian manifold with $\text{Ric}_f \geq -(N-1)K$, $K \geq 0$. Suppose that $u(x, t) \leq D$ is a positive solution to (1.1) in $Q_{R,T}$, $T > 0$, $R \geq 2$ for some constant D . Then there exists a constant $C(N)$ such that*

(1) *If $\beta \geq 1$*

$$\begin{aligned} \frac{|\nabla u|}{u} \leq C(N) & \left(\sqrt{\frac{1+|\mu|}{R}} + \frac{1}{\sqrt{t-(t_0-T)}} + \sqrt{K} + \|q^+\|^{1/2} + \|\nabla q\|^{1/3} \right. \\ & \left. + \sqrt{\beta} D^{\frac{1}{2}(\beta-1)} \|p^+\|^{1/2} + D^{\frac{1}{3}(\beta-1)} \|\nabla p\|^{1/3} \right) \left(1 + \ln \frac{D}{u} \right). \end{aligned} \quad (1.5)$$

(2) *If $\beta < 1$*

$$\begin{aligned} \frac{|\nabla u|}{u} \leq C(N) & \left(\sqrt{\frac{1+|\mu|}{R}} + \frac{1}{\sqrt{t-(t_0-T)}} + \sqrt{K} + \|q^+\|^{1/2} + \|\nabla q\|^{1/3} \right. \\ & \left. + M^{\frac{1}{2}(\beta-1)} (\|(\beta-1)p^+\| + \|p^+\|)^{\frac{1}{2}} + M^{\frac{1}{3}(\beta-1)} \|\nabla p\|^{1/3} \right) \left(1 + \ln \frac{D}{u} \right) \end{aligned} \quad (1.6)$$

for all $(x, t) \in Q_{R/2,T}$ with $t \neq 0$, where $p^+(x, t) = \max\{p(x, t), 0\}$, $q^+ = \max\{q(x, t), 0\}$ and $\mu := \max_{\{x | d(x, x_0)=1\}} \Delta_f r(x)$.

Remark 1.2. Note that if $q \equiv 0$ estimates (1.5) and (1.6) are equal to the estimates in Theorem 1.1 of [28]. Note also that (1.6) includes (1.5). It is therefore obvious that Theorem 1.1 is a generalization of Theorem 1.1 in [28]. See [33] for similar results when p and q are some constants.

The aforementioned estimates have numerous applications. We only mention three of them here: namely, we derive Harnack-type inequalities in Section 4.1, Liouville-type theorems in Section 4.2 and weighted Heat kernel bound in Section 5. In Section 2, we prove a lemma which is fundamental to the proof of main theorem and then introduce a cutoff function depending on space and time. We then apply the lemma and properties of the cutoff function to prove Theorem 1.1 in Section 3.

2 Basic lemma

Consider (1.1) where p and q are continuous differentiable functions and $\beta \in \mathbb{R}$. Introducing a new function $h = \ln u/D$ and assuming $0 < u \leq D$, then $h \leq 0$ satisfies

$$h_t := \frac{\partial h}{\partial t} = \Delta_f h + |\nabla h|^2 + p(x)(De^h)^{\beta-1} + q. \quad (2.1)$$

The following Lemma is key to the proof of Theorem 1.1.

Lemma 2.1. *Let $(M^N, g, e^{-f}dv)$ be an N -dimensional complete weighted manifold with $\text{Ric}_f \geq -(N-1)K$, $K \geq 0$. If $h = h(x, t)$ smoothly satisfies (2.1) in $Q_{R,T}$, $R > 0$, then the function $G = |\nabla \ln(1-h)|^2$ satisfies*

$$\begin{aligned} \left(\Delta_f - \frac{\partial}{\partial t} \right) G &\geq \frac{2h}{1-h} \langle \nabla h, \nabla G \rangle + 2(1-h)G^2 - 2(N-1)KG - \frac{2}{1-h}qG \\ &\quad - 2 \left[\beta + \frac{h}{1-h} \right] (De^h)^{\beta-1} pG - \frac{2(De^h)^{\beta-1}}{(1-h)^2} \langle \nabla h, \nabla p \rangle - \frac{2}{(1-h)^2} \langle \nabla h, \nabla q \rangle \end{aligned} \quad (2.2)$$

for all (x, t) in $Q_{R,T}$.

Proof. By the weighted Bochner formula (1.3) we have

$$\Delta_f G = 2|\nabla^2 \ln(1-h)|^2 + 2\langle \nabla \Delta_f \ln(1-h), \nabla \ln(1-h) \rangle + 2\text{Ric}_f(\nabla \ln(1-h), \nabla \ln(1-h)).$$

Using the fact that $\text{trace}(\nabla^2 u) = \Delta u = \Delta_f u + \langle \nabla f, \nabla u \rangle$ and some elementary inequalities, one can obtain [19]

$$\Delta_f G \geq \frac{2}{m+N} (\Delta_f \ln(1-h))^2 + 2\langle \nabla \Delta_f \ln(1-h), \nabla \ln(1-h) \rangle + 2\text{Ric}_f^m(\nabla \ln(1-h), \nabla \ln(1-h)).$$

Since $\text{Ric}_f^m = \text{Ric}_f$ as $m \rightarrow \infty$ and $\text{Ric}_f \geq -(N-1)K$ by the hypothesis, m can be sent to ∞ and

$$\Delta_f G \geq 2\langle \nabla \Delta_f \ln(1-h), \nabla \ln(1-h) \rangle - 2(N-1)KG. \quad (2.3)$$

By direct computation, we have

$$\begin{aligned} \Delta_f \ln(1-h) &= (\ln(1-h))_{jj} - f_j (\ln(1-h))_j \\ &= -\frac{h_{jj}}{1-h} - \frac{h_j h_j}{(1-h)^2} + \frac{f_j h_j}{1-h} = -\frac{\Delta_f h}{1-h} - G \\ &= \frac{|\nabla h|^2 + p(De^h)^{\beta-1} + q - h_t}{1-h} - G \\ &= (1-h)G + \frac{p(De^h)^{\beta-1} + q}{1-h} - \frac{h_t}{1-h} - G \\ &= \frac{p(De^h)^{\beta-1} + q}{1-h} + (\ln(1-h))_t - hG. \end{aligned}$$

Then

$$\begin{aligned}
 2\langle \nabla \Delta_f \ln(1-h), \nabla \ln(1-h) \rangle &= 2 \left\langle \nabla \left(\frac{p(De^h)^{\beta-1} + q}{1-h} + (\ln(1-h))_t - hG \right), \nabla \ln(1-h) \right\rangle \\
 &= \left\langle \frac{2(De^h)^{\beta-1}}{1-h} \nabla p + \frac{2(\beta-1)p(De^h)^{\beta-1}}{1-h} \nabla h + \frac{2}{1-h} \nabla q + \frac{2(p(De^h)^{\beta-1} + q)}{(1-h)^2} \nabla h \right. \\
 &\quad \left. + 2(\nabla \ln(1-h))_t - 2\nabla hG - 2h\nabla G, \nabla \ln(1-h) \right\rangle \\
 &= -\frac{2(De^h)^{\beta-1}}{(1-h)^2} \langle \nabla p, \nabla h \rangle - 2(\beta-1)(De^h)^{\beta-1} p \frac{|\nabla h|^2}{(1-h)^2} - \frac{2}{(1-h)^2} \langle \nabla q, \nabla h \rangle \\
 &\quad - \frac{2(p(De^h)^{\beta-1} + q)}{(1-h)} \frac{|\nabla h|^2}{(1-h)^2} + G_t + \frac{2|\nabla h|^2}{1-h} G + \frac{2h}{1-h} \langle \nabla h, \nabla G \rangle \\
 &= -\frac{2(De^h)^{\beta-1}}{(1-h)^2} \langle \nabla p, \nabla h \rangle - 2(\beta-1)(De^h)^{\beta-1} pG - \frac{2}{(1-h)^2} \langle \nabla q, \nabla h \rangle \\
 &\quad - \frac{2(De^h)^{\beta-1}}{1-h} pG - \frac{2}{(1-h)} qG + G_t + 2(1-h)G^2 + \frac{2h}{1-h} \langle \nabla h, \nabla G \rangle.
 \end{aligned}$$

Substituting this into (2.3) we obtain (2.2) at once, which completes the proof. $\square$

The last lemma and some localization techniques introduced in [11] will be applied to prove Theorem 1.1. As in [11,22,28], we use a cutoff function satisfying the following properties [28, Lemma 2.2]. The properties will be used to obtain the required bounds in $Q_{R,T}$.

Lemma 2.2. *For any given $\tau \in (t_0 - T, t_0]$, $t_0 \in \mathbb{R}$ is fixed and $T > 0$, there exists a smooth function $\psi : [0, \infty) \times [t_0 - T, t_0] \rightarrow \mathbb{R}$ with the following properties:*

- (i) $\psi = \psi(r, t) = 1$ in $Q_{R/2, T/2}$ and $\psi \in [0, 1]$;
- (ii) ψ is radially decreasing and $\frac{d\psi}{dr} = 0$ in $Q_{R/2, T}$;
- (iii) $\left| \frac{\partial \psi}{\partial r} \right| \frac{1}{\psi^a} \leq \frac{C_a}{R}$ and $\left| \frac{\partial^2 \psi}{\partial r^2} \right| \frac{1}{\psi^a} \leq \frac{C_a}{R^2}$ in $[0, \infty) \times [t_0 - T, t_0]$, where $0 < a < 1$;
- (iv) $\left| \frac{\partial \psi}{\partial t} \right| \frac{1}{\psi^{1/2}} \leq \frac{C}{\tau - (t_0 - T)}$ in $[0, \infty) \times [t_0 - T, t_0]$, $C > 0$ is a constant and $\psi(r, t_0 - T) = 0$, $\forall r \in [0, \infty)$.

In the next section, we use Lemma 2.1 and properties of ψ together with the maximum principle in a local space-time supported set. We follow the usual methodology applied in [28].

3 Gradient estimates

Proof of Theorem 1.1. The idea of the proof is to pick a number $\tau \in (t_0 - T, t_0]$ and define a smooth function $\psi : M \times [t_0 - T, t_0] \rightarrow \mathbb{R}$ to be a cut-function with support in $Q_{R,T}$ and satisfies properties of (i)–(iv) stated in Lemma 2.2. We want to obtain some estimates and do some analysis at a space-time point where ψG reaches maximum. We shall show that inequalities (1.5) and (1.6) hold at the point (x, τ) , $x \in M$ such that $d(x, x_0) < R/2$ and then obtain the conclusion of the theorem at once since τ is arbitrarily chosen.

A straightforward computation yields

$$\left(\Delta_f - \frac{\partial}{\partial t} \right) (\psi G) = \psi \left(\Delta_f - \frac{\partial}{\partial t} \right) G + 2\nabla \psi \nabla G + G \left(\Delta_f - \frac{\partial}{\partial t} \right) \psi. \quad (3.1)$$

Now substituting (2.2) into (3.1) and rearranging give

$$\begin{aligned} & \left(\Delta_f - \frac{\partial}{\partial t} \right) (\psi G) - \left(d + 2 \frac{\nabla \psi}{\psi} \right) \nabla (\psi G) \\ & \geq 2\psi(1-h)G^2 - (d \cdot \nabla \psi)G - 2 \frac{|\nabla \psi|^2}{\psi} G - 2(N-1)K\psi G - \frac{2}{(1-h)^2} (De^h)^{\beta-1} \psi \langle \nabla p, \nabla h \rangle \\ & \quad - 2 \left(\beta + \frac{h}{1-h} \right) (De^h)^{\alpha-1} p \psi G + G \left(\Delta_f - \frac{\partial}{\partial t} \right) \psi - \frac{2}{(1-h)^2} \psi \langle \nabla q, \nabla h \rangle - \frac{2}{1-h} q \psi G, \end{aligned} \quad (3.2)$$

where the following identity has been used

$$(d \cdot \nabla G) \psi + 2 \nabla \psi \nabla G = \left(d + 2 \frac{\nabla \psi}{\psi} \right) \nabla (\psi G) - (d \cdot \nabla \psi) G - 2 \frac{|\nabla \psi|^2}{\psi} G$$

with $d := \frac{2h}{1-h} \nabla h$.

We suppose ψG attains its maximum at the point (x_1, t_1) in $Q_{R,T}$. By Calabi's argument [1], we assume that x_1 is not in the cut locus of M . Then $(\psi G)(x_1, t_1)$ is assumed to be positive, otherwise $G \leq 0$ and the result then holds trivially whenever $d(x, x_0) < R/2$. Then at the point (x_1, t_1) we have

$$\Delta_f(\psi G) \leq 0, \quad (\psi G)_t \geq 0 \quad \text{and} \quad \nabla(\psi G) = 0.$$

By the aforementioned estimate at (x_1, t_1) , (3.1) simplifies as

$$\begin{aligned} 2\psi(1-h)G^2 & \leq \left(d \cdot \nabla \psi + 2 \frac{|\nabla \psi|^2}{\psi} \right) G - (\Delta_f \psi) G + \psi_t G + 2(N-1)K\psi G + \frac{2}{(1-h)^2} \psi \langle \nabla q, \nabla h \rangle \\ & \quad + \frac{2}{1-h} q \psi G + \frac{2}{(1-h)^2} (De^h)^{\beta-1} \psi \langle \nabla p, \nabla h \rangle + 2 \left(\beta + \frac{h}{1-h} \right) (De^h)^{\beta-1} p \psi G. \end{aligned} \quad (3.3)$$

Now, consider the case where $x \notin B(x_0, 1)$. First, we obtain upper bounds on each term at the right hand side of (3.3) at (x_1, t_1) and then do the analysis. Let $C = C(N) > 0$ be a constant whose values may vary from line to line. With repeated use of Young's inequality as in [11,28] and the fact that $1-h \geq 1$ we have

$$\begin{aligned} |(d \cdot \nabla \psi) G| & \leq \frac{2|h|}{1-h} |\nabla h| |\nabla \psi| G \leq 2|h| |\nabla \psi| G^{3/2} \\ & \leq \psi(1-h)G^2 + C \left(\frac{h|\nabla \psi|}{[\psi(1-h)]^{3/4}} \right)^4 \leq \psi(1-h)G^2 + \frac{C}{R^4} \frac{h^4}{(1-h)^3}. \end{aligned} \quad (3.4)$$

$$\frac{2|\nabla \psi|^2}{\psi} G = 2\psi^{1/2} G \cdot \frac{|\nabla \psi|^2}{\psi^{3/2}} \leq \frac{1}{10} (\psi^{1/2} G)^2 + C(\delta) \left(\frac{|\nabla \psi|^2}{\psi^{3/2}} \right)^2 \leq \frac{1}{10} \psi G^2 + \frac{C}{R^4}. \quad (3.5)$$

Using properties (i) and (iv) (Lemma 2.2) of ψ and the comparison theorem for weighted Laplacian [20] $\Delta_f r(x) \leq \mu + (N-1)K(R-1)$, with $\mu := \max_{\{x|d(x,x_0)=1\}} \Delta r(x)$, the third term on the right hand side of (3.3) implies (cf. (3.8) of [28])

$$-(\Delta_f \psi) G \leq \frac{1}{10} \psi G^2 + \frac{C}{R^4} + \frac{C\mu^2}{R^2} + CK^2. \quad (3.6)$$

Using similar arguments to those in [28], the 4th, 5th, 6th and 7th terms of (3.3) are, respectively, estimated as follows:

$$\psi_t G \leq \psi^{1/2} G \frac{|\psi_t|}{\psi^{1/2}} \leq \frac{1}{10} \psi G^2 + C \left(\frac{|\psi_t|}{\psi^{1/2}} \right)^2 \leq \frac{1}{10} \psi G^2 + \frac{C}{(\tau - (t_0 - T))^2}, \quad (3.7)$$

$$2(N-1)K\psi G \leq \frac{1}{10} (\psi^{1/2} G)^2 + C((N-1)K\psi^{1/2})^2 \leq \frac{1}{10} \psi G^2 + CK^2, \quad (3.8)$$

$$\begin{aligned} \frac{2}{(1-h)^2} \psi \langle \nabla q, \nabla h \rangle &\leq \frac{2}{(1-h)^2} \psi |\nabla q| |\nabla h| = \frac{2}{(1-h)} \psi |\nabla q| G \\ &\leq \frac{1}{10} \psi G^2 + C \left(\psi^{3/4} \frac{|\nabla q|}{1-h} \right)^{4/3} \leq \frac{1}{10} \psi G^2 + \frac{C |\nabla q|^{4/3}}{(1-h)^{4/3}} \leq \frac{1}{10} \psi G^2 + C |\nabla q|^{4/3} \end{aligned} \quad (3.9)$$

and

$$\frac{2}{1-h} q \psi G \leq \frac{2}{1-h} q^+ \psi G \leq \frac{1}{10} (\psi^{1/2} G)^2 + C \left(\psi^{1/2} \frac{q^+}{(1-h)} \right)^2 \leq \frac{1}{10} \psi G^2 + \frac{C (q^+)^2}{(1-h)^2} \leq \frac{1}{10} \psi G^2 + C (q^+)^2. \quad (3.10)$$

Case 1: For $\beta \geq 1$.

Since $h \leq 0$ and $0 < e^{h(\beta-1)} \leq 1$, the 8th term on the right hand side of (3.3) can be estimated as follows.

$$\begin{aligned} \frac{2}{(1-h)^2} (De^h)^{(\beta-1)} \psi \langle \nabla p, \nabla h \rangle &\leq \frac{2}{(1-h)^2} (De^h)^{(\beta-1)} \psi |\nabla p| |\nabla h| \leq \frac{2}{(1-h)} D^{\beta-1} \psi |\nabla p| G^{1/2} \\ &\leq \frac{1}{10} (\psi^{1/4} G^{1/2})^4 + C \left(\psi^{3/4} D^{\beta-1} \frac{|\nabla p|}{1-h} \right)^{4/3} \\ &\leq \frac{1}{10} \psi G^2 + C D^{\frac{4}{3}(\beta-1)} \frac{|\nabla p|^{4/3}}{(1-h)^{4/3}} \leq \frac{1}{10} \psi G^2 + C D^{\frac{4}{3}(\beta-1)} |\nabla p|^{4/3}. \end{aligned} \quad (3.11)$$

The 9th term is estimated as follows. Note that $0 < e^{h(\beta-1)} \leq 1$ and $0 \leq \frac{-h}{1-h} = 1 - \frac{1}{1-h} < 1$ because $h \leq 0$. Then

$$0 < \beta + \frac{h}{1-h} = \beta - \frac{-h}{1-h} = \beta - \left(1 - \frac{1}{1-h} \right) \leq \beta.$$

Therefore,

$$\begin{aligned} 2 \left(\beta + \frac{h}{1-h} \right) (De^h)^{(\beta-1)} p \psi G &\leq 2 \left(\beta + \frac{h}{1-h} \right) D^{(\beta-1)} p^+ \psi G \\ &\leq \frac{1}{10} \psi G^2 + C \left(\beta + \frac{h}{1-h} \right)^2 (D^{(\beta-1)} \psi^{1/2} p^+)^2 \\ &\leq \frac{1}{10} \psi G^2 + C \beta^2 D^{2(\beta-1)} (p^+)^2. \end{aligned} \quad (3.12)$$

Putting (3.4)–(3.12) back into the right hand side of (3.3) and rearranging to get

$$\begin{aligned} \psi(1-h)G^2 &\leq \frac{4}{5} \psi G^2 + \frac{C}{R^4} \frac{h^4}{(1-h)^3} + \frac{C}{R^4} + \frac{C\mu^2}{R^2} + CK^2 + \frac{C}{(\tau - (t_0 - T))^2} \\ &\quad + C \|\nabla q\|^{\frac{4}{3}} + C \|q^+\|^2 + C \beta^2 D^{2(\beta-1)} \|p^+\|^2 + C D^{\frac{4}{3}(\beta-1)} \|\nabla p\|^{\frac{4}{3}} \end{aligned} \quad (3.13)$$

at (x_1, t_1) . Since $h \leq 0$, $1-h \geq 1$ and $h^4/(1-h)^4 \leq 1$, (3.13) implies

$$\begin{aligned} (\psi^2 G^2)(x_1, t_1) &\leq (\psi G^2)(x_1, t_1) \\ &\leq \frac{C}{R^4} + \frac{C\mu^2}{R^2} + \frac{C}{(\tau - (t_0 - T))^2} + CK^2 + C \|q^+\|^2 + C \|\nabla q\|^{4/3} \\ &\quad + C \beta^2 D^{2(\beta-1)} \|p^+\|^2 + C D^{\frac{4}{3}(\beta-1)} \|\nabla p\|^{\frac{4}{3}} \end{aligned} \quad (3.14)$$

at (x_1, t_1) . Since $\psi(x, \tau) = 1$ in $Q_{R/2, T/2}$, for all $(x, t) \in Q_{R/2, T}$,

$$\begin{aligned} G(x, \tau) &= (\psi G)(x, \tau) \leq (\psi G)(x_1, t_1) \\ &\leq C \left(\frac{1}{R^2} + \frac{|\mu|}{R} + \frac{1}{(\tau - (t_0 - T))} + K + \|q^+\| + \|\nabla q\|^{2/3} + \beta D^{(\beta-1)} \|p^+\| + D^{\frac{2}{3}(\beta-1)} \|\nabla p\|^{\frac{2}{3}} \right). \end{aligned}$$

Thus, in $Q_{R/2,T}$ we have

$$\frac{|\nabla h|}{(1-h)} \leq C \left(\frac{1}{R} + \sqrt{\frac{|\mu|}{R}} + \frac{1}{\sqrt{(\tau - (t_0 - T))}} + \sqrt{K} + \|q^+\|^{1/2} + \|\nabla q\|^{1/3} + \sqrt{\beta} D^{\frac{1}{2}(\beta-1)} \|p^+\| + D^{\frac{1}{3}(\beta-1)} \|\nabla p\|^{\frac{1}{3}} \right). \quad (3.15)$$

Substituting $h = \ln u/D$ into (3.15) yields (1.5) when $x \notin B(x_0, 1) \subset B(x_0, R)$ for $R \geq 2$.

Case 2: For $\beta < 1$.

In this case, we have $\beta - 1 < 0$ and $e^{h(\beta-1)} > 1$ since $h \leq 0$. Note also that $(De^h)^{\beta-1} = u^{\beta-1} \leq M^{(\beta-1)}$, where $M = \inf_{Q_{R,T}} u$ as defined before.

For the 8th term on the right hand side of (3.3) is estimated as follows:

$$\begin{aligned} \frac{2}{(1-h)^2} (De^h)^{(\beta-1)} \psi \langle \nabla p, \nabla h \rangle &\leq \frac{2}{(1-h)^2} M^{\beta-1} \psi |\nabla p| |\nabla h| \leq \frac{2}{(1-h)} M^{\beta-1} \psi |\nabla p| G^{1/2} \\ &\leq \frac{1}{10} (\psi^{1/4} G^{1/2})^4 + C \left(\psi^{3/4} M^{\beta-1} \frac{|\nabla p|}{1-h} \right)^{4/3} \leq \frac{1}{10} \psi G^2 + C M^{\frac{4}{3}(\beta-1)} \frac{|\nabla p|^{4/3}}{(1-h)^{4/3}} \\ &\leq \frac{1}{10} \psi G^2 + C M^{\frac{4}{3}(\beta-1)} |\nabla p|^{4/3}. \end{aligned} \quad (3.16)$$

The 9th term on the right hand side of (3.3) is estimated as follows. Here $h \leq 0$ and $\beta < 1$, also $0 < \frac{-h}{1-h} = 1 - \frac{1}{1-h} < 1$ and $\frac{1}{1-h} \in (0, 1]$. Then

$$\left(\beta + \frac{h}{1-h} \right) (De^h)^{(\beta-1)} p = \left((\beta - 1) + \frac{1}{1-h} \right) (De^h)^{(\beta-1)} p \leq ([(\beta - 1)p]^+ + p^+) M^{(\beta-1)}$$

and therefore

$$\begin{aligned} 2 \left(\beta + \frac{h}{1-h} \right) (De^h)^{(\beta-1)} p \psi G &= 2 \left((\beta - 1) + \frac{1}{1-h} \right) p (De^h)^{(\beta-1)} \psi G \leq ([(\beta - 1)p]^+ + p^+) M^{(\beta-1)} \psi G \\ &\leq \frac{1}{10} \psi G^2 + C([(\beta - 1)p]^+ + p^+) M^{(\beta-1)} \psi^{1/2})^2 \\ &\leq \frac{1}{10} \psi G^2 + C([(\beta - 1)p]^+ + p^+)^2 M^{2(\beta-1)}, \end{aligned} \quad (3.17)$$

where $[(\beta - 1)p]^+ = \max\{(\beta - 1)p(x, t), 0\}$.

As before, substituting (3.4)–(3.10) and (3.16)–(3.17) into the right hand side of (3.3) and rearranging at (x_1, t_1) yield

$$\begin{aligned} \psi(1-h)G^2 &\leq \frac{4}{5} \psi G^2 + \frac{C}{R^4} \frac{h^4}{(1-h)^3} + \frac{C}{R^4} + \frac{C\mu^2}{R^2} + CK^2 + \frac{C}{(\tau - (t_0 - T))^2} \\ &\quad + C\|\nabla q\|^{\frac{4}{3}} + C\|q^+\|^2 + C(\|[(\beta - 1)p]^+ + \|p^+\|)^{\frac{1}{2}} M^{2(\beta-1)} + C M^{\frac{4}{3}(\beta-1)} \|\nabla p\|^{\frac{4}{3}} \end{aligned}$$

analogous to (3.13) ($\beta \geq 1$). Following similar steps to those in the case of $\beta \geq 1$ we arrive at (1.6). The proof is complete for the case $x \notin B(x_0, 1) \subset B(x_0, R)$ for $R \geq 2$.

Now consider the other case (when $x \in B(x_0, 1)$). In this case, ψ is a constant in space direction in $B(x_0, R/2)$, $R \geq 2$, due to the assumption. Hence, by (3.3) we have for $\beta \geq 1$ that

$$G^2 \leq \frac{\psi_t}{2\psi} G + (N-1)KG + \frac{1}{1-h} qG + \frac{1}{(1-h)^2} \langle \nabla q, \nabla h \rangle + \frac{1}{(1-h)^2} (De^h)^{\beta-1} \langle \nabla p, \nabla h \rangle + \left(\beta + \frac{h}{1-h} \right) (De^h)^{\beta-1} pG,$$

since $1-h \geq 1$ and $0 < e^{h(\beta-1)} \leq 1$. Using estimates (3.4)–(3.12), the last inequality is estimated to be

$$\begin{aligned} G &\leq \frac{\psi_t}{2\psi} + CK + C\|q^+\| + C\|\nabla q\|^{\frac{2}{3}} + C(D)^{\frac{2}{3}(\beta-1)} \|\nabla p\|^{\frac{2}{3}} + C\beta(D)^{\beta-1} \|p^+\| \\ &\leq \frac{C}{\tau - (t_0 - T)} + CK + C\|q^+\| + C\|\nabla q\|^{\frac{2}{3}} + C(D)^{\frac{2}{3}(\beta-1)} \|\nabla p\|^{\frac{2}{3}} + C\beta(D)^{\beta-1} \|p^+\|, \end{aligned}$$

where property (iv) of ψ has been applied. By property (i) of ψ , the last inequality implies

$$\begin{aligned} G(x, \tau) &= (\psi G)(x, \tau) \leq (\psi G)(x_1, t_1) \leq G(x_1, t_1) \\ &\leq \frac{C}{\tau - (t_0 - T)} + CK + C\|q^+\| + C\|\nabla q\|^{\frac{2}{3}} + C(D)^{\frac{2}{3}(\beta-1)}\|\nabla p\|^{\frac{2}{3}} + C\beta(D)^{\beta-1}\|p^+\| \end{aligned}$$

in $Q_{R/2,T}$. By this we arrive at (1.5). Similar procedure yields estimate (1.6) for $\beta < 1$. $\square$

4 Applications of gradient estimates

4.1 Harnack-type inequalities

The estimates obtained in Theorem 1.1 yield the classical Harnack inequalities as an application.

Theorem 4.1. *Given an N -dimensional complete weighted manifold satisfying $\text{Ric}_f \geq -(N-1)K$, $K \geq 0$. Suppose $u \leq D$ is a positive solution to (1.1) for all $(x, t) \in M \times [0, \infty)$, where $\|q^+\|, \|\nabla q\| < \infty$. Then*

$$u(x_2, t) \leq u(x_1, t)^\Gamma (De)^{1-\Gamma} \quad (4.1)$$

for all $x_1, x_2 \in M$, where $\Gamma(r, t) =: \Gamma = \exp\left(-C\left(\frac{1}{\sqrt{t-(t_0-T)}} + \sqrt{K} + \Lambda\right)r\right)$,

$$\Lambda := \max\left\{\|q^+\|^{\frac{1}{2}}, \|\nabla q\|^{\frac{1}{3}}\right\} + \max\left\{\sqrt{\beta} D^{\frac{1}{2}(\beta-1)}\|p^+\|^{1/2}, D^{\frac{1}{3}(\beta-1)}\|\nabla p\|^{\frac{1}{3}}\right\}$$

for $\beta \geq 1$,

$$\Lambda := \max\left\{\|q^+\|^{\frac{1}{2}}, \|\nabla q\|^{\frac{1}{3}}\right\} + \max\left\{M^{\frac{1}{2}(\beta-1)}(\|[(\beta-1)p^+\| + \|p^+\|]^{\frac{1}{2}}, M^{\frac{1}{3}(\beta-1)}\|\nabla p\|^{\frac{1}{3}}\right\}$$

for $\beta < 1$, M and $r = r(x_1, x_2)$ are as defined in Theorem 1.1.

Proof of Theorem 4.1. Let the shortest path connecting x_1 and x_2 be $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Sending $R \rightarrow \infty$ in (1.5) (resp. (1.6)) gives

$$\frac{|\nabla u|}{u(1 + \ln D/u)} = \frac{|\nabla h|}{1-h} \leq C\left(\frac{1}{\sqrt{t-(t_0-T)}} + \sqrt{K} + \Lambda\right),$$

where Λ is as defined in the theorem. We now compute

$$\ln \frac{1-h(x_1, t)}{1-h(x_2, t)} = \int_0^1 \frac{d \ln(1-h(\gamma(s), s))}{ds} ds \leq \int_0^1 |\dot{\gamma}| \frac{|\nabla h|}{1-h} ds \leq C\left(\frac{1}{\sqrt{t-(t_0-T)}} + \sqrt{K} + \Lambda\right)r := B.$$

Define $\Gamma = \Gamma(r(x_1, x_2), t) := e^{-B}$. Then from the last inequality we have

$$\frac{1-h(x_1, t)}{1-h(x_2, t)} \leq \frac{1}{\Gamma}.$$

Hence, some elementary computations imply (4.1), and this completes the proof. $\square$

4.2 Liouville-type theorems

This section presents some Liouville-type theorems for various cases of (1).

Theorem 4.2. *Given an N -dimensional complete weighted manifold satisfying $\text{Ric}_f \geq 0$. Suppose $p = p(x)$ and $q = q(x)$ in (1.1) satisfy the following conditions as $R \rightarrow \infty$*

- (i) $p^+|_{B(x_0, R)} = o(R^{-(\beta-1)})$, $\sup_{B(x_0, R)} |\nabla p| = o(R^{-(\beta-1)})$;
 (ii) $q^+|_{B(x_0, R)} = o(R^{-1})$, $\sup_{B(x_0, R)} |\nabla q| = o(R^{-1})$.

Then

- (a) If $p(x) \neq 0$, (1.1) has no positive ancient solution with $u(x, t) = e^{o([r^{1/2} + |t|^{1/2}])}$ near infinity.
 (b) For $p(x) = 0$, $q(x) \neq 0$ Equation (1.1) has no positive ancient solution with $u(x, t) = e^{o([r^{1/2} + |t|^{1/2}])}$ near infinity.

Furthermore, if $p(x) \equiv 0 \equiv q(x)$, then

- (c) (1.2) has only constant positive ancient solution with $u(x, t) = o([r^{1/2} + |t|^{1/4}])$ near infinity,
 (d) (1.2) has only constant ancient solution with $u(x, t) = o([r^{1/2} + |t|^{1/4}])$ near infinity.

Note that if $p(x) \equiv 0 \equiv q(x)$ (1.1) reduces to (1.2). Any solution which exists for negative time in all space is referred to as the ancient solution.

Proof of Theorem 4.2. We only consider case $\beta \geq 1$ for the proof of (a) since the case $\beta < 1$ is similar. Suppose $u \leq D$ is a positive ancient solution of (1.1) with the assumption $u(x, t) = e^{o([r^{1/2} + |t|^{1/2}])}$ near infinity. For a fixed point (x_0, t_0) , letting $D_R := \sup_{Q_{\sqrt{R}, R}} |u|$ and using the function $U = u + 2D_{2R}$, we have $D_{2R} \leq U(x, t) \leq 3D_{2R}$, for every $(x, t) \in Q_{2\sqrt{R}, 4R}$. Then applying Theorem 1.1 for U in the set $B(x_0, R) \times [t_0 - R^2, t_0]$ yields

$$\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0) + 2D_{2R}} \leq C(N) \left(\sqrt{\frac{1 + |\mu|}{R}} o\left(R^{-\frac{1}{3}}\right) + o\left(R^{-\frac{1}{2}}\right) + \beta \cdot o\left(R^{\frac{1}{2}(\beta-1)}\right) o\left(R^{-\frac{1}{2}(\beta-1)}\right) \right. \\ \left. + o\left(R^{\frac{1}{3}(\beta-1)}\right) o\left(R^{-\frac{1}{3}(\beta-1)}\right) (1 + o(R) - \ln u(x_0, t_0)) \right)$$

near infinity. Since by assumption $D_{2R} = o(R)$, it follows that $|\nabla u(x_0, t_0)| = 0$ as $R \rightarrow \infty$. Then $\nabla u(x, t) \equiv 0$ since (x_0, t_0) is arbitrary. Hence, u must be a constant for all $x \in M$.

Now, suppose $q(x) = 0$, then from (1.1), $p(x) = \tilde{p}$ (a constant) which implies

$$\frac{du(t)}{dt} = \tilde{p}u^\beta(t). \quad (4.2)$$

Integrating (4.2) between the interval t and 0, where $t < 0$ gives

$$u^{1-\beta}(t) = u^{1-\beta}(0) + (1 - \beta)\tilde{p}t. \quad (4.3)$$

By hypothesis (i), we have $\tilde{p} \leq 0$. As $t \rightarrow -\infty$, we get $u^{1-\beta}(t)$, $\beta \geq 1$, which is not possible since u is a positive solution. Hence, $\tilde{p} = 0$ leads to a contradiction to $u = e^{o([r^{1/2} + |t|^{1/2}])}$ near infinity.

Suppose $q(x) \neq 0$, then (1.1) is equivalent to

$$\frac{du(t)}{dt} = p(x)u^\beta(t) + q(x)u(t) \quad (4.4)$$

which, when using the identity $(u/u^\beta)' = (1 - \beta)u'/u^\beta$, becomes

$$\frac{d}{dt}(u^{1-\beta}(t)) = (1 - \beta)p(x) + (1 - \beta)q(x)u^{1-\beta}(t). \quad (4.5)$$

The general solution of (4.5) is given by

$$u^{1-\beta}(t) = -\frac{p(x)}{q(x)} + Ae^{(1-\beta)qt}, \quad (4.6)$$

where A is an arbitrary constant. Since the solution in (4.6) is a function with respect to variable t and by assumptions (a) and (b), we know that $p(x) = p$ and $q(x) = q$ are negative constants. Since $\beta > 1$, $t < 0$ and $u(0) > 0$, letting $t \rightarrow -\infty$, we have $u^{1-\beta}(t) \rightarrow -\frac{p}{q} < 0$ which is not possible since u is a positive solution.

We now prove (c) and (d) of the Theorem. Note that when $p(x) \equiv 0 \equiv q(x)$ (1.1) reduces to equation (1.2). Suppose $u(x, t) \leq D$ is an ancient solution to (1.2) satisfying the assumption $u(x, t) = e^{o([r^{1/2}(x)+|t|^{1/4}])}$ near infinity. Clearly, Theorem 1.1 yields $\nabla u \equiv 0$ such that $u(x, t) = u(t)$. Then by equation (1.2)

$$\frac{du(t)}{dt} = 0,$$

meaning that u is a constant. The case $\beta < 1$ is similar. This concludes the proof. $\square$

Remark 4.3. If we consider the case $\beta = 1$, we can just set $p(x, t) = 0$ and begin to deal with

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)u + q(x)u = 0. \quad (4.7)$$

Using hypothesis (ii) of Theorem 4.2, one can then prove that (4.7) has no positive ancient solution with $u = e^{o(r^{1/2}+|t|^{1/2})}$ near infinity. Indeed, the proof follows the same steps as before by fixing any point (x_0, t_0) and applying Theorem 1.1 in the set $B(x_0, R) \times [t_0 - R^2, t_0]$, with

$$\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0)} \leq C(N) \left(\sqrt{\frac{1+|\mu|}{R}} + o(R^{-\frac{1}{2}}) + o(R^{-\frac{1}{3}}) \right) (1 + o(R) - \ln u(x_0, t_0))$$

for $R > 2$. Letting $R \rightarrow \infty$ implies that $|\nabla u(x_0, t_0)| = 0$. Since (x_0, t_0) is arbitrary, $\nabla u(x, t) \equiv 0$, which implies that $u(x, t) = u(t)$, a constant function of time.

Furthermore, from (1.1), we obtain

$$u'(t) = \lambda u(t), \quad (4.8)$$

where $u(t)$ being independent of space variable implies $q(x) = \lambda$ is a constant. Integrating (4.8) between the interval t and 0, where $t < 0$ gives

$$u(t) = u(0)e^{\lambda t}.$$

It is easy to show that $\lambda = 0$, which contradicts the assumption that $q \neq 0$. Applying the hypothesis (i) of the Theorem gives $\lambda \leq 0$. Assume $\lambda < 0$, $u(t) = u(0)e^{\lambda t}$ is a contradiction to $u = e^{o(r^{1/2}(x)+|t|^{1/4})}$ near infinity. Hence, $\lambda = 0$, which is a contradiction.

Remark 4.4. Theorem 4.2 reduces to Theorems 1.5 and 1.6 of [30] as corollaries. Therefore, it was proved that if $q(x) = -kp(x)$, $k > 0$ is some constant, then $u(x, t) = k^{\frac{1}{\beta-1}}$ on M .

5 Heat kernel estimates

For the case $q \equiv 0 \equiv p$, assuming $\text{Ric}_f \geq 0$ and f is locally bounded, Wu [22, Theorems 3.1 and 3.2] presents some interesting weighted gradient estimates

$$\frac{|\nabla u|}{u} \leq \tilde{C}(N) \left(\frac{1+B(R)}{R} + \frac{1}{\sqrt{t-(t_0-T)}} \right) \left(1 + \ln \frac{D}{u} \right) \quad (5.1)$$

in $Q_{R/2,T}$ with $t \neq t_0 - T$, $B(R) := \sup_{x \in B(x_0, R)} |f(x)|$ and $D := \sup_{(x,t) \in Q_{R,T}} u(x, t)$ and Liouville-type theorem (see also [11]). Similarly, suppose further that function f is assumed to be bounded when both q and p are zero and $\text{Ric}_f \geq 0$, we shall obtain a global gradient estimate

$$\frac{|\nabla u|}{u} \leq \tilde{C}(N) \frac{1}{\sqrt{t-(t_0-T)}} \left(1 + \ln \frac{D}{u(x, t)} \right) \quad (5.2)$$

for positive solution $u(x, t)$, $x \in M$, $t > 0$.

Our result in this direction is the following gradient estimate on the weighted heat kernel.

Theorem 5.1. *Given an N -dimensional complete weighted manifold $(M^N, g, e^{-f} dv)$ satisfying $\text{Ric}_f \geq 0$. Let $H(x, y, t)$ be the weighted heat kernel. There exists a constant C_6 depending on N and $\sup_x |f(x)|$ only, such that, for all $x, y \in M$ and $t > 0$, the following estimate*

$$\frac{|\nabla H(x, y, t)|}{H(x, y, t)} \leq \frac{C_6}{t^{1/2}} \left(1 + \frac{d(x, y)}{t} \right) \quad (5.3)$$

holds, where $d(x, y)$ denotes the geodesic distance from x to y .

In [22], this type of estimate was proved via a global Hamilton-type gradient estimate, which was derived using Bernstein-Shi-type estimate and the weighted Laplacian comparison theorem.

Proof of Theorem 5.1. Let $H(x, y, t)$ be the fundamental solution of the weighted heat equation on $(M^N, g, e^{-f} dv)$ with $\text{Ric}_f \geq 0$. By [21, Theorem 1.2] (see also [24, 25]), we have

$$\frac{C_1}{V_f(B(y, \sqrt{t}))} e^{-C_2 \frac{d^2(x, y)}{t}} \leq H(x, y, t) \leq \frac{C_3}{V_f(B(y, \sqrt{t}))} e^{-C_4 \frac{d^2(x, y)}{t}} \quad (5.4)$$

for all $x, y \in M, t > 0$ with $C_1, C_2, C_3, C_4 > 0$ depending only on N and $\sup_x |f(x)|$, where $V_f(B(x, R))$ is the volume of $B(x, R)$ with respect to the weighted measure.

Fixing $x, y \in M$ and $t > 0$. Setting $q(x, t) \equiv 0 \equiv p(x, t)$, we shall apply (36) to the function $u(z, s) = H(z, y, s)$ in $Q_{R, T}$. We know that $u(z, s)$ is a smooth positive function in $[0, T)$. Then we have

$$\frac{C_1}{V_f(B(y, \sqrt{s}))} e^{-C_2 \frac{d^2(z, y)}{s}} \leq H(z, y, s) \leq \frac{C_3}{V_f(B(y, \sqrt{s}))} \quad (5.5)$$

for all $x, y \in M$ and $s \geq 0$. Letting

$$D := \frac{C_3}{V_f(B(y, \sqrt{s}))},$$

then by the upper bound in (5.5) $u \leq D$ for all x and s , and by the weighted volume doubling property [24], there exists a positive constant C_5 depending on N and $\sup_x |f(x)|$ such that

$$u(z, s) \leq \frac{C_3}{V_f(B(z, \sqrt{s}))} \leq \frac{C_5}{V_f(B(x, \sqrt{t}))}$$

for all $(z, s) \in Q$. Then by (5.2) and the lower bound of (5.5) we have

$$\frac{|\nabla H(x, y, t)|}{u(x, y, t)} \leq \frac{\tilde{C}(N)}{t^{1/2}} \left(1 + \ln \frac{\frac{C_5}{V_f(B(x, \sqrt{t}))}}{\frac{C_1}{V_f(B(x, \sqrt{t}))} e^{-C_2 \frac{d^2(x, y)}{t}}} \right),$$

which implies

$$\frac{|\nabla H(x, y, t)|}{u(x, y, t)} \leq \frac{C_6}{t^{1/2}} \left(1 + \frac{d(x, y)}{t} \right).$$

This completes the proof. $\square$

6 Concluding remark

We have adopted the classical approach of Souplet and Zhang [11] to obtain elliptic gradient estimates on positive solutions to nonlinear parabolic equations on weighted manifolds with lower boundedness condition on the Bakry-Émery tensor. Our result is an improvement of some known results. The gradient estimates derived have been used to establish Harnack-type inequalities (Section 4.1) and Liouville-type theorems (Section 4.2). The weighted heat kernel estimates obtained in Section 5 is interesting on its own.

Furthermore, the gradient estimates obtained can be used to study the existence and nonexistence of Yamabe-type problems on weighted manifolds. In this direction, Wu [30, Theorem 5.1] has used some parabolic gradient estimates to prove that

$$\Delta_f u(x) + pu^\beta(x) = 0, \quad \beta < 1, p \geq 0 \quad (6.1)$$

does not have any positive solution. This result immediately implies nonexistence of a positive solution to

$$\Delta_f u(x) - \frac{m + N - 2}{4(m + N - 1)} \mathcal{R}_f^m u = 0.$$

This further implies that if the weighted scalar curvature $\mathcal{R}_f^m$ is a nonpositive constant, then there does not exist a positive volume normalized minimizer $u(x)$ such that the weighted Yamabe constant is zero. Meanwhile, Gidas and Spruck [34] have earlier shown for constant f and $p(x) \in C^2$ satisfying

- (1) $p(x) \geq 0$, $\Delta p(x) \geq 0$ and $1 \leq \beta \leq (N + 2)/(N - 2)$, $N \geq 4$ and
- (2) $|\nabla \log p| \leq C/r(x)$, $p(x) \geq C(r(x))^\alpha$ with $\alpha \geq -2/(N - 3)$ for $r(x)$ large, where $r(x)$ is the geodesic distance between x and some fixed point, that if $u(x)$ is a nonnegative solution of (6.1), then $u(x) \equiv 0$.

Li [35] obtained Gidas-Spruck's result for $\beta \leq (N + 2)/(N - 2)$, $N \geq 4$, by removing condition (2) on $p(x)$.

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