Research Article

Liviu-Constantin Holdon*

The prime and maximal spectra and the reticulation of residuated lattices with applications to De Morgan residuated lattices

<https://doi.org/10.1515/math-2020-0061>

received August 24, 2019; accepted July 1, 2020

Abstract: In this paper, by using the ideal theory in residuated lattices, we construct the prime and maximal spectra (Zariski topology), proving that the prime and maximal spectra are compact topological spaces, and in the case of De Morgan residuated lattices they become compact T_0 topological spaces. At the same time, we define and study the reticulation functor between De Morgan residuated lattices and bounded distributive lattices. Moreover, we study the I -topology (I comes from ideal) and the stable topology and we define the concept of pure ideal. We conclude that the I -topology is in fact the restriction of Zariski topology to the lattice of ideals, but we use it for simplicity. Finally, based on pure ideals, we define the normal De Morgan residuated lattice (L is normal iff every proper ideal of L is a pure ideal) and we offer some characterizations.

Keywords: residuated lattice, De Morgan laws, filter, ideal, prime spectrum, maximal spectrum, reticulation, bounded distributive lattices, pure ideal, stable topology

MSC 2020: 08A72, 03B22, 03G05, 03G25, 06A06

1 Introduction

Residuation is a fundamental concept of ordered structures and categories. The origin of residuated lattices is in Mathematical Logic without contraction. In 1939, Ward and Dilworth were the first who introduced the concept of a residuated lattice [1,2] as a generalization of ideal lattices of rings. The general definition of a residuated lattice was given by Galatos et al. (2007) [3]. They first developed the structural theory of this kind of algebra about residuated lattices.

Residuated lattices comprise the class of lattice-ordered monoids for which the monoid operation has upper adjoints with respect to the underlying lattice order. These algebraic structures have been the subject of quite a lot of research in recent years, mostly from the perspective of algebraic logic (where they give algebraic semantics of various substructural logics) but also from a purely algebraic point of view (e.g., in the theory of relation algebras). Like groups and rings, the congruences of a residuated lattice may be captured by considering certain subsets of the residuated lattice. In the case of commutative, integral residuated lattices, these subsets are lattice filters that are closed under the monoid operation. When a commutative, integral residuated lattice has a least element 0, one may define additional operations and $*$ by $x^* = x \rightarrow 0$ and $x \oplus y = (x^* \odot y^*)^*$, where $\oplus$ is the monoid operation of the residuated lattice and is its adjoint. When a residuated lattice satisfies $x^{**} = x$, these operations take on great significance, and $\odot$ becomes definable in terms in $\oplus$. Motivated from this point of view, one may define an ideal of

* **Corresponding author: Liviu-Constantin Holdon**, Department of Systems Engineering-Automatic and Informatics Applied, Technical University of Civil Engineering, Lacul Tei Bvd., no. 122, 020396, Bucharest, Romania; International Theoretical High School of Informatics Bucharest, 648, Colentina st., 021187, Bucharest, Romania, e-mail: holdon-liviu@yahoo.com, holdon.liviu@ichb.ro, tel: +40-212-40-3035

a commutative, integral residuated lattice to be a lattice ideal that is closed under $\oplus$. When the identity $x^{**} = x$ is satisfied, ideals in this sense are in bijective correspondence with congruences. This is the case of involution residuated lattices (L is an involution residuated lattice if $x^{**} = x$, for all $x \in L$). However, in 2016, Holdon [4] presented the class of De Morgan residuated lattices (which is larger than involution residuated lattices) and proved: if I is an ideal of a De Morgan residuated lattice L , then the binary relation θ_I on L ($(x, y) \in \theta_I$ iff $x^* \odot y \in I$ and $x \odot y^* \in I$) is a congruence on the reduct $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of L . From this point of view, the ideal theory on De Morgan residuated lattices becomes interesting. In what follows, we would like to present some works on ideals and filters in varieties of residuated lattices, which helps us to develop the present work.

Ideal theory plays an important rule in studying residuated lattices. From a logical point of view, various ideals correspond to various sets of provable formulae. In residuated lattices, the filters and ideals are not dual; in consequence, the ideals have a proper meaning in residuated lattices, from a purely algebraic point of view. Recently, a lot of work has been done with respect to the ideals in varieties of residuated lattices. We note that in MV-algebras the ideals and filters are dual. In 2013, Lele and Nganou [5] constructed some examples to show that, unlike in MV-algebras, ideals and filters are dual but behave differently in BL-algebras. In 2014, in the framework of MV-algebras, Forouzesh et al. [6] published a study on obstinate ideals; they investigated some relationships between the obstinate ideals and the other ideals of an MV-algebra. In 2012, Borumand Saeid and Pourkhatun [7] investigated some properties of obstinate filters in residuated lattices. In 2015, Buşneag et al. [8] investigated the variety of Stonean residuated lattices from the view of ideal theory. In 2016, Zou et al. [9] published a study on ideals and annihilator ideals in BL-algebras. In 2016, Paad [10] published a study on integral ideals and maximal ideals in BL-algebras. In 2017, Holdon [4] investigated a class of residuated lattices called De Morgan residuated lattices (L is called De Morgan residuated lattice if the De Morgan law $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$ holds) and investigated it from the view of ideal theory, and the author defined $\odot$ -prime ideals and studied them in order to establish the relationship between ideals and filters in residuated lattices; moreover, the concept of annihilator ideal had been defined and some interesting properties of them had been investigated. We note that in De Morgan residuated lattices the notions of ideals and filters are not dual. In 2018, Holdon [11] constructed a new topology based on upsets (filters) in residuated lattices, in a dual manner, using downsets (ideals), we can study a dual topology.

In the literature of residuated lattices, there is an increased interest on the prime, maximal spectra, the stable topology and pure filters. For example, in 2003, Leustean [12] investigated from a filter theory point of view the prime and maximal spectra and the reticulation of BL-algebras. In 2007, Haveschi et al. [13] constructed a topology induced by uniformity on BL-algebras based on the filter theory. In 2009, Eslami and Haghani [14] investigated stable topology and F -topology on the set of all prime filters of a BL-algebra and showed that the set of all prime filters (namely $\text{Prime}(L)$) with the stable topology is a compact space but not T_0 . Moreover, by means of stable topology, they defined and studied pure filters of a BL-algebra. In 2012, Buşneag and Piciu [15] investigated and studied another topology on $\text{Prime}(L)$ (the set of all prime filters of L) the so-called stable topology, which turns out to be coarser than the spectral one. Consequently, they introduced the notions of pure i -filter for a residuated lattice and the notion of normal residuated lattice and studied their properties. In 2019, Holdon and Borumand Saeid [16] investigated from a filter theory point of view the regularity in residuated lattices.

In this paper, motivated by the previous research on ideals in MV-algebras [5,6], BL-algebras [5,9,10], Stonean residuated lattices [8] and De Morgan residuated lattices [4], also by the research (based on filter theory) on the prime, maximal spectra, the stable topology and pure filters in BL-algebras [12,14] and residuated lattices [15], we think that it will be interesting to study from ideal theory point of view the prime, maximal spectra, the stable topology and pure ideals and reticulation in varieties of residuated lattices. In the general case of residuated lattices, the development of an ideal theory can face many difficulties, for example, the concepts of prime ideals and $\cap$ -irreducible ($\cap$ -prime) ideals are not equivalent, $\text{Max}(L) \not\subseteq \text{Prime}(L)$ and The Prime Ideal Theorem may not always hold (see Section 3.2). We consider the variety of De Morgan residuated lattices suitable for the study of ideal theory for the following two main reasons: first, the concepts of prime ideals and $\cap$ -irreducible ($\cap$ -prime) ideals are equivalent,

$\text{Max}(L) \subseteq \text{Prime}(L)$ and The Prime Ideal Theorem always hold, and, second, it includes important subvarieties of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices. The relationship between the class of De Morgan residuated lattices and other classes of residuated lattices has been established in [4]. We believe that the present study will enrich and develop the theory of ideals in De Morgan residuated lattices and will help to develop an ideal theory in the general case of residuated lattices.

This paper consists of four sections, and it is organized as follows: in Section 2, we give some preliminaries including the basic definitions, some examples of residuated lattices, rules of calculus and theorems that are needed in the sequel. In Section 3, we present general information about ideals in residuated lattices, and, in Section 3.2, we establish that the concepts of prime ideals and $\cap$ -irreducible ($\cap$ -prime) ideals are not equivalent, $\text{Max}(L) \not\subseteq \text{Prime}(L)$ and The Prime Ideal Theorem may not always hold. These properties are important in order to develop an ideal theory, and this is the reason why, in the next section, we pay attention to the variety of De Morgan residuated lattices. Consequently, in Section 4, we focus on the study of ideals in De Morgan residuated lattices, we extend in a natural sense the study of De Morgan residuated lattices from an ideal theory point of view, and we begin by offering a relevant example of a De Morgan residuated lattice. Section 4 contains four subsections. In Section 4.1, we study the prime (maximal) spectrum and we prove that it is a compact topological space, and in the case of De Morgan residuated lattices it is a compact T_0 topological space; in Section 4.2, we define a congruence relation and we study the reticulation functor between the category of De Morgan residuated lattices and bounded distributive lattices, and prove some important results. In Section 4.3, we define the stable topology. Moreover, in Section 4.4, by means of stable topology, we define and investigate pure ideals of a De Morgan residuated lattice. Finally, we define the normal De Morgan residuated lattice (L is normal iff every proper ideal of L is a pure ideal) and we offer some characterizations.

One of the problems that we face in this work is that “what results hold in residuated lattices (in general) or in De Morgan residuated lattices (in particular)?” In order to avoid these misunderstandings, we clearly specify when to use De Morgan residuated lattices.

2 Preliminaries

Definition 1. [3] A residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is an algebra of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ such that

LR₁: $(L, \vee, \wedge, 0, 1)$ is a bounded lattice relative to $\leq$;

LR₂: $(L, \odot, 1)$ is a commutative ordered monoid;

LR₃: $\odot$ and $\rightarrow$ form an adjoint pair, i.e., $a \odot x \leq b$ iff $x \leq a \rightarrow b$, for all $x, a, b \in L$.

For examples of residuated lattices see [3–22]. We denote residuated lattices by RL.

In what follows, by L we denote the universe of a residuated lattice (unless otherwise specified). For $x \in L$ and $n \geq 0$, we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

For $x, y, z \in L$, we have the following rules of calculus [3–22]:

(c₁) $x \rightarrow x = 1, x \rightarrow 1 = 1, 1 \rightarrow x = x$;

(c₂) $x \leq y$ iff $x \rightarrow y = 1$;

(c₃) If $x \leq y$, then $z \odot x \leq z \odot y, z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$;

(c₄) If $x \leq y$, then $y^* \leq x^*, x^{**} \leq y^{**}$;

(c₅) $x \odot x^* = 0, x \leq x^{**}, x^{***} = x^*$;

(c₆) $x \odot y = 0$ iff $x \leq y^*$;

(c₇) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;

(c₈) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$;

(c₉) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;

(c₁₀) $x \odot (x \rightarrow y) \leq x \wedge y$;

(c₁₁) $(x \vee y)^* = x^* \wedge y^*$;

(c₁₂) $x^* \odot y^* \leq (x \odot y)^*, x^{**} \odot y^{**} \leq (x \odot y)^{**}$;

- (c₁₃) $(x \rightarrow y^{**})^{**} = x \rightarrow y^{**}$;
 (c₁₄) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$, $(x \odot y)^* = x \rightarrow y^* = y \rightarrow x^*$.

Following the aforementioned literature, we consider the identities:

- (i₁) $x \wedge y = x \odot (x \rightarrow y)$ (divisibility);
 (i₂) $(x^* \wedge y^*)^* = [x^* \odot (x^* \rightarrow y^*)]^*$ (semi-divisibility);
 (i₃) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity);
 (i₄) $x^* \vee x^{**} = 1$ (Stone property);
 (i₅) $x^2 = x$ (idempotence);
 (i₆) $x = x^{**}$ (involution);
 (i₇) $(x^2)^* = x^*$;
 (i₈) $(x \wedge y)^* = x^* \vee y^*$.

Then the *residuated lattice* L is called:

- (i) *Divisible* if L verifies (i₁);
 (ii) *Semi-divisible* if L verifies (i₂);
 (iii) *MTL-algebra* if L verifies (i₃);
 (iv) *BL-algebra* if L verifies (i₁) and (i₃);
 (v) *Stonean* if L verifies (i₄);
 (vi) *G-algebra (Heyting algebra)* if L verifies (i₅);
 (vii) *Involution* if L verifies (i₆);
 (viii) *Semi-G-algebra* if L verifies (i₇);
 (ix) *De Morgan* if L verifies (i₈).

For every $x, y \in L$ we define:

$$x \oplus y = (x^* \odot y^*)^* \stackrel{(c_{14})}{=} x^* \rightarrow y^{**}. \quad (1)$$

For $x \in L$ and $n \geq 0$, we define $0 \cdot x = 0$ and $n \cdot x = [(n-1) \cdot x] \oplus x$ for $n \geq 1$. For simplicity, we denote $nx := n \cdot x$.

For every $x, y, z, t \in L$ and $m, n \geq 1$, we have the following rules of calculus [4,8]:

- (c₁₅) $x \oplus 0 = x^{**}$, $x \oplus 1 = 1$, $x \oplus x^* = 1$;
 (c₁₆) $x \oplus y = y \oplus x$, $x, y \leq x \oplus y$;
 (c₁₇) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
 (c₁₈) If $x \leq y$, then $x \oplus z \leq y \oplus z$;
 (c₁₉) If $x \leq y$, $z \leq t$, then $x \oplus z \leq y \oplus t$;
 (c₂₀) If $m \leq n$, then $mx \leq nx$;
 (c₂₁) If $x \leq y$, then $mx \leq my$.

We conclude that the operator $\oplus$ is commutative, associative and compatible with the order relation.

Definition 2. [18] A filter is a non-empty subset F of L such that

- (F₁) if $x \leq y$ and $x \in F$, then $y \in F$;
 (F₂) if $x, y \in F$, then $x \odot y \in F$.

We denote by $\mathcal{F}(L)$ the set of all filters of L .

Also, $(\mathcal{F}(L), \subseteq)$ is the lattice of filters with respect to the inclusion of sets.

3 Ideals in RL

In this section, we present some results with respect to ideals, prime and maximal ideals in residuated lattices. We will establish, in the general case of residuated lattices, the relationship between the concepts of prime ideals, $\cap$ -prime ($\cap$ -irreducible) and maximal ideals.

3.1 General information

Definition 3. [4,8] A nonempty subset I will be called an *ideal* of L if

- (I_1) If $x \leq y$ and $y \in I$, then $x \in I$;
 (I_2) If $x, y \in I$, then $x \oplus y \in I$.

A nonempty subset Q of a join-semilattice A is called a *lattice ideal* if satisfies: from $a \leq b$ and $b \in I$, then $a \in I$; and from $a, b \in Q$ implies $a \vee b \in Q$.

Remark 1. [8] Every ideal is a lattice ideal in the lattice $(L, \wedge, \vee, 0, 1)$, but the converse is not true. It is easy to see that ideals and lattice ideals coincide iff $x \vee y = x \oplus y$. In Boolean algebras, these notions are equivalent. Moreover, the intersection of any set of ideals is an ideal.

We denote by $I(L)$ the set of all ideals of L . Clearly, if $I \in I(L)$, then $I = L$ iff $1 \in I$.

Proposition 1. If I is an ideal of L , then $x \in I$ iff $x^{**} \in I$ iff $nx \in I$, for all $n \geq 1$.

Proof. Let I be an ideal of L . Now, we prove that $x \in I$ iff $x^{**} \in I$. Since I is an ideal and $0, x \in I$, it follows that $x^{**} = 0 \oplus x \in I$. Therefore, $x^{**} \in I$. Conversely, if $x^{**} \in I$, by (c_5), $x \leq x^{**} \in I$, we conclude that $x \in I$.

Now, we prove that $x \in I$ iff $nx \in I$, for all $n \geq 1$. Since I is an ideal and $x \in I$, it follows that $nx \in I$. Conversely, if $nx \in I$, by (c_{20}), $x \leq nx \in I$, we conclude that $x \in I$. The proof is complete. $\square$

Remark 2. If $x, y \in L$ and $I \in I(L)$, then $x \oplus y \in I$ iff $x \vee y \in I$ iff $x, y \in I$.

For a nonempty subset S of L , we denote by (S) the ideal of L generated by S (i.e., $(S) = \cap\{I \in I(L) : S \subseteq I\}$) and for an element $a \in L$ by (a) the ideal generated by $\{a\}$. If $I \in I(L)$ and $a \in L$, we denote $I(a) = (I \cup \{a\})$. Clearly, $I(a) = I$ iff $a \in I$.

Proposition 2. [8] Let L be a residuated lattice, $S \subseteq L$ a nonempty subset, $a \in L$ and $I \in I(L)$. Then

- (i) $(S) = \{x \in L : x \leq s_1 \oplus \dots \oplus s_n, \text{ for some } n \geq 1 \text{ and } s_1, \dots, s_n \in S\}$;
 (ii) $(a) = \{x \in L : x \leq na \text{ for some } n \geq 1\}$;
 (iii) $I(a) = \{x \in L : x \leq i \oplus na \text{ for some } i \in I \text{ and } n \geq 1\}$.

Proposition 3. [4]

- (i) $(I(L), \subseteq)$ is a complete lattice, where for $I_1, I_2 \in I(L)$, $I_1 \vee I_2 = (I_1 \cup I_2) = \{x \in L : x \leq i_1 \oplus i_2 \text{ with } i_1 \in I_1 \text{ and } i_2 \in I_2\}$.
 (ii) If Λ is an index set and $(I_i)_{i \in \Lambda}$ is a family of ideals of L , then the infimum of this family is $\bigwedge_{i \in \Lambda} I_i = \cap_{i \in \Lambda} I_i$ and the supremum is $\bigvee_{i \in \Lambda} I_i = (\cup_{i \in \Lambda} I_i) = \{x \in L : x \leq x_{i_1} \oplus x_{i_2} \oplus \dots \oplus x_{i_m}, \text{ where } i_1, \dots, i_m \in \Lambda, x_{i_j} \in I_{i_j}, 1 \leq j \leq m, \text{ for some } m \geq 1\}$.

Example 1. We consider the set $L = \{0, a, b, c, d, e, f, g, 1\}$ with $0 < a < b < e < 1$, $0 < a < d < e < 1$, $0 < a < d < g < 1$, $0 < c < d < e < 1$, $0 < c < d < g < 1$, $0 < c < f < g < 1$ and elements $\{a, c\}$, $\{b, d\}$, $\{d, f\}$, $\{e, g\}$ and $\{b, f\}$ are pairwise incomparable.

Then [20, p. 166] L becomes a residuated lattice relative to the following operations:

$\rightarrow$	0	a	b	c	d	e	f	g	1	$\odot$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
a	g	1	1	g	1	1	g	1	1	a	0	0	a	0	0	a	0	0	a
b	f	g	1	f	g	1	f	g	1	b	0	a	b	0	a	b	0	a	b
c	e	e	e	1	1	1	1	1	1	c	0	0	0	0	0	0	c	c	c
d	d	e	e	g	1	1	g	1	1	d	0	0	a	0	0	a	c	c	d
e	c	d	e	f	g	1	f	g	1	e	0	a	b	0	a	b	c	d	e
f	b	b	b	e	e	e	1	1	1	f	0	0	0	c	c	c	f	f	f
g	a	b	b	d	e	e	g	1	1	g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1	1	0	a	b	c	d	e	f	g	1

It is easy to ascertain that the following sets $I_1 = \{0\}$, $I_2 = \{0, a, b\}$ and $I_3 = \{0, c, f\}$ are proper ideals of L . We conclude that $I_1 \vee I_2 = I_2$, $I_1 \vee I_3 = I_3$ and $I_2 \vee I_3 = L$.

3.2 Prime and maximal ideals in RL

An ideal $I \in \mathcal{I}(L)$ is called *proper* if $I \neq L$ (equivalently, $I \in \mathcal{I}(L)$ is proper iff $1 \notin I$). In the following definition, we present well-known concepts of ideals in residuated lattices in order to establish the relationships between them.

Definition 4.

- (i) A proper ideal $I \in \mathcal{I}(L)$ is called *prime* iff, for all $a, b \in L$, if $a \wedge b \in I$, then $a \in I$ or $b \in I$. We denote by $\text{Prime}(L)$ the set of all prime ideals of L .
- (ii) A proper ideal I is called $\cap$ -*prime* iff, for all $I_1, I_2 \in \mathcal{I}(L)$ if $I_1 \wedge I_2 \subseteq I$, then $I_1 \subseteq I$ or $I_2 \subseteq I$.
- (iii) A proper ideal I is called $\cap$ -*irreducible* iff, for all $I_1, I_2 \in \mathcal{I}(L)$ if $I_1 \wedge I_2 = I$, then $I_1 = I$ or $I_2 = I$.
- (iv) An ideal $M \in \mathcal{I}(L)$ is called *maximal* if M is not strictly contained in a proper ideal of L . We denote by $\text{Max}(L)$ the set of all maximal ideals of L .

In the residuated lattice L , from Example 1, it is easy to ascertain that I_2 and I_3 are prime ideals of L . But the ideal I_1 is not a prime ideal because $a \wedge c = 0$ and $a, c \notin I_1$.

For any residuated lattice L , it is known that any maximal filter of L is a prime filter, but in the case of ideals that may not hold (see Examples 2 and 3).

Example 2. In this example, we present a residuated lattice with a prime ideal which is not maximal ideal. Let $L = \{0, n, a, b, c, d, 1\}$, with $0 < n < a < b < c, d < 1$ and elements c and d are incomparable.

Then [20, p. 229] L is a distributive residuated lattice with respect to the following operations:

$\rightarrow$	0	n	a	b	c	d	1	$\odot$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
n	d	1	1	1	1	1	1	n	0	0	0	0	n	0	n
a	n	n	1	1	1	1	1	a	0	0	a	a	a	a	a
b	n	n	a	1	1	1	1	b	0	0	a	b	b	b	b
c	0	n	a	d	1	d	1	c	0	n	a	b	c	b	c
d	n	n	a	c	c	1	1	d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1	1	0	n	a	b	c	d	1

It is easy to ascertain that $I_1 = (0) = \{0\}$ and $I_2 = (n) = \{0, n\}$ are prime ideals of L . Since $I_1 \subset I_2$, it follows that I_1 is a prime ideal which is not a maximal ideal.

Example 3. In this example, we present a residuated lattice with a maximal ideal which is not a prime ideal.

Let $L = \{0, a, b, c, d, e, f, m, 1\}$ with $0 < a < c < m < 1$, $0 < a < e < m < 1$, $0 < b < c < m < 1$, $0 < b < f < m < 1$, $0 < d < f < m < 1$, $0 < d < e < m < 1$ and elements $\{a, b\}$, $\{a, f\}$, $\{a, d\}$, $\{b, d\}$, $\{b, e\}$, $\{d, c\}$, $\{c, e\}$, $\{c, f\}$ and $\{e, f\}$ are pairwise incomparable.

Then [20, p. 252] L is a residuated lattice with respect to the following operations:

$\rightarrow$	0	a	b	c	d	e	f	m	1	$\odot$	0	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
a	m	1	m	1	m	1	m	1	1	a	0	0	0	0	0	0	0	0	a
b	m	m	1	1	m	m	1	1	1	b	0	0	0	0	0	0	0	0	b
c	m	m	m	1	m	m	m	1	1	c	0	0	0	0	0	0	0	0	c
d	m	m	m	m	1	1	1	1	1	d	0	0	0	0	0	0	0	0	d
e	m	m	m	m	m	1	m	1	1	e	0	0	0	0	0	0	0	0	e
f	m	m	m	m	m	m	1	1	1	f	0	0	0	0	0	0	0	0	f
m	m	m	m	m	m	m	m	1	1	m	0	0	0	0	0	0	0	0	m
1	0	a	b	c	d	e	f	m	1	1	0	a	b	c	d	e	f	m	1

It is easy to ascertain that the only proper ideal of L is $I_1 = (0) = \{0\}$, but $a \wedge b = 0 \in I_1$ and $a, b \notin I_1$. Therefore, I_1 is a maximal ideal and it is not a prime ideal of L .

Remark 3. In residuated lattices, the notions of ideals and filters are not dual. Indeed, in the residuated lattice L from Example 3 we have that $I = \{0\}$ is an ideal of L , but $F = L \setminus I = \{a, b, c, d, e, f, m, 1\}$ is not a filter (because $a \odot d = 0$ and $0 \notin F$). Hence, in residuated lattices, the study of ideals has a proper meaning.

Proposition 4. Any prime ideal is an $\cap$ -irreducible ($\cap$ -prime) ideal, but the converse may not hold.

Proof. Let $I_1, I_2 \in \mathcal{I}$ and $I \in \text{Prime}(L)$ be such that $I_1 \cap I_2 = I$. By contrary, we assume that $I_1 \neq I$ and $I_2 \neq I$, then there are elements $x \in I_1 \setminus I$ and $y \in I_2 \setminus I$. Since $x \wedge y \in I_1 \cap I_2 = I$ and I is prime, we conclude that $x \in I$ or $y \in I$, which is a contradiction. For the converse, we consider the residuated lattice L from Example 3. It

is easy to ascertain that the only proper ideal of L is $I_1 = \{0\} = \{0\}$ and I is an $\cap$ -irreducible ($\cap$ -prime) ideal of L . But I is not a prime ideal (because $a \wedge b = 0 \in I$ and $a, b \notin I$). The proof is similar for $\cap$ -prime ideals. $\square$

Remark 4. One of the most important separability properties for the ideal theory in residuated lattices is The Prime Ideal Theorem, that is, for any $I \in \mathcal{I}(L)$, and $S \subseteq L$ a nonempty $\wedge$ -closed subset of L such that $S \cap I = \emptyset$, then there is a prime ideal P of L such that $I \subseteq P$ and $P \cap S = \emptyset$. In this remark, we point that The Prime Ideal Theorem for residuated lattices may not always hold. Indeed, if we consider the residuated lattice L from Example 3, the only proper ideal is $I = \{0\}$. We consider the set $S = \{d, e, f, m\}$ to be the $\wedge$ -closed subset of L with the property that $S \cap I = \emptyset$. But there is no prime ideal P of L such that $I \subseteq P$ and $P \cap S = \emptyset$.

Remark 5. In conclusion, in the general case of residuated lattices, the concepts of prime ideals and $\cap$ -irreducible ($\cap$ -prime) ideals are not equivalent, $\text{Max}(L) \not\subseteq \text{Prime}(L)$ and The Prime Ideal Theorem may not always hold. We think it will be interesting to find a suitable subvariety of residuated lattices (in which the properties from above hold) in order to study the prime and maximal spectra, the reticulation and stable topology. We consider the variety of De Morgan residuated lattices suitable for two main reasons: first, the concepts of prime ideals and $\cap$ -irreducible ($\cap$ -prime) ideals are equivalent $\text{Max}(L) \subseteq \text{Prime}(L)$ and The Prime Ideal Theorem always hold, and, second, it includes important subvarieties of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices (as we will see in the next results).

We recall [4] that a residuated lattice L is called *De Morgan* if it satisfies the identity $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$ (see (i₈)). The variety of De Morgan residuated lattices includes important subvarieties of residuated lattices such as Boolean algebras, MV-algebras, BL-algebras, Stonean residuated lattices, MTL-algebras and involution residuated lattices, indeed, by (c₁₁), it follows that $(x \wedge y)^* = (x^{**} \wedge y^{**})^* = [(x^* \vee y^*)^*]^* = (x^* \vee y^*)^{**} = x^* \vee y^*$. We note that the classes of De Morgan residuated lattices and divisible residuated lattices (or semi-G-algebras) are different [4].

Example 4. We consider the residuated lattice L from Example 1. It is easy to ascertain that L is an involution residuated lattice, and so L is a De Morgan residuated lattice. The following sets $I_1 = \{0\}$, $I_2 = \{0, a, b\}$ and $I_3 = \{0, c, f\}$ are proper ideals of L .

Now, we present some important results concerning ideals in De Morgan residuated lattices. The following two results are characterizations for prime ideals in De Morgan RL:

Theorem 1. [4] *Let L be a De Morgan residuated lattice. For $P \in \mathcal{I}(L)$, the following conditions are equivalent:*

- (i) P is $\cap$ -prime;
- (ii) P is $\cap$ -irreducible;
- (iii) P is prime.

We present the Prime Ideal Theorem in De Morgan RL:

Theorem 2. [4, Theorem 4.14] *Let $I \in \mathcal{I}(L)$, $S \subseteq L$ a nonempty $\wedge$ -closed subset of L such that $S \cap I = \emptyset$. Then there is a prime ideal P of L such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Theorem 3. [4] *Let L be a De Morgan residuated lattice, $I \in \mathcal{I}(L)$ and $a \in L \setminus I$. Then*

- (i) *There is $P \in \text{Prime}(L)$ such that $I \subseteq P$ and $a \notin P$;*
- (ii) *I is the intersection of those prime ideals which contain I ;*
- (iii) $\cap \text{Prime}(L) = \{0\}$.

We present a characterization of maximal ideals for residuated lattices:

Theorem 4. [4] *Let L be a residuated lattice. If I is a proper ideal of L , then the following conditions are equivalent:*

- (i) $I \in \text{Max}(L)$;
- (ii) For any $x \notin I$, there exist $a \in I$, $n \geq 1$ such that $a \oplus (nx) = 1$;
- (iii) For any $x \in L$, $x \notin I$ iff $(nx)^* \in I$, for some $n \geq 1$.

4 Zariski topology on residuated lattices

The Zariski topology and reticulation of a BL-algebra were studied by L. Leuştean [12], using the filter theory. Since filters and ideals are different notions in residuated lattice (see Remark 3), using the ideal theory we study the prime spectrum $\text{Prime}(L)$, the maximal spectrum $\text{Max}(L)$, the Zariski topology and reticulation of a residuated lattice. It turns out that $\text{Prime}(L)$ and $\text{Max}(L)$ are compact T_0 topological spaces. Also, we present some applications on De Morgan residuated lattices.

4.1 The prime (maximal) spectra

In this section, some results may hold in residuated lattices, we will use comments to specify this fact.

A prime ideal P of L is called *minimal prime ideal belonging to ideal I* if $I \subseteq P$ and there is no prime ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

A minimal prime ideal belonging to $\{0\}$ is called a *minimal prime ideal*.

Corollary 1. *Let L be a De Morgan residuated lattice. Then*

- (i) *Let S be a nonempty $\wedge$ -closed subset of L and I be an ideal of L such that $S \cap I = \emptyset$. Then there exists a minimal prime ideal Q of L such that $I \subseteq Q$ and $S \cap Q = \emptyset$.*
- (ii) *Let S be a nonempty $\wedge$ -closed subset of L such that $0 \notin S$. Then there exists a minimal prime ideal Q such that $S \cap Q = \emptyset$.*

Proof.

- (i) Let I be a proper ideal of L . Consider $X = \{P \in \text{Prime}(L) : I \subseteq P, S \cap P = \emptyset\}$. By Theorem 2, we have that X is nonempty and by Zorn Lemma, X has a minimal element.
- (ii) Take $I = \{0\}$ and apply item (i). □

For each subset X of L , we define

$$V(X) = \{P \in \text{Prime}(L) : X \subseteq P\}.$$

Proposition 5. *Let L be a residuated lattice, X, Y be non-empty subsets of L . Then*

- (i) $X \subseteq Y \subseteq L$ implies $V(Y) \subseteq V(X) \subseteq \text{Prime}(L)$;
- (ii) $V(\{1\}) = \emptyset$ and $V(\emptyset) = V(\{0\}) = \text{Prime}(L)$;
- (iii) $V(X) = \emptyset$ iff $[X] = L$;
- (iv) for L a De Morgan residuated lattice we have: $V(X) = \text{Prime}(L)$ iff $X = \emptyset$ or $X = \{0\}$;
- (v) if $\{X_i\}_{i \in I}$ is a family of subsets of L , then $V(\cup_{i \in I} X_i) = \cap_{i \in I} V(X_i)$;
- (vi) $V([X]) = V(X)$;
- (vii) $V(X) \cup V(Y) = V([X] \cap [Y])$;
- (viii) for L a De Morgan residuated lattice we have: if $X, Y \subseteq L$, then $[X] = [Y]$ iff $V(X) = V(Y)$;
- (ix) for L a De Morgan residuated lattice we have: if I, J are ideals of L , then $I = J$ iff $V(I) = V(J)$.

Proof.

- (i) Obviously.
- (ii) For any P a prime ideal of L , P is a proper ideal, so $1 \notin P$, that is, $P \notin V(\{1\})$. Hence, $V(\{1\}) = \emptyset$. It is obvious that $V(\emptyset) = \text{Prime}(L)$. Since 0 is an element of any ideal of L , it follows that 0 is an element of any prime ideal of L , that is, $V(\{0\}) = \text{Prime}(L)$.
- (iii) " $\Rightarrow$." Suppose that $(X) \neq L$, that is, (X) is a proper ideal of L . Then there is a prime ideal P of L that includes the proper ideal (X) . Since $X \subseteq (X)$, it follows that $X \subseteq P$, so $P \in V(X)$. Thus, $V(X) \neq \emptyset$.
" $\Leftarrow$." If $V(X) \neq \emptyset$, then there is a prime ideal $P \in V(X)$. Since P is an ideal including X and (X) is the least ideal of L with this property, it follows that $L = (X) \subseteq P$, that is, $P = L$, a contradiction.
- (iv) " $\Leftarrow$." By (ii).
" $\Rightarrow$." Suppose that $X \neq \emptyset$ and $X \neq \{0\}$. Then there is $a \in X$, $a \neq 0$. Then there is a prime ideal P of L such that $a \notin P$. Thus, $X \not\subseteq P$, so $P \notin V(X)$. Therefore, $V(X) \neq \text{Prime}(L)$.
- (v) " $\subseteq$." We have that $X_i \subseteq \cup_{i \in I} X_i$ for all $i \in I$. Applying (i), it follows that $V(\cup_{i \in I} X_i) \subseteq V(X_i)$ for all $i \in I$, hence $V(\cup_{i \in I} X_i) \subseteq \cap_{i \in I} V(X_i)$.
" $\supseteq$." If $P \in \cap_{i \in I} V(X_i)$, then $X_i \subseteq P$ for all $i \in I$. We get that $\cup_{i \in I} X_i \subseteq P$, that is, $P \in V(\cup_{i \in I} X_i)$.
- (vi) " $\subseteq$." Let $P \in V(X)$, so $X \subseteq P$. It follows that $(X) \subseteq P$, that is, $P \in V((X))$.
" $\supseteq$." Since $X \subseteq (X)$ and applying (i) we get that $V((X)) \subseteq V(X)$.
- (vii) " $\subseteq$." Apply (i).
" $\supseteq$." Let $P \in V((X) \cap (Y))$ and suppose that $P \notin V(X) \cup V(Y)$. Hence, $P \notin V(X) = V((X))$ and $P \notin V(Y) = V((Y))$, that is, $(X) \not\subseteq P$ and $(Y) \not\subseteq P$. Thus, there are $x \in (X)$, $y \in (Y)$ such that $x, y \notin P$. Since $x \wedge y \leq x, y$ and $(X), (Y)$ are ideals of L , we get that $x \wedge y \in (X) \cap (Y) \subseteq P$. Hence, we have obtained $x, y \in L$ such that $x \wedge y \in P$ and $x, y \notin P$, a contradiction with the fact that P is a prime ideal of L .
- (viii) " $\Rightarrow$." Applying (vi), we get that $V(X) = V((X)) = V((Y)) = V(Y)$.
" $\Leftarrow$." If $(X) = L$, then $V(X) = \emptyset$, by (iii). Thus, $V(Y) = \emptyset$, so, applying again (iii), we get that $(Y) = L$. Therefore, $(X) = (Y) = L$. Now, we suppose that $(X), (Y)$ are proper ideals of L . Applying Theorem 3 and (vi), it follows that $(X) = \cap\{P \in \text{Prime}(L) : P \in V((X))\} = \cap\{P \in \text{Prime}(L) : P \in V(X)\} = \cap\{P \in \text{Prime}(L) : P \in V(Y)\} = \cap\{P \in \text{Prime}(L) : P \in V((Y))\} = (Y)$.
- (ix) Applying (viii) and the fact that, since I, J are ideals of L , we have $(I) = I$ and $(J) = J$. □

In Proposition 5, we note that items (iv), (viii) and (ix) hold in De Morgan residuated lattices because in their proofs we used Theorem 3, but the other items may hold in residuated lattices.

For any $X \subseteq L$, let us denote the complement of $V(X)$ by $U(X)$. Hence,

$$U(X) = \{P \in \text{Prime}(L) : X \not\subseteq P\}.$$

It follows that the family $\{U(X)\}_{X \subseteq L}$ is the family of the open sets of a topology on $\text{Prime}(L)$, called the Zariski topology. By duality, from Proposition 5 we get the following results.

Proposition 6. *Let X, Y be non-empty subsets of L . Then*

- (i) $X \subseteq Y \subseteq L$ implies $U(X) \subseteq U(Y) \subseteq \text{Prime}(L)$;
- (ii) $U(\{1\}) = \text{Prime}(L)$ and $U(\emptyset) = U(\{0\}) = \emptyset$;
- (iii) $U(X) = \text{Prime}(L)$ iff $(X) = L$;
- (iv) for L a De Morgan residuated lattice we have: $U(X) = \emptyset$ iff $X = \emptyset$ or $X = \{0\}$;
- (v) if $\{X_i\}_{i \in I}$ is a family of subsets of L , then $U(\cup_{i \in I} X_i) = \cup_{i \in I} U(X_i)$;
- (vi) $U((X)) = U(X)$;
- (vii) $U(X) \cup U(Y) = U((X) \cup (Y))$;
- (viii) for L a De Morgan residuated lattice we have: if $X, Y \subseteq L$, then $(X) = (Y)$ iff $U(X) = U(Y)$;
- (ix) for L a De Morgan residuated lattice we have: if I, J are ideals of L , then $I = J$ iff $U(I) = U(J)$.

In Proposition 6, we note that items (iv), (viii) and (ix) are dual to items (iv), (viii) and (ix) from Proposition 5, consequently, they hold in De Morgan residuated lattices. The other items from Proposition 6 may hold in residuated lattices.

For any $x \in L$, let us denote $V(\{x\})$ by $V(x)$ and $U(\{x\})$ by $U(x)$. Then, $V(x) = \{P \in \text{Prime}(L) : x \in P\}$ and $U(x) = \{P \in \text{Prime}(L) : x \notin P\}$.

Proposition 7. *Let $x, y \in L$. Then*

- (i) $U(x) = \text{Prime}(L)$ iff $[x] = L$;
- (ii) $U(x) = \emptyset$ iff $x = 0$;
- (iii) for L a De Morgan residuated lattice we have: $[x] = [y]$ iff $U(x) = U(y)$;
- (iv) $V(x) \subseteq U(x^*)$;
- (v) if $x \leq y$, then $U(x) \subseteq U(y)$;
- (vi) $U(x) \cap U(y) = U(x \wedge y)$;
- (vii) $U(x) \cup U(y) = U(x \oplus y) = U(x \vee y)$.

Proof. (i), (ii), (iii). Apply Proposition 6(iii), (iv) and (viii).

- (iv) Let $P \in V(x)$, hence $x \in P$. If $x^* \in P$, then $x \oplus x^* = 1 \in P$, so P is not proper. Thus, we must have $x^* \notin P$, that is, $P \in U(x^*)$.
- (v) Let $P \in U(x)$, so $x \notin P$. If $P \notin U(y)$, then $y \in P$ and from $x \leq y$ we get that $x \in P$, a contradiction.
- (vi) For any prime ideal P of L , we have that $x \wedge y \notin P$ iff $x \notin P$ and $y \notin P$. Hence, $P \in U(x \wedge y)$ iff $x \wedge y \notin P$ iff $x \notin P$ and $y \notin P$ iff $P \in U(x)$ and $P \in U(y)$ iff $P \in U(x) \cap U(y)$.
- (vii) Applying Remark 2, we get that for any proper ideal I of L , $x \oplus y \notin I$ iff $x \vee y \notin I$ iff $x \notin I$ or $y \notin I$. It follows that for any prime ideal P of L , $P \in U(x) \cup U(y)$ iff $P \in U(x \oplus y)$ iff $P \in U(x \vee y)$. $\square$

In Proposition 7, we note that item (iii) holds in De Morgan residuated lattices because in his proof we used item (viii) from Proposition 6, but the other items hold in residuated lattices.

Theorem 5. *Let L be a De Morgan residuated lattice and $x \in L$. Then*

- (i) the family $\{U(x)\}_{x \in L}$ is a basis for the topology of $\text{Prime}(L)$;
- (ii) $U(x)$ is compact in $\text{Prime}(L)$;
- (iii) the compact open subsets of $\text{Prime}(L)$ are exactly the finite unions of basic open sets;
- (iv) $\text{Prime}(L)$ is a compact T_0 topological space.

Proof.

- (i) Let $X \subseteq L$ and $U(X)$ an open subset of $\text{Prime}(L)$. Then $U(X) = U(\cup_{x \in X} \{x\}) = \cup_{x \in X} U(x)$, by Proposition 6(v). Hence, any open subset of $\text{Prime}(L)$ is the union of subsets from the family $\{U(x)\}_{x \in L}$.
- (ii) It is enough to prove that any cover of $U(x)$ with basic open sets contains a finite cover of $U(x)$. Let Λ be an index set. Let $U(x) = \cup_{i \in \Lambda} U(x_i) = U(\cup_{i \in \Lambda} \{x_i\})$. By Proposition 6(viii), we get that $[x] = (\cup_{i \in \Lambda} [x_i])$, so $x \in (\cup_{i \in \Lambda} [x_i])$. Hence, there are $n \geq 1$ and $i_1, i_2, \dots, i_n \in \Lambda$ such that $x \leq x_{i_1} \oplus x_{i_2} \oplus \dots \oplus x_{i_n}$. We shall prove that $U(x) = U(x_{i_1}) \cup U(x_{i_2}) \cup \dots \cup U(x_{i_n})$. Using Proposition 7(v) and (vii), we obtain that $U(x) \subseteq U(x_{i_1} \oplus \dots \oplus x_{i_n}) = U(x_{i_1}) \cup \dots \cup U(x_{i_n})$. The other inclusion is obvious, since $U(x_{i_1}) \cup \dots \cup U(x_{i_n}) \subseteq \cup_{i \in \Lambda} U(x_i) = U(x)$.
- (iii) Since any basic open set is compact open, a finite union of basic open sets is also compact open. Let now $U(X)$, with $X \subseteq L$, be a compact open subset of $\text{Prime}(L)$. Since $U(X)$ is open, we get that $U(X)$ is a union of basic open sets. Since $U(X)$ is compact, it follows that $U(X)$ is a finite union of basic open sets.
- (iv) Using Proposition 6(ii), we have that $\text{Prime}(L) = U(\{1\})$. Apply now (ii) to get that $\text{Prime}(L)$ is compact. It remains to prove that $\text{Prime}(L)$ is a T_0 space, which means that for any two distinct prime ideals $P \neq Q \in \text{Prime}(L)$ there is an open set U of $\text{Prime}(L)$ such that $P \in U$, $Q \notin U$ or $P \notin U$, $Q \in U$. Since $P \neq Q$, we have that $P \not\subseteq Q$ or $Q \not\subseteq P$. Assume that $P \not\subseteq Q$, so there is $x \in P$ such that $x \notin Q$. Take $U = U(x)$. Then $Q \in U$ and $P \notin U$. Similarly, if $Q \not\subseteq P$. $\square$

In Theorem 5, we note that the only item which holds in residuated lattices is (i), the items (ii), (iii) and (iv) hold in De Morgan residuated lattices.

Now, we study the maximal spectra in De Morgan residuated lattices, because we use several times the property: $\text{Max}(L) \subseteq \text{Prime}(L)$. We recall that an ideal $M \in \mathcal{I}(L)$ is called *maximal* if M is not strictly contained in a proper ideal of L . We denote by $\text{Max}(L)$ the set of all maximal ideals of L .

In the residuated lattice L , from Example 1, it is easy to ascertain that I_2 and I_3 are maximal ideals of L . From [4, Proposition 4.17], we get that $\text{Max}(L) \subseteq \text{Prime}(L)$. Then we consider on $\text{Max}(L)$ the topology induced by the Zariski topology. Thus, we obtain a topological space called *maximal spectrum* of L .

For any $X \subseteq L$ and $a \in L$, we define

$$V_{\text{Max}}(X) = V(X) \cap \text{Max}(L) = \{M \in \text{Max}(L) : X \subseteq M\},$$

$$U_{\text{Max}}(X) = U(X) \cap \text{Max}(L) = \{M \in \text{Max}(L) : X \not\subseteq M\}$$

and

$$V_{\text{Max}}(x) = V(x) \cap \text{Max}(L) = \{M \in \text{Max}(L) : x \in M\},$$

$$U_{\text{Max}}(x) = U(x) \cap \text{Max}(L) = \{M \in \text{Max}(L) : x \notin M\}.$$

It follows that the family $\{V_{\text{Max}}(X)\}_{X \subseteq L}$ is the family of closed sets of the maximal spectrum, the family $\{U_{\text{Max}}(X)\}_{X \subseteq L}$ is the family of open sets of the maximal spectrum and the family $\{U_{\text{Max}}(x)\}_{x \in L}$ is a basis for the topology of $\text{Max}(L)$.

Theorem 6. Let L be a De Morgan residuated lattice, X, Y be non-empty subsets of L and $a, b \in L$. Then

- (i) $X \subseteq Y \subseteq L$ implies $U_{\text{Max}}(X) \subseteq U_{\text{Max}}(Y) \subseteq \text{Max}(L)$;
- (ii) $U_{\text{Max}}(\{1\}) = \text{Max}(L)$ and $U_{\text{Max}}(\emptyset) = U_{\text{Max}}(\{0\}) = \emptyset$;
- (iii) $U_{\text{Max}}(X) = \text{Max}(L)$ iff $[X] = L$;
- (iv) $U_{\text{Max}}(X) = \emptyset$ iff $X = \emptyset$ or $X = \{0\}$;
- (v) if $\{X_i\}_{i \in \Lambda}$ is a family of subsets of L , then $U_{\text{Max}}(\cup_{i \in \Lambda} X_i) = \cup_{i \in \Lambda} U_{\text{Max}}(X_i)$;
- (vi) $U_{\text{Max}}([X]) = U_{\text{Max}}(X)$;
- (vii) $U_{\text{Max}}(X) \cup U_{\text{Max}}(Y) = U_{\text{Max}}([X] \cup [Y])$;
- (viii) $V_{\text{Max}}(a) \subseteq U_{\text{Max}}(a^*)$;
- (ix) if $a \leq b$, then $U_{\text{Max}}(a) \subseteq U_{\text{Max}}(b)$;
- (x) $U_{\text{Max}}(a) \cap U_{\text{Max}}(b) = U_{\text{Max}}(a \wedge b)$;
- (xi) $U_{\text{Max}}(a) \cup U_{\text{Max}}(b) = U_{\text{Max}}(a \oplus b) = U_{\text{Max}}(a \vee b)$;
- (xii) $\text{Max}(L)$ is a compact T_0 topological space.

Proof. We have only to prove (iii) and (xii), the other ones being immediate consequence of the corresponding properties for $\text{Prime}(L)$.

(iii). " $\Rightarrow$ " If $[X] \neq L$, then $[X]$ is a proper ideal of L , and there is a maximal ideal M such that $[X] \subseteq M$. It follows that $X \subseteq M$, that is, $M \notin U_{\text{Max}}(X)$. This contradicts the fact that $U_{\text{Max}}(X) = \text{Max}(L)$.

" $\Leftarrow$ " If $[X] = L$, then $U(X) = \text{Prime}(L)$, by Proposition 6(iii). Therefore, $U_{\text{Max}}(X) = \text{Max}(L)$.

(xii). We prove that $\text{Max}(L)$ is compact. Let Λ be an index set. Let $\text{Max}(L) = \cup_{i \in \Lambda} U_{\text{Max}}(x_i) = U_{\text{Max}}(\cup_{i \in \Lambda} \{x_i\})$, by

(v). By (iii), we get that $L = (\cup_{i \in \Lambda} [x_i])$, so $1 \in (\cup_{i \in \Lambda} [x_i])$. Hence, there are $n \geq 1$ and $i_1, i_2, \dots, i_n \in \Lambda$ such that $1 \leq x_{i_1} \oplus x_{i_2} \oplus \dots \oplus x_{i_n}$. We will prove that $U_{\text{Max}}(1) = U_{\text{Max}}(x_{i_1}) \cup U_{\text{Max}}(x_{i_2}) \cup \dots \cup U_{\text{Max}}(x_{i_n})$. By (ii) and (ix), we obtain that $\text{Max}(L) = U_{\text{Max}}(1) \subseteq U_{\text{Max}}(x_{i_1} \oplus \dots \oplus x_{i_n}) = U_{\text{Max}}(x_{i_1}) \cup \dots \cup U_{\text{Max}}(x_{i_n})$. The other inclusion is obvious, since $U_{\text{Max}}(x_{i_1}) \cup \dots \cup U_{\text{Max}}(x_{i_n}) \subseteq \cup_{i \in \Lambda} U_{\text{Max}}(x_i) = \text{Max}(L) = U_{\text{Max}}(1)$. Hence, $\text{Max}(L)$ is compact.

Since $\text{Max}(L) \subseteq \text{Prime}(L)$ and $\text{Prime}(L)$ is a T_0 topological space, we deduce that $\text{Max}(L)$ is a T_0 topological space. Therefore, $\text{Max}(L)$ is a compact T_0 topological space. $\square$

4.2 The reticulation functor

In this section, we study the reticulation functor $\beta : \text{DeMorgan} \rightarrow \text{LD01}$ between the categories of De Morgan residuated lattices and bounded distributive lattices. The reason why we consider De Morgan residuated lattices instead of residuated lattices (in general) is the fact that Proposition 10, Theorem 8 and Corollary 2 hold for De Morgan residuated lattices (in their proofs is used the Prime Ideal Theorem [see Theorem 2]).

For any $x, y \in L$ define the binary relation $x \equiv y$ iff $U(x) = U(y)$. Hence, $x \equiv y$ iff for any prime ideal P , $(x \notin P \text{ iff } y \notin P)$ iff for any prime ideal P , $(x \in P \text{ iff } y \in P)$.

Theorem 7. *For any residuated lattice L , the binary relation $\equiv$ is a congruence relation on L with respect to $\odot$, $\wedge$, $\vee$ and $\oplus$.*

Proof. It is routine to prove that the binary relation $\equiv$ is an equivalence relation on L . Let $x, y, z, t \in L$ be such that $x \equiv y$ and $z \equiv t$. Now, we prove that the binary relation $\equiv$ is a congruence relation on L with respect to $\wedge$, $\odot$, $\oplus$ and $\vee$, equivalently, if $x \equiv y$ and $z \equiv t$, then $x \wedge z \equiv y \wedge t$, $x \odot z \equiv y \odot t$, $x \oplus z \equiv y \oplus t$, respectively, $x \vee z \equiv y \vee t$. Since $x \equiv y$ and $z \equiv t$, it follows that $U(x) = U(y)$ and $U(z) = U(t)$. Let $P \in \text{Prime}(L)$. Since P is a prime ideal, we get that $x \wedge z \in P$ iff $x \in P$ or $z \in P$ iff $y \in P$ or $t \in P$ iff $y \wedge t \in P$. Hence, $x \wedge z \equiv y \wedge t$. We obtain similarly that $x \odot z \equiv y \odot t$. Since P is an ideal of L , we get that $x \oplus z \in P$ iff $x \in P$ and $z \in P$ iff $y \in P$ and $t \in P$ iff $y \oplus t \in P$. Hence, $x \oplus z \equiv y \oplus t$. We obtain similarly that $x \vee z \equiv y \vee t$. $\square$

Let us denote by $[x]$ the equivalence class of $x \in L$ and let $L/\equiv$ be the quotient set. We also denote by $\beta : L \rightarrow L/\equiv$ the canonical surjection defined by $\beta(x) = [x]$. The algebra $(L/\equiv, \wedge, \vee, 0, 1)$ becomes a bounded distributive lattice, where $[x] \vee [y] = [x \vee y]$ and $[x] \wedge [y] = [x \wedge y]$. We denote by $\beta(L) = L/\equiv$ and it will be called the *reticulation* of L .

An element $e \in L$ is called the Boolean element iff there is an element $e^* \in L$ such that $e \vee e^* = 1$. We denote by $B(L)$ the set of all Boolean elements of L .

Proposition 8. *Let L be a residuated lattice and $x, y \in L$. Then*

- (i) $[x] \leq [y]$ iff $U(x) \subseteq U(y)$;
- (ii) if $x \leq y$, then $[x] \leq [y]$;
- (iii) $[x] = [y]$ iff $(x) = (y)$;
- (iv) $[x] = [0]$ iff $x = 0$;
- (v) $[x] = [1]$ iff $n \cdot x = 1$ for some $n \geq 1$;
- (vi) $[x] = [n \cdot x]$ for any $n \geq 1$;
- (vii) $[x \vee y] = [x \oplus y]$;
- (viii) if $e \in B(L)$, then $[e] \leq [x]$ iff $e \leq x$.

Proof.

- (i) Following Proposition 7(vi), $[x] \leq [y]$ iff $[x] = [x] \wedge [y]$ iff $[x] = [x \wedge y]$ iff $U(x) = U(x \wedge y) = U(x) \cap U(y)$ iff $U(x) \subseteq U(y)$.
- (ii) By Proposition 7(v), if $x \leq y$, then $U(x) \subseteq U(y)$, and now we apply (i), hence $[x] \leq [y]$.
- (iii) Applying (i) we deduce that $[x] = [y]$ iff $U(x) = U(y)$. Now, applying Proposition 7(iii), we get that $[x] = [y]$ iff $U(x) = U(y)$ iff $(x) = (y)$.
- (iv) By (iii) and Proposition 2(ii), we have that $[x] = [0]$ iff $(x) = (0) = \{0\}$ iff $x = 0$.
- (v) By (iii) and Proposition 2(ii), we have that $[x] = [1]$ iff $(x) = (1) = L$ iff $1 \in (x)$ iff $n \cdot x = 1$ for some $n \geq 1$.
- (vi) By (iii) and Proposition 2(ii), we have that $[x] = [n \cdot x]$ iff $(x) = (n \cdot x)$ for any $n \geq 1$.
- (vii) Obviously, by (iii) and Proposition 7(vii).

(viii) If $[e] \leq [x]$, then $[e] = [e] \wedge [x] = [e \wedge x]$ iff $U(e) = U(e \wedge x)$ iff $(e) = (e \wedge x)$. Since $e \in B(L)$, we conclude that $e = e \wedge x$, that is, $e \leq x$. If $e \leq x$, by (ii), we get that $[e] \leq [x]$. $\square$

For any residuated lattices L and L' , a *morphism* is a function $f : L \rightarrow L'$ such that $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee y) = f(x) \vee f(y)$, $f(x \odot y) = f(x) \odot f(y)$, $f(x \rightarrow y) = f(x) \rightarrow f(y)$, $f(0) = 0$ and $f(1) = 1$. An *isomorphism* is a bijective morphism. The kernel of f is the set $\text{Ker}(f) = \{x \in L : f(x) = 0\}$.

Lemma 1. Let L, L' be two residuated lattices and $f : L \rightarrow L'$ be a morphism. Then

- (i) if I is an (proper) ideal of L' , then $f^{-1}(I)$ is an (proper) ideal of L . Thus, in particular, $\text{Ker}(f)$ is an ideal of L ;
- (ii) if P is a prime (maximal) ideal of L' , then $f^{-1}(P)$ is a prime (maximal) ideal of L ;
- (iii) if f is onto then, for any (prime, maximal) ideal P of L , $f(P)$ is a (prime, maximal) ideal of L' .

Proposition 9. Let L, L' be two residuated lattices and $f : L \rightarrow L'$ be a morphism. For any $x, y \in L$, $U(x) = U(y)$ implies $U(f(x)) = U(f(y))$.

Proof. Let P be a prime ideal of L' . Following Lemma 1(ii) we have that $f^{-1}(P)$ is a prime ideal of L . It follows that $P \in U(f(x))$ iff $f(x) \notin P$ iff $x \notin f^{-1}(P)$ iff $f^{-1}(P) \in U(x)$ iff $f^{-1}(P) \in U(y)$ iff $y \notin f^{-1}(P)$ iff $f(y) \notin P$ iff $P \in U(f(y))$. Therefore, $U(f(x)) = U(f(y))$. $\square$

Proposition 10. Let L be a De Morgan residuated lattice. If I is an ideal of L and $x, y \in L$ such that $[x] = [y]$, then $x \in I$ iff $y \in I$.

Proof. If $I = L$, it is obvious. Now, assume I is a proper ideal of L . By contrary, we suppose that $x \in I$ and $y \notin I$. Following the Prime Ideal Theorem (see Theorem 2) with I and $S = \{y\}$, there is a prime ideal P such that $I \subseteq P$ and $y \notin P$. So, $P \in U(y)$, but $P \notin U(x)$, since $x \in I \subseteq P$. We conclude that $U(x) \neq U(y)$, that is, $[x] \neq [y]$, a contradiction. $\square$

Let L, L' be two residuated lattices. If $f : L \rightarrow L'$ is a morphism, we define $\beta(f) : \beta(L) \rightarrow \beta(L')$ by $\beta(f(x)) = [f(x)]$.

Proposition 11. $\beta(f)$ is a bounded lattice morphism.

Proof. Let $x, y \in L$. Then we obtain successively $\beta(f)([x] \wedge [y]) = \beta(f)([x \wedge y]) = [f(x \wedge y)] = [f(x) \wedge f(y)] = [f(x)] \wedge [f(y)] = \beta(f)([x]) \wedge \beta(f)([y])$. In a similar manner, we get that $\beta(f)([x] \vee [y]) = \beta(f)([x]) \vee \beta(f)([y])$. Finally, $\beta(f)([0]) = [f(0)] = [0]$ and $\beta(f)([1]) = [f(1)] = [1]$. $\square$

We defined a functor $\beta : \text{DeMorgan} \rightarrow \text{LD01}$ between the categories of De Morgan residuated lattices and bounded distributive lattices called the *reticulation functor*. For any ideal $I \in \mathcal{I}(L)$, we consider the set $\beta(I) = \{[x] : x \in I\}$, and for any lattice ideal J of $\beta(L)$, $J \in \mathcal{I}(\beta(L))$, we consider the set $J_* = \{\beta^{-1}(J)\}$. It is clear that by $\mathcal{I}(L)$ we understand the set of all ideals of L , and by $\mathcal{I}(\beta(L))$ we understand the set of all lattice ideals of the distributive lattice $\beta(L)$. Clearly, if $I \subseteq J$, then $\beta(I) \subseteq \beta(J)$.

We consider the maps $i : \mathcal{I}(L) \rightarrow \mathcal{I}(\beta(L))$, $i(I) = \beta(I)$, for any ideal I of L and $q : \mathcal{I}(\beta(L)) \rightarrow \mathcal{I}(L)$, $q(J) = J_*$, for every ideal J of $\beta(L)$.

Theorem 8. Let L be a De Morgan residuated lattice. Then

- (i) if $I \in \mathcal{I}(L)$, then for any $x \in L$, $[x] \in \beta(I)$ iff $x \in I$;
- (ii) if $I \in \mathcal{I}(L)$, then $\beta(I) \in \mathcal{I}(\beta(L))$;
- (iii) if $J \in \mathcal{I}(\beta(L))$, then $J_* \in \mathcal{I}(L)$;
- (iv) if $I \in \mathcal{I}(L)$, then $(\beta(I))_* = I$;
- (v) if $J \in \mathcal{I}(\beta(L))$, then $\beta(J_*) = J$;

- (vi) if $I, J \in \mathcal{I}(L)$, then $I \subseteq J$ iff $\beta(I) \subseteq \beta(J)$;
- (vii) the mapping $I \rightarrow \beta(I)$ is an isomorphism between the lattices $\mathcal{I}(L)$ and $\mathcal{I}(\beta(L))$.

Proof.

- (i) Assume that $[x] \in \beta(I)$, then there is $y \in I$ such that $[x] = [y]$. Now, following Proposition 10 we get that $x \in I$.
- (ii) Since $0 \in I$, so $[0] \in \beta(I)$, hence, $\beta(I)$ is nonempty. Let $x, y \in L$ such that $[x], [y] \in \beta(I)$. By (i), we have that $x, y \in I$. Since I is an ideal and $x, y \in I$, we get that $x \vee y \in I$, that is, $[x \vee y] \in \beta(I)$. Let $x, y \in L$ such that $[x] \leq [y]$ and $[y] \in \beta(I)$. We must prove that $[x] \in \beta(I)$. Since $[y] \in \beta(I)$, $y \in I$, so $x \wedge y \in I$. Since $[x \wedge y] = [x] \wedge [y] = [x]$ and $x \wedge y \in I$, we obtain that $[x] \in \beta(I)$. Hence, $\beta(I) \in \mathcal{I}(\beta(L))$.
- (iii) Since $[0] \in J$, $0 \in J_*$. Let $x, y \in J_*$, so $[x], [y] \in J$. By Proposition 8(vii), $[x \oplus y] = [x \vee y] = [x] \vee [y] \in J$, that is, $x \oplus y \in J_*$. Let $y \in J_*$ and $x \in L$ such that $x \leq y$. We must prove that $x \in J_*$. Following Proposition 8(ii), we obtain that $[x] \leq [y]$ and, since $[y] \in J$, we get that $[x] \in J$, that is, $x \in J_*$.
- (iv) Let $x \in L$. By (i), we get that $x \in (\beta(I))_*$ iff $[x] \in \beta(I)$ iff $x \in I$. Hence, $(\beta(I))_* = I$.
- (v) Let $x \in L$. By (i), we get that $[x] \in \beta(J_*)$ iff $x \in J_*$ iff $[x] \in J$. Hence, $\beta(J_*) = J$.
- (vi) By (i), we get $I \subseteq J$ iff for any $x \in L$, $x \in I$ implies $x \in J$ iff for any $x \in L$, $[x] \in \beta(I)$ implies $[x] \in \beta(J)$ iff $\beta(I) \subseteq \beta(J)$.
- (vii) Following (ii) and (iii) we deduce that i, q are well-defined. Now, following (iv) and (v), we deduce that i is a bijection and its inverse is q . Finally, from (vi) we obtain that i is a lattice morphism. Therefore, i is a bijective morphism of lattices, that is, an isomorphism of lattices. $\square$

Since $(\beta(L), \subseteq)$ is a distributive lattice, we will denote by $\text{Spec}_{\text{Id}}(\beta(L))$ the set of all its lattice prime ideals, respectively, by $\text{Max}_{\text{Id}}(\beta(L))$ the set of all its lattice maximal ideals.

Corollary 2. Let L be a De Morgan residuated lattice. Then

- (i) I is a proper ideal of L iff $\beta(I)$ is a proper ideal of $\beta(L)$;
- (ii) $P \in \text{Prime}(L)$ iff $\beta(P) \in \text{Spec}_{\text{Id}}(\beta(L))$;
- (iii) $M \in \text{Max}(L)$ iff $\beta(M) \in \text{Max}_{\text{Id}}(\beta(L))$;
- (iv) the mapping $P \rightarrow \beta(P)$ is a homeomorphism between the topological spaces $\text{Prime}(L)$ and $\text{Spec}_{\text{Id}}(\beta(L))$;
- (v) the mapping $M \rightarrow \beta(M)$ is a homeomorphism between the topological spaces $\text{Max}(L)$ and $\text{Max}_{\text{Id}}(\beta(L))$.

Proof. By Theorem 8(vii), it follows that $I \in \mathcal{I}(L)$ iff $\beta(I) \in \mathcal{I}(\beta(L))$. In the proof, we will apply several times Theorem 8(i).

- (i) I is a proper ideal of L iff $1 \notin I$ iff $[1] \notin \beta(I)$ iff $\beta(I)$ is a proper ideal of $\beta(L)$.
- (ii) $P \in \text{Prime}(L)$ iff P is a proper ideal and for any $x, y \in L$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$ iff $\beta(P)$ is a proper ideal and for any $x, y \in L$, $[x] \wedge [y] = [x \wedge y] \in \beta(P)$ iff $[x] \in \beta(P)$ or $[y] \in \beta(P)$ iff $\beta(P) \in \text{Spec}_{\text{Id}}(\beta(L))$.
- (iii) By (i) and Theorem 8(vii), we get that $M \in \text{Max}(L)$ iff M is proper and for any proper ideal I of L , $M \subseteq I$ implies $I = M$ iff $\beta(M)$ is proper and for any proper ideal $\beta(I)$ of $\beta(L)$, $\beta(M) \subseteq \beta(I)$ implies $\beta(I) = \beta(M)$ iff $\beta(M) \in \text{Max}_{\text{Id}}(\beta(L))$.
- (iv) Let us consider the restriction of i to $\text{Prime}(L)$, denoted by $\bar{i}$. By (ii) and Theorem 8(vii), we deduce that $\bar{i} : \text{Prime}(L) \rightarrow \text{Spec}_{\text{Id}}(\beta(L))$ is bijective. In order to prove that $\bar{i}$ is a homeomorphism, we will prove that $\bar{i}$ is continuous and open. Let $x \in L$. Then we obtain successively $\bar{i}^{-1}(U([x])) = \{P \in \text{Prime}(L) : \bar{i}(P) \in U([x])\} = \{P \in \text{Prime}(L) : \beta(P) \in U([x])\} = \{P \in \text{Prime}(L) : [x] \notin \beta(P)\} = \{P \in \text{Prime}(L) : x \notin P\} = U(x)$. Therefore, $\bar{i}$ is continuous.

Now, we obtain successively $\bar{i}(U(x)) = \{\beta(P) : P \in \text{Prime}(L), P \in U(x)\} = \{\beta(P) : P \in \text{Prime}(L), x \notin P\} = \{\beta(P) : P \in \text{Prime}(L), [x] \notin \beta(P)\} = \{Q \in \text{Spec}_{\text{Id}}(\beta(L)) : [x] \notin Q\} = U([x])$. Therefore, $\bar{i}$ is open.

- (v) Let us consider the restriction of i to $\text{Max}(L)$, denoted by $\bar{i}$. By (iii) and Theorem 8(vii), we deduce that $\bar{i} : \text{Max}(L) \rightarrow \text{Max}_{\text{Id}}(\beta(L))$ is bijective. By the proof of (iv), it follows that for any $x \in L$, $\bar{i}^{-1}(U_{\text{Max}}([x])) =$

$\bar{i}^{-1}(U([x]) \cap \text{Max}_{\text{Id}}(\beta(L))) = \bar{i}^{-1}(U([x])) \cap \text{Max}(L) = U_{\text{Max}}(x)$ and $\bar{i}(U_{\text{Max}}(x)) = \bar{i}(U(x) \cap \text{Max}(L)) = \bar{i}(U(x)) \cap \text{Max}_{\text{Id}}(\beta(L)) = U([x]) \cap \text{Max}_{\text{Id}}(\beta(L)) = U_{\text{Max}}([x])$. Therefore, $\bar{i}$ is continuous and open. $\square$

Corollary 3. *Let L be a De Morgan residuated lattice. Then the lattice $(\mathcal{I}(L), \subseteq)$ is distributive.*

Proof. The proof is a direct consequence of Theorem 8(vii), since the lattice $(\mathcal{I}(\beta(L)), \subseteq)$ is distributive. The same result had been proved in [4, Proposition 4.11]. $\square$

4.3 The stable topology

In this section, we study stable topology in the general case of residuated lattices. If we present results for De Morgan residuated lattices, we will clearly specify that.

The open sets in $\text{Prime}(L)$ are of the form $U(X) = \{P \in \text{Prime}(L) : X \not\subseteq P\}$, where X is a nonempty subset of L . The set $U(X)$ is stable under ascent if $P \in U(X)$ and $Q \in \text{Prime}(L)$ with $P \subseteq Q$, then $Q \in U(X)$. The set $U(X)$ is stable under descent if $P \in U(X)$ and $Q \in \text{Prime}(L)$ with $Q \subseteq P$, then $Q \in U(X)$. $U(X)$ is said to be stable if it is stable under ascent and under descent. The stable topology for L is the collection of stable open subsets of $\text{Prime}(L)$.

Now, we are planning to introduce I -topology on $\text{Prime}(L)$.

Theorem 9. *Let L be a residuated lattice, with I an ideal of L and Λ an index set. For $V(I) = \{P \in \text{Prime}(L) : I \subseteq P\}$, the following assertions hold:*

- (i) $V(\{0\}) = \text{Prime}(L)$ and $V(L) = \{\emptyset\}$;
- (ii) if $\{I_i\}_{i \in \Lambda}$ is a family of ideals of L , then $V(\cup_{i \in \Lambda} I_i) = \cap_{i \in \Lambda} V(I_i)$;
- (iii) if I_1, I_2 are ideals of L , then $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$.

Proof. Let I be an ideal of L . By Proposition 2(i), we have that $I = [I]$. Now, we apply Proposition 5(ii), (iii), (v) and (vii). $\square$

For any ideals I of L , the complement of $V(I)$ is the set $U(I) = \{P \in \text{Prime}(L) : I \not\subseteq P\}$. We note that for a residuated lattice L , the I -topology is in fact the Zariski topology restricted to $\text{Id}(L)$. Based on Proposition 6, Theorems 5 and 9, we have the following.

Corollary 4. *The collection $\{V(I) : I \text{ is an ideal of } L\}$ defines a topology on $\text{Prime}(L)$ whose closed sets are of the form $V(I)$ for some ideal I in L . We call the resulting topology, I -topology. The family $\{U(x)\}_{x \in L}$, where $U(x) = \{P \in \text{Prime}(L) : x \notin P\}$, is a basis for the I -topology. Any open subset of $\text{Prime}(L)$ with I -topology (Zariski topology) is the union of subsets from the family $\{U(x)\}_{x \in L}$. Moreover, if L is a De Morgan residuated lattice, then $\text{Prime}(L)$ with I -topology is a compact T_0 topological space.*

Proposition 12. *Let L be a residuated lattice and I, J two proper ideals of L . Then $U(I) \cap U(J) = U((I \cup J)) = U(I \vee J)$.*

Proof. Let I, J be proper ideals of L . By Proposition 2, we have that $(I \cup J) = I \vee J$. Then $U((I \cup J)) = U(I \vee J)$. Now, we prove that $U(I) \cap U(J) = U((I \cup J))$. Since $I, J \subseteq I \vee J$, it follows that $U(I) \cap U(J) \subseteq U((I \cup J))$. For the converse, let $x \in U((I \cup J))$. Then $x \notin (I \cup J)$, $x \notin I \vee J$, $x \notin I$ and $x \notin J$, consequently, $x \in U(I)$ and $x \in U(J)$, $x \in U(I) \cap U(J)$. $\square$

Remark 6.

- (i) If I is an ideal of L , then $V(I)$ is stable under ascent. Indeed, if $P \in V(I)$ and $Q \in \text{Prime}(L)$ be such that $P \subseteq Q$, then $Q \in V(I)$ (because $Q \in \text{Prime}(L)$ and $I \subseteq P \subseteq Q$).

- (ii) We also have that $U(I) = \text{Prime}(L) \setminus V(I) = \{P \in \text{Prime}(L) : I \not\subseteq P\}$ is stable under descent. Indeed, if $P \in U(I)$ and $Q \in \text{Prime}(L)$ be such that $Q \subseteq P$, then $Q \in U(I)$ (because $Q \in \text{Prime}(L)$ and $I \not\subseteq P, Q \subseteq P$, so $I \not\subseteq Q$).
- (iii) Let L be a De Morgan residuated lattice. For every ideals I, J of L , $I = J$ iff $U(I) = U(J)$. Since $I = [I]$, for any ideal I of L , by Proposition 6(viii), we obtain the result.

We say that $U(I)$ is stable if $U(I)$ is stable under ascent and descent, for any ideal I of L . Since $U(I)$ is always stable under descent, being stable it is enough that $U(I)$ is stable under ascent. In the following result, we introduce the stable topology on $\text{Prime}(L)$.

Theorem 10. *The collection of all stable open subsets of $\text{Prime}(L)$ satisfies the axioms for open sets in a topological space. The resulting topology is called stable topology on $\text{Prime}(L)$. In other words, $\{U : U \text{ is open with } I\text{-topology and stable}\}$ is the collection of open sets for stable topology.*

Proof. Let O be the set of all stable open subsets of $\text{Prime}(L)$. Clearly, $\emptyset$ and $\text{Prime}(L) \in O$. Now, let O_1 and O_2 be in O . Then $O_1 = U(I_1)$ and $O_2 = U(I_2)$ for some I_1, I_2 ideals of L . Since $U(I_1) \cap U(I_2) = U((I_1 \cup I_2))$, it follows that $O_1 \cap O_2$ is open. For stability, it suffices to prove that $O_1 \cap O_2$ is stable under ascent. Let $P \in U((I_1 \cup I_2))$, $Q \in \text{Prime}(L)$ and $P \subseteq Q$. Then $P \in U(I_1)$ and $P \in U(I_2)$ and hence by stability of $U(I_1), U(I_2)$, we conclude that $Q \in U(I_1), U(I_2)$. Therefore, $Q \in U(I_1) \cap U(I_2)$. So, we obtain that $Q \in U((I_1 \cup I_2))$, that is, $O_1 \cap O_2$ is stable under ascent.

Let $\{O_i\}_{i \in \Lambda}$ be a family of stable open subsets of $\text{Prime}(L)$, with Λ an index set. Then for each $i \in \Lambda$, there is an ideal I_i of L such that $O_i = U(I_i)$. We have that $\cup_{i \in \Lambda} O_i = \cup_{i \in \Lambda} U(I_i) = U(\cup_{i \in \Lambda} I_i)$. For stability, let $P \in U(\cup_{i \in \Lambda} I_i)$, $Q \in \text{Prime}(L)$ and $P \subseteq Q$, then $P \in U(I_i)$ for some $i \in \Lambda$ and by stability of $U(I_i)$ we have that $Q \subseteq U(I_i) \subseteq \cup_{i \in \Lambda} U(I_i)$. Therefore, $Q \in \cup_{i \in \Lambda} U(I_i) = U(\cup_{i \in \Lambda} I_i)$. $\square$

Until this moment, the aforementioned results on I -topology and stable topology may hold for residuated lattices, in general. We note that the I -topology is in fact the Zariski topology restricted to $I(L)$ the set of all ideals of L .

For De Morgan residuated lattices, in the next result, we show that there is a distinction between topological property of $\text{Prime}(L)$ with stable topology and I -topology. In fact, $\text{Prime}(L)$ with I -topology is a compact T_0 topological space, but with stable topology it is a compact space but not a T_0 topological space.

Proposition 13. *With the stable topology, $\text{Prime}(L)$ is a compact topological space but not T_0 and hence neither T_1 nor T_2 .*

Proof. It is easy to ascertain that every stable open set is also open in I -topology. By Corollary 4, we have that $\text{Prime}(L)$ is compact in I -topology, hence it is also compact in stable topology. Now, let $P, Q \in \text{Prime}(L)$ be such that $P \subsetneq Q$. We know that all open sets $U(I)$, for some ideal I of L , are stable under descent, hence every $U(I)$ that contains Q will contain P . Now, we assume that $U(I)$ is stable, and $P \in U(I)$. Since Q contains P , and $U(I)$ is stable, it follows that $Q \in U(I)$. We conclude that P and Q cannot be separated by stable open sets. Therefore, $\text{Prime}(L)$ with stable topology is a compact topological space, but it is not T_0 and neither T_1 nor T_2 . $\square$

We note that in the residuated lattice L from Example 2, if we consider $P = \{0\}$ and $Q = \{0, n\}$, it is easy to ascertain that P and Q are prime ideals such that $P \subsetneq Q$.

Remark 7. If we consider L to be a Boolean residuated lattice (that is, $x \vee x^* = 1$ for all $x \in L$), then with the stable topology, $\text{Prime}(L)$ is a compact topological space, T_0, T_1 or T_2 . Indeed, in the case of Boolean residuated lattices the topologies (Zariski, I -topology and stable topology) coincide, so the conclusion is obvious.

4.4 The pure ideals

We recall [14] that a filter F of L is *pure* iff $U(F)$ is stable. Some details on the concept of pure filter can be found in [14,15]. In this section, we introduce the notion of pure ideal in a De Morgan residuated lattice. We show that pure ideals and pure filters are not dual notions and we study some specific properties of them. The results from this section may not hold in residuated lattices (in general) because we use several times the property that $\text{Max}(L) \subseteq \text{Prime}(L)$ and the fact that any annihilator is an ideal [4, Proposition 4.24]. Consequently, in this section, by L we refer to a De Morgan residuated lattice.

Definition 5. We say that an ideal I of L is *pure* iff $U(I)$ is stable.

Clearly, $\{0\}$ and L are pure ideals. The fact that pure ideals and pure filters are not dual notions is a simple consequence of the fact that ideals and filters are not dual, as we can see in the following example:

Example 5. In this example, we present proper ideals which are pure ideals and we show that pure ideals and pure filters are not dual notions in De Morgan residuated lattices.

In the residuated lattice L , from Example 1, the proper ideals of L are $I_1 = \{0\}$, $I_2 = \{0, a, b\}$ and $I_3 = \{0, c, f\}$. It is easy to ascertain that $I_2, I_3 \in \text{Prime}(L)$ and I_1, I_2 , and I_3 are pure ideals of L . Moreover, $I_1 = \{0\}$ is a pure ideal and its complement set denoted by $F = L \setminus I_1 = \{a, b, c, d, e, f, g, 1\}$ is not a pure filter of L (because $a \odot c = 0 \notin F$, that is, F is not a filter).

Proposition 14. Let I be a pure ideal of L , $P \in \text{Prime}(L)$, $M \in \text{Max}(L)$ such that $P, I \subseteq M$. Then $I \subseteq P$.

Proof. Let I be a pure ideal of L , $P \in \text{Prime}(L)$, $M \in \text{Max}(L)$ and $P, I \subseteq M$. By the contrary, we suppose that $I \not\subseteq P$. Then $P \in U(I)$. Since L is a De Morgan residuated lattice, it follows that $\text{Max}(L) \subseteq \text{Prime}(L)$. Since $P \subseteq M$, $M \in \text{Max}(L)$ and $\text{Max}(L) \subseteq \text{Prime}(L)$, by stability of $U(I)$, we conclude that $M \in U(I)$, that is, $I \not\subseteq M$, which is a contradiction. $\square$

The following result is a direct consequence of Proposition 14:

Theorem 11. Every prime and pure ideal of L is a minimal prime ideal.

We recall [4] that if I is an ideal of L , then we say the set $I^\perp = \{x \in L : a \wedge x = 0 \text{ for all } a \in I\}$ is an *annihilator* of I . For any element $x \in I$, the annihilator generated by element x is denoted by $x^\perp = \{a \in L : x \wedge a = 0\}$ and it is an ideal of L .

Lemma 2. Let I be a pure ideal of L and $x \in I$. Then we have $x^\perp \vee I = L$.

Proof. Let I be a pure ideal of L , that is, $U(I)$ is stable. By contrary, we suppose that $x^\perp \vee I \neq L$, for some $x \in I$, that is, $x^\perp \vee I \subset L$. In other words, $x^\perp \vee I$ is a proper ideal of L . So, there is a maximal ideal M of L such that $x^\perp \vee I \subseteq M$. Then we have $x^\perp \subseteq M$. We define $X := \{x \wedge y : y \notin M\}$. Since $x = x \wedge 1$, it follows that $x \in X$ and X is nonempty. Now, let $x \wedge y$ and $x \wedge z$ be two elements of X . Then we have $(x \wedge y) \wedge (x \wedge z) = x \wedge (y \wedge z) \in X$. Since $M \in \text{Max}(L) \subseteq \text{Prime}(L)$, it follows that M is a prime ideal and $y \notin M$ and $z \notin M$. So, we have $y \wedge z \notin M$. Hence, X is an $\wedge$ -closed subset of L . Now, we claim that $0 \notin X$ (otherwise, if $0 \in X$, $x \wedge y = 0$ for $y \notin M$, that is, $y \in x^\perp$ and since $x^\perp \subseteq M$, it follows that $y \in M$, which is a contradiction). Hence, X is an $\wedge$ -closed subset of L with $0 \notin X$, by Corollary 1, there is a minimal prime ideal Q such that $Q \cap X = \emptyset$. There are two possibilities: first, we consider that $Q \subseteq M$. Since $x = x \wedge 1 \in X \cap I$, we have $I \not\subseteq Q$, which means $Q \in U(I)$. Since $Q \in U(I)$ and $Q \subseteq M$, by stability of $U(I)$, we conclude that $M \in U(I)$, that is, $I \not\subseteq M$. Since $I \subseteq x^\perp \vee I \subseteq M$, it follows that $I \subseteq M$, which is a contradiction. Second, we consider that $Q \not\subseteq M$. Let $t \in Q \setminus M$. Since $0 \in M$ and $x^\perp \subseteq M$, it follows that

$t \neq 0$ and $t \notin x^\perp$. Consequently, $x \wedge t \neq 0$. Since $t \notin M$, we get $x \wedge t \in X$. But $x \wedge t \in X \cap Q = \emptyset$, which is a contradiction. Therefore, $x^\perp \vee I = L$. $\square$

Lemma 3. *Let I be an ideal of L such that for each $x \in L$, $x^\perp \vee I = L$. Then I is a pure ideal of L .*

Proof. Let I be an ideal of L such that for each $x \in L$, $x^\perp \vee I = L$. In order to prove that I is a pure ideal of L , it is sufficient to prove that $U(I)$ is stable. We consider $P \in U(I)$, $P \subseteq Q$ and $Q \in \text{Prime}(L)$. We need to prove that $Q \in U(I)$. By contrary, we suppose that $Q \notin U(I)$, which implies $I \subseteq Q$. We consider a minimal prime ideal J such that $J \subseteq P$. Then $I \not\subseteq J$. In fact, if $I \subseteq J \subseteq P$, then $I \subseteq P$ and hence $P \notin U(I)$, which is a contradiction. Hence $I \not\subseteq J$. Thus, there is an element $x \in I \setminus J$ such that $x^\perp \subseteq J$ (otherwise, if for each $x \in I \setminus J$, $x^\perp \not\subseteq J$, consequently, there exists $t \in x^\perp$, $t \notin J$ and we have $t \wedge x = 0 \in J$, but $x \notin J$ and $t \notin J$, which is impossible). On the other hand, we have that $J \subseteq P \subseteq Q$. Then $x^\perp \subseteq Q$. Since $I \subseteq Q$, it follows that $(x^\perp \vee I) \subseteq Q$, consequently, $x^\perp \vee I \subseteq Q$. Thus, $L = Q$, which is a contradiction. Therefore, $Q \in U(I)$. $\square$

In the following result, we propose a characterization for pure ideals.

Theorem 12. *Let I be an ideal of L and $x \in I$. Then I is a pure ideal of L iff $x^\perp \vee I = L$.*

Proof. The result follows by Lemmas 2 and 3. $\square$

Corollary 5. *Let I be an ideal of L . Then I is a pure ideal iff $U(I) = \bigcup_{x \in I} V(x^\perp)$.*

Proof. Let I be an ideal of L . It suffices to prove that $U(I)$ is stable iff $U(I) = \bigcup_{x \in I} V(x^\perp)$. We consider $U(I) = \bigcup_{x \in I} V(x^\perp)$, $P \in U(I)$, $Q \in \text{Prime}(L)$ and $P \subseteq Q$. Then $I \not\subseteq P$. Therefore, there is an element $x \in I$ such that $x^\perp \subseteq P \subseteq Q$ and hence $P \in V(x^\perp)$. Since $x^\perp \subseteq Q$, it follows that $Q \in V(x^\perp) \subseteq \bigcup_{x \in I} V(x^\perp) = U(I)$, consequently, $U(I)$ is stable.

Now, let $P \in U(I)$, that is, $I \not\subseteq P$. Therefore, there is an element $x \in I \setminus P$ such that $x^\perp \subseteq P$, that is, $P \in V(x^\perp) \subseteq \bigcup_{x \in I} V(x^\perp)$.

Conversely, let $P \in \bigcup_{x \in I} V(x^\perp)$. Then there is an element $y \in I$ such that $P \in V(y^\perp)$, that is, $y^\perp \subseteq P$. We see that $I \not\subseteq P$. In fact, if $I \subseteq P$, since $y^\perp \subseteq P$, we conclude that $y^\perp \vee I \subseteq P$. By the stability of $U(I)$ and Theorem 12, we have $L \subseteq P$, which is a contradiction. Therefore, $P \in U(I)$. $\square$

Definition 6. A De Morgan residuated lattice L is called *normal* iff each ideal I of L is a pure ideal.

If we consider the residuated lattice L , from Example 1, since $I_1 = \{0\}$, $I_2 = \{0, a, b\}$ and $I_3 = \{0, c, f\}$ are all pure ideals of L , we conclude that L is a normal lattice. Boolean algebras and MV-algebras are examples of normal residuated lattices. The next result is a direct consequence of Theorem 12.

Corollary 6. *The following assertions are equivalent:*

- (i) L is normal;
- (ii) for each proper ideal I of L and $x \in I$, $x^\perp \vee I = L$;
- (iii) for each proper ideal I of L , $U(I) = \bigcup_{x \in I} V(x^\perp)$.

Remark 8. We note that if L is normal then every prime ideal is contained in a unique maximal ideal. Let $P \in \text{Prime}(L)$. If by contrary, we suppose that there are two distinct maximal ideals M_1 and M_2 of L such that $P \subseteq M_1$ and $P \subseteq M_2$. Since L is normal, we conclude that $U(M_1)$ and $U(M_2)$ are stable, that is, $P \in U(M_1) \cap U(M_2)$, $P \in U((M_1 \cup M_2))$, $P \notin (M_1 \cup M_2)$, which is a contradiction. Therefore, $M_1 = M_2$. The converse remains an open problem.

5 Conclusions

In this paper, we studied prime and maximal spectra, the reticulation functor, the stable topology and pure ideals in varieties of residuated lattices. In the general case of residuated lattices, the development of an ideal theory faced some difficulties such as $\text{Max}(L) \not\subseteq \text{Prime}(L)$ and the absence of The Prime Ideal Theorem, we paid attention to the class of De Morgan residuated lattices (L is called De Morgan residuated lattice if the De Morgan law $(x \wedge y)^* = x^* \vee y^*$, for all $x, y \in L$ holds). We extended the study to the prime and maximal spectra, the reticulation functor, the stable topology and pure ideals. We recalled some important facts: the open sets in $\text{Prime}(L)$ are of the form $U(X) = \{P \in \text{Prime}(L) : X \not\subseteq P\}$, where X is a nonempty subset of L and $U(x) = \{P \in \text{Prime}(L) : x \notin P\}$ is a basis for the topology of $\text{Prime}(L)$. Also, $U_{\text{Max}}(X) = U(X) \cap \text{Max}(L) = \{M \in \text{Max}(L) : X \not\subseteq M\}$ is the family of open sets of the maximal spectrum and the family $U_{\text{Max}}(x) = U(x) \cap \text{Max}(L) = \{M \in \text{Max}(L) : x \notin M\}$ is a basis for the topology of $\text{Max}(L)$. We proved that the prime and maximal spectra are compact T_0 topological spaces, and if the residuated lattice L is De Morgan, then they are compact T_0 topological spaces. We defined the reticulation functor between the category of De Morgan residuated lattices and bounded distributive residuated lattices and studied its properties. Moreover, we defined the stable sets and based on them we introduced the stable topology as follows. The set $U(X)$ is stable under ascent if $P \in U(X)$ and $Q \in \text{Prime}(L)$ with $P \subseteq Q$, then $Q \in U(X)$. The set $U(X)$ is stable under descent if $P \in U(X)$ and $Q \in \text{Prime}(L)$ with $Q \subseteq P$, then $Q \in U(X)$. $U(X)$ is said to be stable if it is stable under ascent and under descent. The stable topology for L is the collection of stable open subsets of $\text{Prime}(L)$. In Corollary 4, we stated the stable topology. Moreover, by means of stable topology, we defined pure ideals and studied their properties. Also, we introduced the notion of normal De Morgan residuated lattice.

In our future work, we will continue our study of algebraic properties of ideals in residuated lattices. We will use these ideals to define congruence relations on L and to study the properties of the quotient residuated lattice of L . We will study the uniform topology induced by this congruence relations. It seems that the residuated lattices can be studied from ideal theory view in a very nice way.

Acknowledgments: The author is extremely grateful to the editors and the anonymous reviewers for giving him many valuable comments and helpful suggestions which helps to improve the presentation of this paper.

References

- [1] M. Ward, *Residuated distributive lattices*, Duke Math. J. **6** (1940), no. 3, 641–651.
- [2] M. Ward and R. P. Dilworth, *Residuated lattices*, Trans. Am. Math. Soc. **45** (1939), 335–354.
- [3] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, 1st edition, Volume 151, Studies in Logic and the Foundations of Mathematics, Elsevier, 2007.
- [4] L. C. Holdon, *On ideals in De Morgan residuated lattices*, Kybernetika **54** (2018), no. 3, 443–475.
- [5] C. Lele and J. B. Nganou, *MV-algebras derived from ideals in BL-algebra*, Fuzzy Sets and Systems **218** (2013), 103–113.
- [6] F. Forouzesh, E. Eslami, and A. Borumand Saeid, *On obstinate ideals in MV-algebras*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **76** (2014), no. 2, 53–62.
- [7] A. Borumand Saeid and M. Pourkhatun, *Obstinate filters in resituated lattices*, Bull. Math. Soc. Sci. Math. Roumanie **55** (2012), no. 4, 413–422.
- [8] D. Buşneag, D. Piciu, and L. C. Holdon, *Some properties of ideals in Stonean residuated lattice*, J. Mult-Valued Log. S. **24** (2015), no. 5–6, 529–546.
- [9] Y. X. Zou, X. L. Xin, and P. F. He, *On annihilators in BL-algebras*, Open Math. **14** (2016), 324–337.
- [10] A. Paad, *Integral ideals and maximal ideals in BL-algebras*, An. Univ. Craiova Ser. Mat. Inform. **43** (2016), no. 2, 231–242.
- [11] L. C. Holdon, *New topology in residuated lattices*, Open Math. **16** (2018), 1104–1127.
- [12] L. Leuştean, *The prime and maximal spectra and the reticulation of BL-algebras*, Cent. Eur. J. Math. **1** (2003), no. 3, 382–397.
- [13] M. Haveshki, E. Eslami, and A. Borumand Saeid, *A Topology Induced by Uniformity on BL-algebras*, MLQ Math. Log. Q. **53** (2007), no. 2, 162–169.

- [14] E. Eslami and F. Kh. Haghani, *Pure filters and stable topology on BL-algebras*, Kybernetika **45** (2009), no. 3, 491–506.
- [15] C. Buşneag and D. Piciu, *The stable topologies for residuated lattices*, Soft Comput. **16** (2012), 1639–1655.
- [16] L. C. Holdon and A. Borumand Saeid, *Regularity in residuated lattices*, Iran. J. Fuzzy Syst. **16** (2019), no. 6, 107–126.
- [17] D. Buşneag, D. Piciu, and J. Paralescu, *Divisible and semi-divisible residuated lattices*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. **LXI** (2015), 287–318.
- [18] P. Hájek, *Mathematics of Fuzzy Logic*, Kluwer Acad. Publ., Dordrecht, 1998.
- [19] L. C. Holdon, L. M. Niţu, and G. Chiriac, *Distributive residuated lattices*, An. Univ. Craiova Ser. Mat. Inform. **39** (2012), 100–109.
- [20] A. Iorgulescu, *Algebras of logic as BCK algebras*, Academy of Economic Studies Bucharest, Romania, 2008.
- [21] D. Piciu, *Algebras of Fuzzy Logic*, Editura Universitaria, Craiova, 2007.
- [22] E. Turunen, *Mathematics Behind Fuzzy logic*, Physica-Verlag Heidelberg, New York, 1999.